HE2001 Microeconomics II

Tutorial 6 / Mock Exam

Academic Year 2025/2026, Semester 2

Quantitative Research Society @NTU

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Overview

This mock exam reviews core tools from consumer theory and asset pricing:

- revealed preference and WARP (direct and strict revealed preference);
- normal goods, cross-price effects, and ambiguity of comparative statics;
- labor supply: substitution vs income effects;
- present value and no-arbitrage decisions;
- first-order stochastic dominance (FOSD);
- Cobb–Douglas demands and Slutsky decomposition;
- present value of multi-period assets and perpetuities; no-arbitrage rates of return.

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Part I: Short Questions

Question 1 (Revealed preference)

Problem

Bob maximizes an increasing utility function U. Bob buys $x = (5, 5)$ when prices are $p = (1, 1)$. Is the bundle $(8, 2)$ revealed preferred to the bundle $(1, 6)$?

Solution

With only one observation (p, x), define the *(direct) revealed preference relation* R by

$$xRy \quad \Longleftrightarrow \quad p \cdot y \leq p \cdot x,$$

i.e. y is affordable at prices p with expenditure $m = p \cdot x$.

Here $m = p \cdot x = 1 \cdot 5 + 1 \cdot 5 = 10$.

Method 1: Definition-based (what is actually revealed?)

To conclude "$(8, 2)$ is revealed preferred to $(1, 6)$", we would need an observation where $(8, 2)$ is *chosen* and $(1, 6)$ is affordable at that same observation. But $(8, 2)$ is *not* observed as a chosen bundle; the only observed chosen bundle is $(5, 5)$.

From the single observation, we can infer only statements of the form:

$$(5, 5)Ry \quad \text{for every } y \text{ with } p \cdot y \leq 10.$$

Thus the data reveal $(5, 5)$ is (weakly) preferred to both $(8, 2)$ (since $p \cdot (8, 2) = 10$) and $(1, 6)$ (since $p \cdot (1, 6) = 7$), but the data reveal *nothing* about the preference ordering between $(8, 2)$ and $(1, 6)$.

No: from the given observation, we cannot conclude that $(8, 2)$ is revealed preferred to $(1, 6)$.

Method 2: Construct two increasing preferences consistent with the data but giving opposite rankings

We show the ranking between $(8, 2)$ and $(1, 6)$ is not identified even under increasing preferences.

Preference 1 (increasing, favors good 1). Let $U_1(x_1, x_2) = x_1 + 0.01x_2$ (strictly increasing). Then

$$U_1(8, 2) = 8 + 0.02 = 8.02, \qquad U_1(1, 6) = 1 + 0.06 = 1.06,$$

so $(8, 2) \succ_1 (1, 6)$.

Preference 2 (increasing, favors good 2). Let $U_2(x_1, x_2) = 0.01x_1 + x_2$ (strictly increasing). Then

$$U_2(8, 2) = 0.08 + 2 = 2.08, \qquad U_2(1, 6) = 0.01 + 6 = 6.01,$$

so $(1, 6) \succ_2 (8, 2)$.

Both utilities are increasing, and the single observation $x = (5, 5)$ at $p = (1, 1)$ does not force a unique ranking between $(8, 2)$ and $(1, 6)$. Therefore no revealed-preference conclusion between these two bundles is warranted.

Not enough information: the data do not reveal whether $(8, 2)R(1, 6)$ or vice versa.

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Question 2 (Normal goods and a cross-price change)

Problem

There are two goods ($L = 2$) and both goods are normal. The price of good 1 increases. What can we conclude about the change in demand for good 2 (increase, decrease, or cannot say)?

