Tutorial 2

1 GARP

Bob makes purchases to maximize an increasing utility function and therefore his choices must satisfy GARP. We observe Bob buy $x = (x_1, x_2)$ when prices are $p = (p_1, p_2)$ where $x_1 > 0$ and $x_2 > 0$ and he spends $m = p \cdot x$.

1. I told you that GARP and WARP are equivalent when there are only two goods. Suppose there are three goods and the consumer buys $x^1 = (1, 0, 0)$, $x^2 = (0, 1, 0)$, and $x^3 = (0, 0, 1)$ at prices p^1, p^2, and p^3. Can you come up with price vectors p^1, p^2, and p^3 so that the consumer's behavior satisfies WARP but not GARP?

   Answer. Can take $p^1 = (2, 4, 1)$, $p^2 = (1, 2, 4)$, and $p^3 = (4, 1, 2)$. We have $p^1 \cdot x^1 = p^2 \cdot x^2 = p^3 \cdot x^3 = 2$ and $p^2 \cdot x^1 = p^3 \cdot x^2 = p^1 \cdot x^3 = 1$ and $p^3 \cdot x^1 = p^1 \cdot x^2 = p^2 \cdot x^3 = 4$. So, $x^3 \ P \ x^2 \ P \ x^1 \ P \ x^3$ so GARP is violated. Also, $x^1 \not R \ x^2$, $x^2 \not R \ x^3$, $x^3 \not R \ x^1$.

2. A utility function $U : \mathbb{R}^L_+ \to \mathbb{R}$ is *strictly concave* if

$$U(\alpha x + (1 - \alpha)x') > \alpha U(x) + (1 - \alpha)U(x') \tag{1}$$

   for all $x' \neq x$ and all α taking values strictly between 0 and 1. Show that if U is strictly concave then each budget set $B(p, m)$ can have at most one consumption bundle which maximizes the utility function. In other words, there is some $x \in B(p, m)$ so that $U(x) > U(x')$ for all $x' \in B(p, m)$ so that $x \neq x'$. Hint: suppose there are two bundles x and x' which maximize utility U over all bundles in $B(p, m)$ and show that this contradicts (1).

   Answer. Suppose x and x' are in $B(p, m)$ and they both maximize utility. Let $\bar{x} = \frac{1}{2}x + \frac{1}{2}x'$. Note that $p \cdot \bar{x} \leqslant m$ and so $\bar{x}$ is affordable. Note that $U(\bar{x}) > \frac{1}{2}U(x) + \frac{1}{2}U(x')$ and so neither x nor x' maximizes utility.

3. A consumer satisfies the Strong Axiom of Revealed Preferences (SARP) if $x \ R^* \ x'$ and $x' \ R \ x$ then $x = x'$. Suppose that a consumer maximizes a strictly concave utility

function (that is, for each t we have $U(x^t) \geqslant U(x)$ for all $x \in B(p^t, p^t \cdot x^t)$) and that $x^1 \; R \; x^2 \; R \; x^3 \; R \; x^4 \; R \ldots R \; x^T \; R \; x^1$. Show that $x^1 = x^2 = x^3 = \ldots = x^T$.

Answer. As $x^t \; R \; x^{t+1}$ we have $U(x^t) \geqslant U(x^{t+1})$. Thus, $U(x^1) \geqslant U(x^2) \geqslant U(x^3) \geqslant \ldots \geqslant U(x^T) \geqslant U(x^1)$. Therefore, $U(x^1) = U(x^2) = \ldots = U(x^T)$. As U is strictly concave we see (from previous question) that $x^1 = x^2 = x^3 = \ldots = x^T$.

4. When prices are $p = (p_1, p_2)$ Bob purchases $x = (x_1, x_2)$ where $x_1 > 0$. Let $m = p \cdot x$. Prices change to $p' = (p_1', p_2)$ with $p_1' > p_1$ (note the price of good 2 is unchanged). Bob buys some bundle x' which satisfies $p' \cdot x' = m$. Show that his behavior satisfies GARP.

