HE2001 Microeconomics II

Tutorial 5

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Overview

This tutorial covers:

- taxes and no-arbitrage rates of return;
- contingent consumption, spanning, and portfolio feasibility (with/without short sales);
- state prices and replication pricing;
- insurance as a contingent-claims market restriction and budget geometry;
- expected-utility (EU) maximisation and demand for state-contingent consumption.

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Unofficial Tutorial Solutions

1 Taxes

1.1 After-tax vs pre-tax returns

There are two assets A and B. Asset A is not taxed whereas asset B is taxed on its returns. Let r be the interest rate and τ be the tax rate. Neither asset pays out any money in period 0.

Calculate rates of return $\frac{p_1^A}{p_0^A} - 1$ and $\frac{p_1^B}{p_0^B} - 1$ (where p_t^k is the price of asset k in period t).

Solution

Method 1: No-arbitrage using after-tax returns

Asset A. Asset A has no taxes. Its (gross) return is $\frac{p_1^A}{p_0^A}$, so its net rate of return is

$$R_A = \frac{p_1^A}{p_0^A} - 1.$$

Since r is the (relevant) interest rate, no-arbitrage requires $R_A = r$, hence

$$\frac{p_1^A}{p_0^A} - 1 = r.$$

Asset B. Asset B is taxed on its *returns*. Interpreting this as a proportional tax τ on the net return, the *after-tax* return is

$$(1-\tau)\left(\frac{p_1^B}{p_0^B} - 1\right).$$

No-arbitrage requires the after-tax return equals the interest rate r:

$$(1-\tau)\left(\frac{p_1^B}{p_0^B} - 1\right) = r \quad \Rightarrow \quad \frac{p_1^B}{p_0^B} - 1 = \frac{r}{1-\tau}.$$

$$\boxed{\frac{p_1^A}{p_0^A} - 1 = r, \qquad \frac{p_1^B}{p_0^B} - 1 = \frac{r}{1-\tau}}$$

Method 2: Replication by borrowing/lending and tax wedge

If asset A is untaxed, it can be used as the benchmark risk-free asset (or equivalently saving/borrowing at rate r). Thus A must satisfy $\frac{p_1^A}{p_0^A} = 1 + r$, i.e. $\frac{p_1^A}{p_0^A} - 1 = r$.

For B, an investor who buys B receives a pre-tax net return $\frac{p_1^B}{p_0^B} - 1$ but keeps only a fraction $1-\tau$. Therefore its after-tax return is $(1-\tau)\left(\frac{p_1^B}{p_0^B} - 1\right)$. To avoid arbitrage relative to the risk-free rate r, we must have

$$(1-\tau)\left(\frac{p_1^B}{p_0^B} - 1\right) = r,$$

yielding the same result.

$$\boxed{\frac{p_1^A}{p_0^A} - 1 = r, \qquad \frac{p_1^B}{p_0^B} - 1 = \frac{r}{1-\tau}}$$

1.2 Dividend on untaxed asset

Now, suppose asset A pays out y_0 in period 0. This dividend is un-taxed. Calculate $\frac{p_1^A}{p_0^A} - 1$.

Solution

Method 1: Return definition with a period-0 dividend

Buying A at price p_0^A and receiving an untaxed dividend y_0 in period 0 means the *net* outlay at $t = 0$ is

$$p_0^A - y_0.$$

There is no other payoff until resale at $t = 1$ for p_1^A. Thus the (net) rate of return is

$$r = \frac{p_1^A - (p_0^A - y_0)}{p_0^A - y_0}.$$

Rearrange:

$$p_1^A = (1+r)(p_0^A - y_0).$$

Hence

$$\frac{p_1^A}{p_0^A} - 1 = \frac{(1+r)(p_0^A - y_0)}{p_0^A} - 1 = (1+r)\left(1 - \frac{y_0}{p_0^A}\right) - 1 = r - \frac{(1+r)y_0}{p_0^A}.$$

$$\boxed{\frac{p_1^A}{p_0^A} - 1 = r - \frac{(1+r)y_0}{p_0^A}}$$

Method 2: Present-value identity then convert to the requested return

No-arbitrage implies the time-0 price equals the present value of time-1 resale plus the time-0 dividend (which is already in present value):

$$p_0^A = y_0 + \frac{p_1^A}{1+r} \quad \Rightarrow \quad p_1^A = (1+r)(p_0^A - y_0).$$

Then

$$\frac{p_1^A}{p_0^A} - 1 = \frac{(1+r)(p_0^A - y_0)}{p_0^A} - 1 = r - \frac{(1+r)y_0}{p_0^A},$$

as above.

$$\boxed{\frac{p_1^A}{p_0^A} - 1 = r - \frac{(1+r)y_0}{p_0^A}}$$

1.3 Dividend on taxed asset

Now, suppose asset B pays out y_0 in period 0. This dividend is un-taxed. Calculate $\frac{p_1^B}{p_0^B} - 1$.