Solution

Method 1: Slutsky decomposition (substitution vs income effect)

Let Marshallian demand be $x(p, m)$. Consider an increase in p_1 holding p_2, m fixed. For good 2, the total change can be decomposed into:

$$\Delta x_2 = \underbrace{\Delta x_2^s}_{\text{substitution effect}} + \underbrace{\Delta x_2^n}_{\text{income effect}} .$$

- The substitution effect Δx_2^s can be positive or negative depending on whether goods are net substitutes or net complements.
- If goods are normal, a rise in p_1 reduces real purchasing power (with m fixed), making the income effect for good 2 *non-positive*:

$$\Delta x_2^n \leq 0.$$

Since Δx_2^s has ambiguous sign, the sum is ambiguous.

Cannot say: x_2 may increase or decrease depending on substitution vs income effects.

Method 2: Counterexamples with normal goods

We exhibit two standard preference classes where both goods are normal but the cross-price effect differs.

Case A: Substitutes (cross-price effect positive). For Cobb–Douglas $U = x_1^\alpha x_2^{1-\alpha}$ ($\alpha \in (0, 1)$), both goods are normal and

$$x_2(p, m) = \frac{(1-\alpha)m}{p_2},$$

which does not decrease with p_1; in fact the substitution effect tends to raise x_2 when p_1 rises.

Case B: Complements (cross-price effect negative). For Leontief $U = \min\{x_1, x_2\}$, both goods are normal and demands satisfy $x_1 = x_2 = m/(p_1 + p_2)$, so as p_1 rises, x_2 *falls*.

Thus even with normality, the sign is not pinned down.

Cannot say.

Question 3 (Labor supply with normal consumption and leisure)

Problem

Bob supplies 60 hours/week of labor. Bob gets utility from consumption and leisure, which are both normal goods. Bob's wage rises from \$10/hour to \$20/hour. What can we conclude about Bob's labor supply (increase, decrease, or cannot say)?

Solution

Let total time endowment be T, labor be ℓ, and leisure be $L = T - \ell$. Consumption equals labor income (for simplicity): $c = w\ell$. Utility $U(c, L)$ with both c and L normal.

Method 1: Substitution vs income effect

A wage increase has two forces:

- **Substitution effect:** leisure becomes more expensive (higher opportunity cost), so the consumer tends to substitute away from leisure toward labor:

$$\ell \text{ tends to increase.}$$

- **Income effect:** higher wage increases potential income for any given ℓ; with leisure normal, higher real income increases leisure L, which reduces labor $\ell = T - L$:

$$\ell \text{ tends to decrease.}$$

Because these effects work in opposite directions, the net change is ambiguous.

$$\boxed{\text{Cannot say: labor supply may increase or decrease.}}$$

Method 2: Provide two standard examples with opposite responses

Example where labor rises. Take preferences with weak income effect relative to substitution (e.g. quasi-linear in consumption), so the substitution effect dominates: wage up $\Rightarrow$ labor up.

Example where labor falls (backward-bending supply). With sufficiently strong income effects (e.g. Cobb–Douglas over c and L and large initial income), the income effect can dominate: wage up $\Rightarrow$ leisure up $\Rightarrow$ labor down.

Hence no definite prediction follows from normality alone.

$$\boxed{\text{Cannot say.}}$$

Question 4 (Present value decision)

Problem

The interest rate is $r = 0.1$. There are two periods $t = 0, 1$. Farmer Ann's cow produces \$0 today and \$109 next period, then dies. You have \$100 today you want to save. Ann offers to sell you the cow for \$100 today. Should you buy the cow (yes/no/not enough information)? How do you maximize the present value?

Solution

Method 1: Compare present values

If you keep your \$100 and save at interest rate $r = 0.1$, then next period you have

$$100(1 + r) = 100(1.1) = 110.$$

If you buy the cow for \$100 today, you receive \$109 next period. Compare next-period payoffs: $109 < 110$, so saving dominates buying the cow.

Equivalently compare present value (PV) at $t = 0$:

$$PV(\text{cow}) = \frac{109}{1.1} = \frac{1090}{11} \approx 99.09 < 100 = PV(\text{keep cash}).$$

No. Do not buy the cow; save the \$100 at interest rate 0.1.