   Answer. As $p_1' > p_1$ and $x_1 > 0$ we see $p' \cdot x > m$. Therefore, $x' \not R \; x$ and so no matter which x' the consumer buys his behavior still satisfies GARP.

5. As in the previous question suppose that prices change to $p' = (p_1', p_2)$ with $p_1' > p_1$ but suppose that expenditure changes to $m' = p' \cdot x$. Suppose the consumer satisfies GARP. Show that his demand x' at the new price p' must satisfy $p \cdot x' \geqslant p \cdot x$.

   Answer. As $p' \cdot x' = m' = p' \cdot x$ we see that $x' \; R \; x$. So, to satisfy GARP it must be that $x \not P \; x'$. Therefore, $p \cdot x' \geqslant p \cdot x$.

2 Demand example

Consider the utility function

$$U(x) = \sqrt{x_1} + \sqrt{x_2}$$

This U generates the demand function

$$x(p_1, p_2, m) = \frac{m}{p_1 + p_2}\left(\frac{p_2}{p_1}, \frac{p_1}{p_2}\right)$$

1. What consumption bundle does the consumer demand when $p = (1, 1)$ and $m = 1$.

   Answer. $x(1, 1, 1) = (\frac{1}{2}, \frac{1}{2})$.

2. Suppose the price of good 1 rises. Does the quantity demanded for good 2 rise / fall

or cannot say? Hint: you can write the demand for good 2 as

$$x_2(p_1, p_2, m) = \left(\frac{p_1}{p_1 + p_2}\right)\frac{m}{p_2}$$

Answer. Price p_1 rises and so $\frac{p_1}{p_1+p_2}$ rises and therefore x_2 rises.

3. Are the goods normal, inferior, or neither?

   Answer. Normal.

4. The original price and expenditure is $p = (1, 1)$ and $m = 2$. Suppose the price of good 1 rises to 2 and so prices are $p' = (2, 1)$. Calculate the Slutsky substitution and income effects for good 1. That is, calculate

$$\Delta x_1^s \qquad \text{and} \qquad \Delta x_1^m$$

   What can you say about the sign of these terms?

   Answer. $x(p, m) = (1, 1)$, $x(p', m) = (1/3, 4/3)$, $m_s = p' \cdot x(p, m) = 3$, and $x(p', m_s) = (\frac{1}{2}, 2)$. So, $\Delta x_1^s = \frac{1}{2} - 1 = -\frac{1}{2}$ and $\Delta x_1^n = 1/3 - 1/2 = -1/6$.

3 Perfect Complements

Consider the utility function

$$U(x_1, x_2) = \min(x_1, x_2)$$

So, $U(2, 3) = 2$, $U(6, 1) = 1$, $U(100, 3) = 3$ and so on.

1. Draw some indifference curves for this utility function.

   Answer. Pick a point on the 45 degree line. The indifference curve is everything north and east of this point.

2. Find the demand function for this utility function. Hint: demand ought to satisfy $p \cdot x(p, m) = m$ and $x_1(p, m) = x_2(p, m)$.

Answer. $p_1x_1 + p_2x_2 = m$ and $x_1 = x_2$ and so $(p_1 + p_2)x_1 = (p_1 + p_2)x_2 = m$ and so

$$x(p, m) = \left(\frac{m}{p_1 + p_2}, \frac{m}{p_1 + p_2}\right)$$

3. Are the goods normal, inferior, or neither?

   Answer. Normal.

4. The original price and expenditure is $p = (1, 1)$ and $m = 2$. Suppose the price of good 1 rises to 2 and so prices are $p' = (2, 1)$. Calculate the Slutsky substitution and income effects. That is, calculate

   $$\Delta x_1^s \qquad \text{and} \qquad \Delta x_1^m$$

   Answer. $x(p, 2) = (1, 1)$ and $x(p', 2) = (2/3, 2/3)$. Also, $m_s = 3$ and $x(p', 3) = (1, 1)$. And so $\Delta x_1^s = 0$ and $\Delta x_1^n = -1/3$.