Solution

Method 1: After-tax return equals r with untaxed dividend

Dividend y_0 at $t = 0$ is untaxed, so the net outlay is $p_0^B - y_0$. Asset B is taxed on its returns, so the after-tax net return on the net investment is

$$(1-\tau)\left(\frac{p_1^B - (p_0^B - y_0)}{p_0^B - y_0}\right).$$

No-arbitrage requires this after-tax return equals r:

$$(1-\tau)\left(\frac{p_1^B - (p_0^B - y_0)}{p_0^B - y_0}\right) = r.$$

Solve:

$$\frac{p_1^B - (p_0^B - y_0)}{p_0^B - y_0} = \frac{r}{1-\tau} \quad \Rightarrow \quad p_1^B = \left(1 + \frac{r}{1-\tau}\right)(p_0^B - y_0).$$

Therefore

$$\frac{p_1^B}{p_0^B} - 1 = \left(1 + \frac{r}{1-\tau}\right)\left(1 - \frac{y_0}{p_0^B}\right) - 1.$$

$$\boxed{\frac{p_1^B}{p_0^B} - 1 = \left(1 + \frac{r}{1-\tau}\right)\left(1 - \frac{y_0}{p_0^B}\right) - 1}$$

Method 2: Rewrite using pre-tax return wedge

Let the pre-tax rate of return on the net investment be

$$R_B^{\text{pre}} = \frac{p_1^B - (p_0^B - y_0)}{p_0^B - y_0}.$$

Taxation implies after-tax return $R_B^{\text{aft}} = (1-\tau)R_B^{\text{pre}}$. No-arbitrage gives $R_B^{\text{aft}} = r$, hence

$$R_B^{\text{pre}} = \frac{r}{1-\tau}.$$

Thus $p_1^B = (1 + R_B^{\text{pre}})(p_0^B - y_0) = \left(1 + \frac{r}{1-\tau}\right)(p_0^B - y_0)$, and

$$\frac{p_1^B}{p_0^B} - 1 = \left(1 + \frac{r}{1-\tau}\right)\left(1 - \frac{y_0}{p_0^B}\right) - 1.$$

$$\boxed{\frac{p_1^B}{p_0^B} - 1 = \left(1 + \frac{r}{1-\tau}\right)\left(1 - \frac{y_0}{p_0^B}\right) - 1}$$

2 Contingent Consumption and Assets

Suppose there are two equally likely states: $S = 2$ and $\pi = \left(\frac{1}{2}, \frac{1}{2}\right)$.

2.1 Expected values in two states

What is the expected value of $x = (5, 3)$ and $x^1 = (2, 8)$?

Solution

Method 1: Direct computation

$$\mathbb{E}[x] = \frac{1}{2} \cdot 5 + \frac{1}{2} \cdot 3 = \frac{8}{2} = 4, \qquad \mathbb{E}[x^1] = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 8 = \frac{10}{2} = 5.$$

$$\boxed{\mathbb{E}[x] = 4, \qquad \mathbb{E}[x^1] = 5}$$

Method 2: Use the midpoint property under equal probabilities

With two equally likely states, the expectation is the midpoint of the two state values:

$$\mathbb{E}[(a, b)] = \frac{a + b}{2}.$$

So $\mathbb{E}[(5, 3)] = (5 + 3)/2 = 4$ and $\mathbb{E}[(2, 8)] = (2 + 8)/2 = 5$.

$$\boxed{\mathbb{E}[x] = 4, \qquad \mathbb{E}[x^1] = 5}$$

2.2 Affordability with state prices

Suppose can buy contingent consumption directly at prices $p = (1, 4)$. Suppose the consumer can spend 6. Can the consumer afford $(2, 1)$? What about $(1, 2)$?

Solution

Method 1: Compute costs

Cost of $(2, 1)$:

$$p \cdot (2, 1) = 1 \cdot 2 + 4 \cdot 1 = 6 \leq 6 \quad \Rightarrow \quad \text{affordable.}$$

Cost of $(1, 2)$:

$$p \cdot (1, 2) = 1 \cdot 1 + 4 \cdot 2 = 9 > 6 \quad \Rightarrow \quad \text{not affordable.}$$

$$\boxed{(2, 1) \text{ is affordable; } (1, 2) \text{ is not affordable.}}$$

Method 2: Budget-line inequality

Affordable set is $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 + 4x_2 \leq 6\}$. Check:

$$2 + 4 \cdot 1 = 6 \leq 6 \Rightarrow (2,1) \in B, \qquad 1 + 4 \cdot 2 = 9 > 6 \Rightarrow (1,2) \notin B.$$

$\boxed{(2,1) \text{ is affordable; } (1,2) \text{ is not affordable.}}$

2.3 Portfolio feasibility without short sales

Now, suppose you cannot buy contingent consumption directly. Suppose there are two assets: a risk free asset $a^f = (1,1)$ and a risky asset $a^r = (4,1)$. The risk free asset costs 1 and the risky asset costs 2. Suppose you cannot short and suppose the consumer has 3 to spend. Can the consumer afford the contingent consumption bundle $(5,2)$? What about $(4,4)$? What about $(2,3)$?