Method 2: No-arbitrage / dominance argument

If a risk-free investment of \$100 yields \$110 at $t = 1$, any alternative that costs \$100 at $t = 0$ must deliver $\geq$ \$110 at $t = 1$ to be weakly preferred (since there is no other benefit/cost stated). The cow delivers only \$109 at $t = 1$, hence it is strictly dominated by saving.

No: the cow is dominated by the risk-free saving technology.

Question 5 (First-order stochastic dominance)

Problem

There are 3 states with probabilities $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Consider contingent consumption bundles

$$x = (1, 3, 3), \qquad x' = (5, 1, 2).$$

Does either bundle first-order stochastically dominate the other?

Solution

Treat each bundle as a random variable taking the three listed values with equal probability.

Method 1: Compare cumulative distribution functions (CDFs)

Let X take values $\{1, 3, 3\}$ and X' take values $\{5, 1, 2\}$, each with probability 1/3. Sort outcomes:

$$X: \ 1, 3, 3; \qquad X': \ 1, 2, 5.$$

Compute CDFs $F_X(t) = \mathbb{P}(X \leq t)$, $F_{X'}(t) = \mathbb{P}(X' \leq t)$:

- For $1 \leq t < 2$: $F_X(t) = 1/3$, $F_{X'}(t) = 1/3$ (tie).
- For $2 \leq t < 3$: $F_X(t) = 1/3$ (only the 1), but $F_{X'}(t) = 2/3$ (values $1, 2$). Thus $F_X(t) < F_{X'}(t)$ here (which favors X).
- For $3 \leq t < 5$: $F_X(t) = 1$ (all values ≤ 3), but $F_{X'}(t) = 2/3$. Thus $F_X(t) > F_{X'}(t)$ here (which favors X').

Since the CDFs cross, neither distribution has F everywhere weakly below the other. Therefore no FOSD holds.

Neither x nor x' first-order stochastically dominates the other.

Method 2: Construct increasing-utility reversals (characterisation of FOSD)

Recall: X FOSD X' iff $\mathbb{E}[u(X)] \geq \mathbb{E}[u(X')]$ for all increasing u.

We show there exist two increasing utility functions ranking the bundles in opposite ways.

Utility that values high outcomes strongly (favoring x'). Let $u_1(t) = t^2$ (increasing). Then

$$\mathbb{E}[u_1(X)] = \tfrac{1}{3}(1^2 + 3^2 + 3^2) = \tfrac{1}{3}(1 + 9 + 9) = \tfrac{19}{3},$$
$$\mathbb{E}[u_1(X')] = \tfrac{1}{3}(5^2 + 1^2 + 2^2) = \tfrac{1}{3}(25 + 1 + 4) = 10.$$

So $\mathbb{E}[u_1(X')] > \mathbb{E}[u_1(X)]$, favoring x'.

Utility that is very steep near low outcomes (favoring x). Let $u_2(t) = \min\{t, 2.5\}$ (nondecreasing, hence increasing in the weak sense used for FOSD tests). Then

$$u_2(X): \ (1, 2.5, 2.5) \ \Rightarrow \ \mathbb{E}[u_2(X)] = \tfrac{1}{3}(1 + 2.5 + 2.5) = 2,$$

$$u_2(X'): \ (2.5, 1, 2) \ \Rightarrow \ \mathbb{E}[u_2(X')] = \tfrac{1}{3}(2.5 + 1 + 2) = \tfrac{5.5}{3} \approx 1.833.$$

So $\mathbb{E}[u_2(X)] > \mathbb{E}[u_2(X')]$, favoring x.

Since different increasing utilities prefer different bundles, neither FOSD relation can hold.

No FOSD: preferences can reverse across increasing utility functions.