Solution

Let holdings be $\theta = (\theta_f, \theta_r)$ with $\theta_f \geq 0, \theta_r \geq 0$ (no shorting). Cost constraint:

$$\theta_f + 2\theta_r \leq 3.$$

Delivered consumption:

$$\theta_f a^f + \theta_r a^r = (\theta_f + 4\theta_r,\ \theta_f + \theta_r).$$

Method 1: Solve replication system and check constraints

Target $(5,2)$**.** Solve

$$\theta_f + 4\theta_r = 5, \qquad \theta_f + \theta_r = 2.$$

Subtract: $3\theta_r = 3 \Rightarrow \theta_r = 1$, hence $\theta_f = 1$. Cost $= 1 + 2 = 3 \leq 3$ and nonnegativity holds, so affordable.

Target $(4,4)$**.** Solve

$$\theta_f + 4\theta_r = 4, \qquad \theta_f + \theta_r = 4.$$

Subtract: $3\theta_r = 0 \Rightarrow \theta_r = 0$, hence $\theta_f = 4$. Cost $= 4 > 3$, not affordable.

Target $(2,3)$**.** Solve

$$\theta_f + 4\theta_r = 2, \qquad \theta_f + \theta_r = 3.$$

Subtract: $3\theta_r = -1 \Rightarrow \theta_r = -\frac{1}{3}$, which violates $\theta_r \geq 0$. Not affordable.

$\boxed{(5,2) \text{ affordable; } (4,4) \text{ not affordable; } (2,3) \text{ not affordable.}}$

Method 2: Feasible set geometry in payoff space

Payoff set from $\theta_f, \theta_r \geq 0$ with $\theta_f + 2\theta_r \leq 3$ is the convex hull of the payoffs at the budget corners:

$$(\theta_f, \theta_r) = (0,0) \Rightarrow (0,0), \quad (3,0) \Rightarrow (3,3), \quad (0,1.5) \Rightarrow (6,1.5).$$

Thus feasible contingent consumption bundles are all convex combinations of $(0,0), (3,3), (6,1.5)$.

Check:

- $(5,2)$ lies on the segment between $(3,3)$ and $(6,1.5)$: indeed $(5,2) = (3,3) + \frac{2}{3}(3,-1.5)$, corresponding to $(\theta_f, \theta_r) = (1,1)$.
- $(4,4)$ has second component 4, but the feasible set has $x_2 \leq 3$ when $\theta_f + 2\theta_r \leq 3$ and $\theta \geq 0$ (since $x_2 = \theta_f + \theta_r \leq \theta_f + 2\theta_r \leq 3$). So impossible.
- $(2,3)$ requires $x_2 = 3$ but $x_1 = 2$; with $\theta_f, \theta_r \geq 0$, $x_1 = \theta_f + 4\theta_r \geq \theta_f + \theta_r = x_2$, so $x_1 < x_2$ is impossible. Hence not affordable.

$$\boxed{(5,2) \text{ affordable; } (4,4) \text{ not affordable; } (2,3) \text{ not affordable.}}$$

2.4 State prices from spanning with short sales

Now, suppose that the assets can be held short. Find the state price vector $p = (p_1, p_2)$. That is, p is the price of contingent consumption so that, for any $\theta = (\theta_f, \theta_r)$ the price of θ is the same as the price of the contingent consumption delivered by θ:

$$\theta_f + 2\theta_r = p \cdot \left(\theta_f a^f + \theta_r a^r\right), \quad \text{for all } \theta_f, \theta_r.$$

Hint: take $\theta = (1,0)$ and $\theta = (0,1)$ and solve for (p_1, p_2). This is two equations and two unknowns.

Solution

Method 1: Use the hint $\theta = (1,0)$ and $\theta = (0,1)$

For $\theta = (1,0)$, LHS price is 1. Delivered bundle is $a^f = (1,1)$, so

$$1 = p \cdot (1,1) = p_1 + p_2. \tag{A}$$

For $\theta = (0,1)$, LHS price is 2. Delivered bundle is $a^r = (4,1)$, so

$$2 = p \cdot (4,1) = 4p_1 + p_2. \tag{B}$$

Subtract (A) from (B): $3p_1 = 1 \Rightarrow p_1 = \frac{1}{3}$. Then $p_2 = 1 - p_1 = \frac{2}{3}$.

$$\boxed{p = \left(\frac{1}{3}, \frac{2}{3}\right)}$$

Method 2: Coefficient matching for all θ_f, θ_r

We require for all $(\theta_f, \theta_r) \in \mathbb{R}^2$:

$$\theta_f + 2\theta_r = p \cdot \big(\theta_f(1,1) + \theta_r(4,1)\big) = \theta_f(p_1 + p_2) + \theta_r(4p_1 + p_2).$$

Matching coefficients of θ_f and θ_r gives the same system:

$$p_1 + p_2 = 1, \qquad 4p_1 + p_2 = 2,$$

so $p_1 = \frac{1}{3}$, $p_2 = \frac{2}{3}$.

$$\boxed{p = \left(\frac{1}{3}, \frac{2}{3}\right)}$$

2.5 Replicating a target bundle and pricing

Suppose you want to buy contingent consumption $(6, 3)$. How much of each asset do you need to purchase? How much does this cost? How much does $(6, 3)$ cost according to the state prices $p = (p_1, p_2)$ you found in the previous question?

Solution

Method 1: Replication by solving for θ_f, θ_r

We need

$$\theta_f + 4\theta_r = 6, \qquad \theta_f + \theta_r = 3.$$

Subtract: $3\theta_r = 3 \Rightarrow \theta_r = 1$. Then $\theta_f = 2$.

Portfolio cost:

$$\theta_f + 2\theta_r = 2 + 2 = 4.$$

State-price cost using $p = (1/3, 2/3)$:

$$p \cdot (6, 3) = \frac{1}{3} \cdot 6 + \frac{2}{3} \cdot 3 = 2 + 2 = 4.$$

$$\boxed{\theta_f = 2,\ \theta_r = 1;\ \text{cost} = 4;\ \text{state-price cost} = 4.}$$

Method 2: Use spanning/pricing identity directly

Since state prices satisfy

$$\theta_f + 2\theta_r = p \cdot (\theta_f a^f + \theta_r a^r),$$

any replicable bundle is priced by p. If $(6, 3)$ is replicated by (θ_f, θ_r), then its cost equals $p \cdot (6, 3)$. Compute

$$p \cdot (6, 3) = \frac{1}{3} \cdot 6 + \frac{2}{3} \cdot 3 = 4.$$

To find holdings, solve the payoff system:

$$\theta_f + \theta_r = 3,\ \theta_f + 4\theta_r = 6 \Rightarrow \theta_r = 1,\ \theta_f = 2,$$

and the portfolio cost is $2 + 2 = 4$, matching $p \cdot (6, 3)$.

$$\boxed{\theta_f = 2,\ \theta_r = 1;\ \text{cost} = 4.}$$

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3 Contingent Consumption and Insurance

Suppose there are two states where the first state has probability $\frac{1}{3}$: $S = 2$ and $\pi = \left(\frac{1}{3}, \frac{2}{3}\right)$.

3.1 Expected values with unequal probabilities

What is the expected value of $x = (6, 3)$ and $x^1 = (9, 12)$.

Solution

Method 1: Direct computation

$$\mathbb{E}[x] = \frac{1}{3} \cdot 6 + \frac{2}{3} \cdot 3 = 2 + 2 = 4, \qquad \mathbb{E}[x^1] = \frac{1}{3} \cdot 9 + \frac{2}{3} \cdot 12 = 3 + 8 = 11.$$

$$\boxed{\mathbb{E}[x] = 4, \qquad \mathbb{E}[x^1] = 11}$$

Method 2: Weighted-average form

For $\pi = (1/3, 2/3)$,

$$\mathbb{E}[(a, b)] = \frac{a + 2b}{3}.$$

So $\mathbb{E}[(6, 3)] = (6 + 2 \cdot 3)/3 = 12/3 = 4$ and $\mathbb{E}[(9, 12)] = (9 + 2 \cdot 12)/3 = 33/3 = 11$.

$$\boxed{\mathbb{E}[x] = 4, \qquad \mathbb{E}[x^1] = 11}$$

3.2 Endowment vs expected value under concave utility

Suppose the consumer has utility function $U(x_1, x_2) = \frac{1}{3}u(x_1) + \frac{2}{3}u(x_2)$ where $u(x) = \sqrt{x}$ and suppose the consumer is endowed with $\omega = (0, 9)$. Does the consumer prefer his endowment or its expected value: $x = (6, 6)$?