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Part II: Long Questions

Question 1 (Revealed preference and WARP)

Problem

Bob buys $x^1 = (1, 4)$ when prices are $p^1 = (4, 2)$ and Bob buys $x^2 = (2, 2)$ when prices are $p^2 = (3, 3)$.

(a) How much money did Bob spend in each period?

(b) Is x^1 directly revealed preferred to x^2?

(c) Is x^1 directly strictly revealed preferred to x^2?

(d) Does Bob satisfy WARP?

(e) Suppose $p^3 = (4, 1)$. Find an $x^3 = (x_1^3, x_2^3)$ so that if Bob bought x^3 when prices were p^3 then Bob would violate WARP.

Solution

Let expenditures be $m^t = p^t \cdot x^t$. Define direct revealed preference:

$$x^t R x^s \iff p^t \cdot x^s \leq p^t \cdot x^t = m^t, \qquad x^t P x^s \iff p^t \cdot x^s < m^t.$$

Method 1: Dot products and the definition

(a) Spending:

$$m^1 = p^1 \cdot x^1 = 4 \cdot 1 + 2 \cdot 4 = 4 + 8 = 12, \qquad m^2 = p^2 \cdot x^2 = 3 \cdot 2 + 3 \cdot 2 = 6 + 6 = 12.$$

$$\boxed{m^1 = 12, \quad m^2 = 12.}$$

(b) Check if x^2 is affordable at (p^1, m^1):

$$p^1 \cdot x^2 = 4 \cdot 2 + 2 \cdot 2 = 8 + 4 = 12 \leq 12 = m^1.$$

So $x^1 R x^2$.

$$\boxed{\text{Yes: } x^1 \text{ is directly revealed preferred to } x^2.}$$

(c) Strictness requires $p^1 \cdot x^2 < m^1$, but equality holds. Hence not strict.

$$\boxed{\text{No: } x^1 \text{ is not directly strictly revealed preferred to } x^2.}$$

(d) WARP: a violation would require $x^1 R x^2$ and $x^2 P x^1$ (or the reverse). We already have $x^1 R x^2$. Check whether $x^2 P x^1$:

$$p^2 \cdot x^1 = 3 \cdot 1 + 3 \cdot 4 = 3 + 12 = 15.$$

Since $15 > m^2 = 12$, x^1 is not affordable in period 2, so $x^2 R x^1$ fails and thus $x^2 P x^1$ fails. Therefore there is no WARP-violating pair.

Yes: Bob satisfies WARP (given only these two observations).

(e) Construct x^3 at $p^3 = (4, 1)$ to create a WARP violation with x^2. We want:

$$x^3 R x^2 \quad \text{and} \quad x^2 P x^3.$$

These conditions are:

$$p^3 \cdot x^2 \leq p^3 \cdot x^3 \quad \text{and} \quad p^2 \cdot x^3 < p^2 \cdot x^2 = m^2 = 12.$$

Compute $p^3 \cdot x^2 = 4 \cdot 2 + 1 \cdot 2 = 10$. Thus require $p^3 \cdot x^3 \geq 10$. Also need $p^2 \cdot x^3 = 3(x_1^3 + x_2^3) < 12 \iff x_1^3 + x_2^3 < 4$.

Choose, for instance,

$$x^3 = (3, \tfrac{1}{2}).$$

Then $x_1^3 + x_2^3 = 3.5 < 4$ so $p^2 \cdot x^3 = 3 \cdot 3.5 = 10.5 < 12$, hence $x^2 P x^3$. Also $p^3 \cdot x^3 = 4 \cdot 3 + 1 \cdot \frac{1}{2} = 12.5 \geq 10 = p^3 \cdot x^2$, hence $x^3 R x^2$. Thus WARP is violated by the pair (x^2, x^3).

$x^3 = (3, \frac{1}{2})$ works (among many possible choices).

Method 2: Geometry and inequalities (budget lines and affordability)

(a) Same dot-product computation gives $m^1 = m^2 = 12$.