Solution

Method 1: Compute expected utility exactly

Compute EU of endowment:

$$U(\omega) = \frac{1}{3}\sqrt{0} + \frac{2}{3}\sqrt{9} = 0 + \frac{2}{3} \cdot 3 = 2.$$

Compute EU of $x = (6, 6)$:

$$U(x) = \frac{1}{3}\sqrt{6} + \frac{2}{3}\sqrt{6} = \sqrt{6}.$$

Since $\sqrt{6} > 2$ (square both sides: $6 > 4$), we have $U(x) > U(\omega)$.

$$\boxed{\text{The consumer prefers } (6, 6) \text{ to } \omega = (0, 9).}$$

Method 2: Jensen's inequality intuition plus verification

Because $u(x) = \sqrt{x}$ is strictly concave, equalising consumption across states increases expected utility relative to a spread, holding mean fixed. Here $(6, 6)$ is fully equalised and the endowment is highly unequal. A direct check confirms:

$$U(6, 6) = \sqrt{6} > 2 = U(0, 9).$$

Prefer the expected value bundle $(6, 6)$.

3.3 Affordable safe bundle using insurance

Suppose the consumer can buy insurance. Specifically, can buy consumption in state 1 at the cost of one unit of consumption in state 2. That is, insurance is an asset $(1, -1)$. Find the safe bundle (k, k) (for some value of k) that the consumer can afford. Does the consumer prefer this bundle or the endowment?

Solution

Starting from $\omega = (0, 9)$, buying θ units of insurance $(1, -1)$ yields

$$x(\theta) = (\omega_1 + \theta,\ \omega_2 - \theta) = (\theta,\ 9 - \theta).$$

A safe bundle satisfies $\theta = 9 - \theta \Rightarrow \theta = \frac{9}{2}$, hence

$$k = \theta = \frac{9}{2}.$$

So the affordable safe bundle is $\left(\frac{9}{2}, \frac{9}{2}\right)$.

Now compare utilities:

$$U\left(\frac{9}{2}, \frac{9}{2}\right) = \frac{1}{3}\sqrt{\frac{9}{2}} + \frac{2}{3}\sqrt{\frac{9}{2}} = \sqrt{\frac{9}{2}} = \frac{3}{\sqrt{2}},$$

$$U(\omega) = 2.$$

Since $\frac{3}{\sqrt{2}} > 2$ (square both sides: $9/2 > 4 \iff 9 > 8$), the safe bundle is preferred.

$$k = \frac{9}{2},\ \text{so the safe bundle is } \left(\frac{9}{2}, \frac{9}{2}\right),\ \text{and it is preferred to } \omega.$$

Method 2: Feasible set midpoint and concavity

The budget set generated by $(1, -1)$ from ω is the line $x_1 + x_2 = 9$. The safe bundle is the midpoint of the segment where $x_1 = x_2$, hence $k = 9/2$. Since u is strictly concave, equalisation increases utility, so the safe bundle dominates ω; the exact inequality check above confirms.

$$k = \frac{9}{2} \text{ and the safe bundle is preferred.}$$

3.4 Budget line with insurance

Draw a picture of all the contingent consumption bundles which are affordable (i.e. draw the budget line).

Solution

Method 1: Parametric form and intercepts

Affordable bundles are $x(\theta) = (\theta, 9 - \theta)$. Eliminating θ gives

$$x_1 + x_2 = 9.$$

If we restrict to nonnegative consumption, then $x_1 \in [0, 9]$ and $x_2 = 9 - x_1$, with intercepts $(0, 9)$ and $(9, 0)$.

Budget line: $x_2 = 9 - x_1$ (in $\mathbb{R}^2_+$, the segment from $(0, 9)$ to $(9, 0)$).

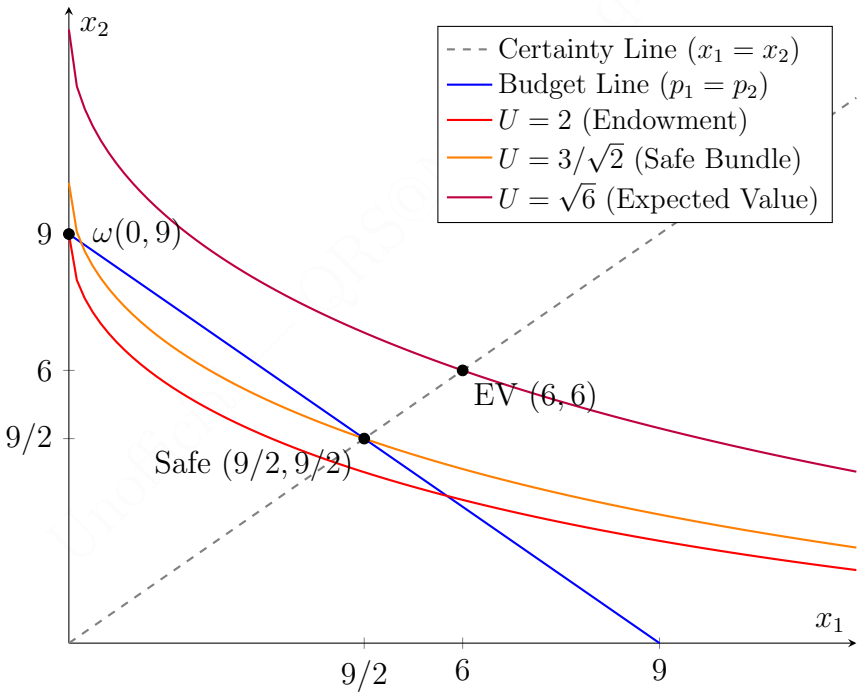


Figure 1: Affordable contingent consumption bundles, the certainty line, and indifference curves illustrating the preference for insurance and expected value over the endowment.

Method 2: Vector-geometry description

The endowment is $\omega = (0, 9)$ and the trade direction is $(1, -1)$, so the affine line of attainable bundles is

$$\omega + \text{span}\{(1, -1)\} = \{(0, 9) + \theta(1, -1) : \theta \in \mathbb{R}\},$$

i.e. $x_1 + x_2 = 9$. Restricting to $\mathbb{R}^2_+$ gives the segment between the axis intercepts.

$$\boxed{x_1 + x_2 = 9 \text{ (restricted to nonnegative quadrant).}}$$

3.5 Price vectors constant on the budget line

Can you find a price vector $p = (p_1, p_2)$ so that the price of a contingent consumption x is the same for every affordable contingent consumption bundle:

$$p \cdot \omega = p \cdot (\omega_1 + \theta, \omega_2 - \theta), \quad \text{for all } \theta?$$

Note that there is more than one price vector p which satisfies this equality.

Solution

We need $p \cdot (\omega_1 + \theta, \omega_2 - \theta)$ to be independent of θ.

Method 1: Coefficient of θ must be zero

Compute:

$$p \cdot (\omega_1 + \theta, \omega_2 - \theta) = p_1(\omega_1 + \theta) + p_2(\omega_2 - \theta) = p_1\omega_1 + p_2\omega_2 + \theta(p_1 - p_2).$$

For this to equal $p \cdot \omega = p_1\omega_1 + p_2\omega_2$ for all θ, we require

$$p_1 - p_2 = 0 \quad \Rightarrow \quad p_1 = p_2.$$

Thus any vector of the form $p = (c, c)$ (with $c \neq 0$, typically $c > 0$) works.

$$\boxed{\text{All solutions are } p = (c, c) \text{ for any } c \neq 0 \text{ (e.g. } p = (1, 1)).}$$

Method 2: Normal vector to the budget line

The affordable set lies on the line $x_1 + x_2 = 9$, whose direction vector is $(1, -1)$. A price vector that assigns the same value to all points on the line must be orthogonal to the direction, i.e.

$$p \cdot (1, -1) = 0 \Rightarrow p_1 - p_2 = 0 \Rightarrow p_1 = p_2.$$

So $p = (c, c)$ for any $c \neq 0$.

$$\boxed{p_1 = p_2 \text{ (e.g. } p = (1, 1)).}$$

4 EU maximization

Suppose there are two states ($S = 2$) and that $\pi = \left(\frac{1}{3}, \frac{2}{3}\right)$. Suppose utility takes the form

$$U(x_1, x_2) = \frac{1}{3}u(x_1) + \frac{2}{3}u(x_2)$$

where $u(x) = \sqrt{x}$. We note that $u'(x) = x^{-1/2}$.

4.1 Marginal rate of substitution

Find $MRS_{1,2}$?

Solution

Method 1: Ratio of marginal utilities

$$\frac{\partial U}{\partial x_1} = \frac{1}{3}u'(x_1) = \frac{1}{3}x_1^{-1/2}, \qquad \frac{\partial U}{\partial x_2} = \frac{2}{3}u'(x_2) = \frac{2}{3}x_2^{-1/2}.$$

Thus

$$MRS_{1,2} = \frac{\partial U/\partial x_1}{\partial U/\partial x_2} = \frac{\frac{1}{3}x_1^{-1/2}}{\frac{2}{3}x_2^{-1/2}} = \frac{1}{2}\cdot\frac{x_2^{-1/2}}{x_1^{-1/2}} = \frac{1}{2}\sqrt{\frac{x_2}{x_1}}.$$

$$\boxed{MRS_{1,2} = \frac{1}{2}\sqrt{\frac{x_2}{x_1}}}$$

Method 2: General formula with probabilities

For EU, $MRS_{1,2} = \frac{\pi_1 u'(x_1)}{\pi_2 u'(x_2)}$. Here $\pi_1/\pi_2 = (1/3)/(2/3) = 1/2$, so

$$MRS_{1,2} = \frac{1}{2}\cdot\frac{u'(x_1)}{u'(x_2)} = \frac{1}{2}\cdot\frac{x_1^{-1/2}}{x_2^{-1/2}} = \frac{1}{2}\sqrt{\frac{x_2}{x_1}}.$$

$$\boxed{MRS_{1,2} = \frac{1}{2}\sqrt{\frac{x_2}{x_1}}}$$

4.2 Demand with state prices

Find the demand function for the consumer. That is, suppose you can buy contingent consumption with price vector $p = (p_1, p_2)$ and expenditure level m. Find how much contingent consumption is demanded for each price vector p and each m.

Solution

We maximise $U(x_1, x_2)$ subject to $p_1x_1 + p_2x_2 = m$, with $x_1, x_2 > 0$.

Method 1: Lagrangian

Lagrangian:

$$\mathcal{L} = \frac{1}{3}\sqrt{x_1} + \frac{2}{3}\sqrt{x_2} + \lambda(m - p_1x_1 - p_2x_2).$$

FOCs:

$$\frac{1}{3}\cdot\frac{1}{2\sqrt{x_1}} - \lambda p_1 = 0, \qquad \frac{2}{3}\cdot\frac{1}{2\sqrt{x_2}} - \lambda p_2 = 0.$$

Equivalently,

$$\frac{1}{6\sqrt{x_1}} = \lambda p_1, \qquad \frac{1}{3\sqrt{x_2}} = \lambda p_2.$$

Divide the first by the second:

$$\frac{1}{6\sqrt{x_1}}\Big/\frac{1}{3\sqrt{x_2}} = \frac{\sqrt{x_2}}{2\sqrt{x_1}} = \frac{p_1}{p_2} \quad\Rightarrow\quad \sqrt{\frac{x_2}{x_1}} = \frac{2p_1}{p_2} \quad\Rightarrow\quad x_2 = x_1\cdot\frac{4p_1^2}{p_2^2}.$$

Use the budget:

$$p_1x_1 + p_2x_2 = p_1x_1 + p_2x_1\cdot\frac{4p_1^2}{p_2^2} = x_1\left(p_1 + \frac{4p_1^2}{p_2}\right) = m.$$

Hence

$$x_1(p,m) = \frac{m}{p_1 + \frac{4p_1^2}{p_2}} = \frac{mp_2}{p_1p_2 + 4p_1^2} = \frac{mp_2}{p_1(p_2 + 4p_1)}.$$

Then

$$x_2(p,m) = \frac{4p_1^2}{p_2^2}\,x_1 = \frac{4p_1^2}{p_2^2}\cdot\frac{mp_2}{p_1(p_2 + 4p_1)} = \frac{4p_1m}{p_2(p_2 + 4p_1)}.$$

$$\boxed{x_1(p,m) = \frac{mp_2}{p_1(p_2 + 4p_1)}, \qquad x_2(p,m) = \frac{4p_1m}{p_2(p_2 + 4p_1)}}$$

Solutions

Method 2: Use MRS = price ratio

Optimality requires

$$MRS_{1,2} = \frac{p_1}{p_2}.$$

From Question 4.1,

$$\frac{1}{2}\sqrt{\frac{x_2}{x_1}} = \frac{p_1}{p_2} \quad \Rightarrow \quad x_2 = x_1 \cdot \frac{4p_1^2}{p_2^2}.$$

Substitute into the budget to solve x_1, then obtain x_2, yielding the same demand functions as Method 1.

$$\boxed{x_1(p,m) = \frac{mp_2}{p_1(p_2+4p_1)}, \qquad x_2(p,m) = \frac{4p_1 m}{p_2(p_2+4p_1)}}$$

4.3 Demand with given endowment and prices

Suppose the consumer has endowment $\omega = (0, 9)$ and faces prices $p = (2, 1)$. What contingent consumption bundle will the consumer buy?

Solution

Expenditure equals the value of endowment:

$$m = p \cdot \omega = 2 \cdot 0 + 1 \cdot 9 = 9.$$

Use the demand from Question 4.2 with $p_1 = 2$, $p_2 = 1$, $m = 9$:

$$x_1 = \frac{mp_2}{p_1(p_2+4p_1)} = \frac{9 \cdot 1}{2(1+8)} = \frac{9}{18} = \frac{1}{2},$$

$$x_2 = \frac{4p_1 m}{p_2(p_2+4p_1)} = \frac{4 \cdot 2 \cdot 9}{1(1+8)} = \frac{72}{9} = 8.$$

$$\boxed{x = \left(\frac{1}{2}, 8\right)}$$

Method 2: Use MRS condition then the budget

At optimum,

$$\frac{1}{2}\sqrt{\frac{x_2}{x_1}} = \frac{p_1}{p_2} = \frac{2}{1} = 2 \quad \Rightarrow \quad \sqrt{\frac{x_2}{x_1}} = 4 \quad \Rightarrow \quad x_2 = 16x_1.$$

Budget is $2x_1 + 1 \cdot x_2 = m = 9$, so $2x_1 + 16x_1 = 18x_1 = 9 \Rightarrow x_1 = 1/2$, hence $x_2 = 8$.

$$\boxed{x = \left(\frac{1}{2}, 8\right)}$$

4.4 Insurance-only trading

Continue to assume that the consumer is endowed with $(0, 9)$ but now the consumer cannot purchase contingent consumption directly. Instead he can only buy insurance. Each unit of insurance costs two units of consumption from state 2 and delivers 1 unit of consumption in state 1. Draw the budget line of all consumption bundles which the consumer can afford. How much contingent consumption will the consumer demand? Hint: use the answer from the previous question.

Solution

One unit of insurance transforms consumption by $(+1, -2)$. Starting from $\omega = (0, 9)$, buying θ units yields

$$x(\theta) = (\theta,\ 9 - 2\theta).$$

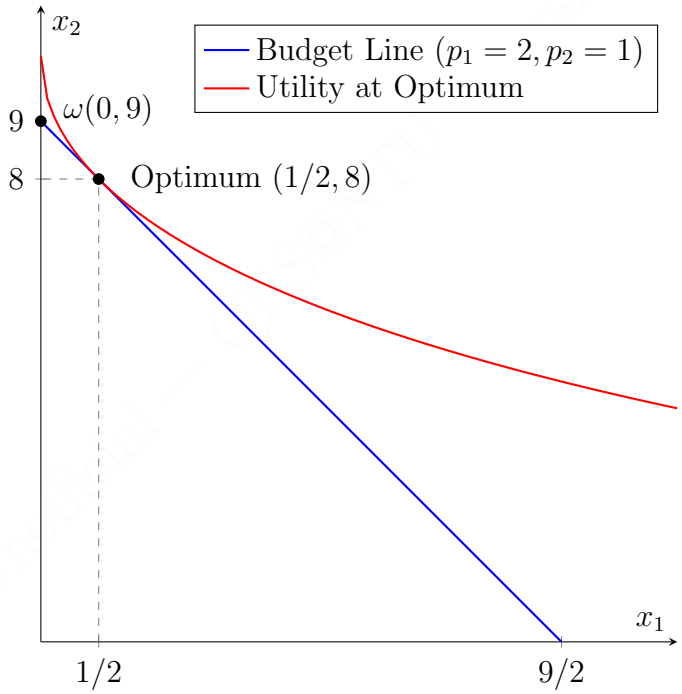


Figure 2: Expected utility maximization with state prices $p = (2, 1)$ and endowment $\omega = (0, 9)$. The consumer trades along the budget line to reach the optimal bundle $(1/2, 8)$.

Method 1: Budget line equation and feasibility; then use the hint

Eliminate θ: since $x_1 = \theta$, we get

$$x_2 = 9 - 2x_1.$$

If we restrict to nonnegative consumption and no short positions in insurance, then $\theta \geq 0$ and $9 - 2\theta \geq 0$, so

$$0 \leq x_1 \leq \frac{9}{2}.$$

Thus the affordable set is the line segment on $x_2 = 9 - 2x_1$ between $(0, 9)$ and $(9/2, 0)$.

Using the hint (Question 4.3), the optimal contingent bundle under prices $(2, 1)$ was $\left(\frac{1}{2}, 8\right)$. Check feasibility on the insurance line:

$$8 = 9 - 2 \cdot \frac{1}{2} = 8,$$

so $\left(\frac{1}{2}, 8\right)$ is on the budget line and hence affordable with $\theta = \frac{1}{2}$. Therefore the insurance-only demand is the same bundle:

Budget line: $x_2 = 9 - 2x_1$ $(0 \leq x_1 \leq 9/2)$, Demand: $\left(\frac{1}{2}, 8\right)$.

Method 2: Interpret insurance as imposing a fixed state-price ratio

Insurance gives up 2 units in state 2 for 1 unit in state 1, so the *shadow relative price* is $p_1/p_2 = 2$. Thus the insurance-only opportunity set corresponds exactly to trading at prices proportional to $(2, 1)$ through the endowment $(0, 9)$, which is precisely the environment in Question 4.3. Hence the chosen bundle is again $\left(\frac{1}{2}, 8\right)$, and the line is $x_2 = 9 - 2x_1$.

Budget line: $x_2 = 9 - 2x_1$, Demand: $\left(\frac{1}{2}, 8\right)$.