(b) At period 1, the budget line is $4x_1 + 2x_2 = 12$. Since $4 \cdot 2 + 2 \cdot 2 = 12$, the point $x^2 = (2, 2)$ lies on the period-1 budget line, hence is affordable. Therefore $x^1 R x^2$.

(c) Because x^2 lies exactly on the line (not strictly inside), strict revealed preference does not hold.

(d) At period 2, the budget line is $3x_1 + 3x_2 = 12$, i.e. $x_1 + x_2 = 4$. But $x^1 = (1, 4)$ has $x_1 + x_2 = 5 > 4$, so it lies strictly outside the budget set and is not affordable; hence no reversal is possible and WARP holds.

(e) For WARP violation with x^2, we need x^3 such that:

$$x^3 \text{ is strictly inside budget 2: } x_1^3 + x_2^3 < 4,$$

yet at prices $p^3 = (4, 1)$ the expenditure on x^3 is at least that on x^2:

$$4x_1^3 + x_2^3 \geq 4 \cdot 2 + 2 = 10.$$

The choice $x^3 = (3, \frac{1}{2})$ satisfies both, so it generates the violation.

Final: $m^1 = m^2 = 12$; $x^1 R x^2$ but not strictly; WARP holds; $x^3 = (3, \frac{1}{2})$ violates WARP.

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Question 2 (Symmetric utility and Slutsky)

Problem

Bob's utility function is

$$U(x_1, x_2) = 5u(x_1) + 5u(x_2)$$

where u is an increasing function.

(a) Does Bob prefer $(1, 2)$ or $(2, 2)$ (or not enough info)?

(b) Does Bob prefer $(1, 2)$ or $(3, 1)$ (or not enough info)?

(c) Suppose $u(x) = \ln(x)$ so $u'(x) = 1/x$. Find Bob's demand function $x(p_1, p_2, m)$.

(d) Prices are $(1, 1)$ and Bob has 4 to spend. How much of each good does Bob buy? The price of good 1 increases to 2. How much of each good does Bob buy?

(e) Calculate the substitution and income effects Δx_1^s and Δx_1^n from the price change in part (d).

Solution

Method 1: Monotonicity + full optimisation when $u = \ln$

(a) Since u is increasing, U is increasing in each argument. Bundle $(2, 2)$ has weakly more of each good and strictly more of good 1 than $(1, 2)$, hence

$$(2, 2) \succ (1, 2).$$

$$\boxed{(2, 2) \text{ is preferred.}}$$

(b) Bundles $(1, 2)$ and $(3, 1)$ are not comparable by componentwise dominance. With only that u is increasing, the curvature is unknown, so we cannot rank them.

$$\boxed{\text{Not enough information.}}$$

(c) If $u(x) = \ln x$, then

$$U(x_1, x_2) = 5 \ln x_1 + 5 \ln x_2 = 5 \ln(x_1 x_2),$$

so maximising U is equivalent to maximising $\ln x_1 + \ln x_2$, i.e. Cobb–Douglas with equal weights.

Set up the Lagrangian:

$$\mathcal{L} = \ln x_1 + \ln x_2 + \lambda(m - p_1 x_1 - p_2 x_2).$$

FOCs:

$$\frac{1}{x_1} - \lambda p_1 = 0, \qquad \frac{1}{x_2} - \lambda p_2 = 0.$$

Thus $\lambda = \frac{1}{p_1 x_1} = \frac{1}{p_2 x_2}$, hence $p_1 x_1 = p_2 x_2$. Using the budget $p_1 x_1 + p_2 x_2 = m$, we get $2p_1 x_1 = m$ and $2p_2 x_2 = m$, so

$$x_1(p, m) = \frac{m}{2p_1}, \qquad x_2(p, m) = \frac{m}{2p_2}.$$

$$\boxed{x(p_1, p_2, m) = \left(\frac{m}{2p_1}, \frac{m}{2p_2}\right).}$$

(d) At $p = (1, 1)$, $m = 4$:

$$x = \left(\frac{4}{2 \cdot 1}, \frac{4}{2 \cdot 1}\right) = (2, 2).$$

If p_1 rises to 2 while $p_2 = 1$ and $m = 4$ stays fixed:

$$x' = \left(\frac{4}{2 \cdot 2}, \frac{4}{2 \cdot 1}\right) = (1, 2).$$

$$\boxed{\text{Old: } (2, 2). \text{ New: } (1, 2).}$$

(e) Slutsky decomposition for good 1 from $p = (1, 1)$ to $p' = (2, 1)$. Initial bundle $x^0 = (2, 2)$, so $x_1^0 = 2$. New Marshallian bundle (with $m = 4$) is $x^1 = (1, 2)$, so $x_1^1 = 1$.

Slutsky (compensated) income:

$$m_s := p' \cdot x^0 = (2, 1) \cdot (2, 2) = 4 + 2 = 6.$$

Compensated demand at new prices:

$$x^c = x(p', m_s) = \left(\frac{6}{2 \cdot 2}, \frac{6}{2 \cdot 1}\right) = \left(\frac{3}{2}, 3\right),$$

so $x_1^c = 3/2$.

Therefore

$$\Delta x_1^s = x_1^c - x_1^0 = \frac{3}{2} - 2 = -\frac{1}{2}, \qquad \Delta x_1^n = x_1^1 - x_1^c = 1 - \frac{3}{2} = -\frac{1}{2}.$$

$$\boxed{\Delta x_1^s = -\tfrac{1}{2}, \qquad \Delta x_1^n = -\tfrac{1}{2}.}$$

Method 2: Expenditure shares + Hicksian confirmation

(a) (a) follows from monotonicity and dominance: $(2, 2)$ strictly dominates $(1, 2)$ in good 1.

(b) (b) cannot be determined because increasing u alone does not specify substitution between goods.

(c) (c) For ln-Cobb–Douglas with equal exponents, the optimal spending shares are equal:

$$p_1 x_1 = p_2 x_2 = \frac{m}{2},$$

so $x_1 = m/(2p_1)$, $x_2 = m/(2p_2)$.

(d) (d) Plugging gives the same numerical bundles as Method 1.

(e) (e) Hicksian (utility-compensated) viewpoint: Initial utility (dropping constant factor 5) is $\ln 2 + \ln 2 = \ln 4$. To keep utility at $\ln 4$ under $p' = (2, 1)$, minimise $2x_1 + x_2$ s.t. $x_1 x_2 \geq 4$. At optimum $x_1 x_2 = 4$ and tangency gives $MRS = x_2/x_1 = p_1/p_2 = 2$, hence $x_2 = 2x_1$. Then $x_1(2x_1) = 4 \Rightarrow x_1 = \sqrt{2} \approx 1.414$, which is *Hicksian* compensation. Slutsky compensation uses the cost of the original bundle at new prices, yielding the *Slutsky* compensated bundle $x^c = (3/2, 3)$ computed above; this directly gives $\Delta x_1^s = -1/2$ and $\Delta x_1^n = -1/2$.

Final: $(2, 2) \succ (1, 2)$; $(1, 2)$ versus $(3, 1)$ is ambiguous; $x = \left(\frac{m}{2p_1}, \frac{m}{2p_2}\right)$; $(2, 2) \to (1, 2)$; and $\Delta x_1^s = \Delta x_1^n = -\frac{1}{2}$.

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Question 3 (Present value and no-arbitrage)

Problem

People can save and borrow at an interest rate $r > 0$.

(a) If I save \$5 today, how much will this become next period? If I borrow \$5 today, how much will I owe next period?

(b) What is the present value of the asset $a = (0, 1, 5)$, i.e. it pays 0 in period 0, 1 in period 1, and 5 in period 2?

(c) What is the present value of the asset which pays out \$5 each period starting in period 1? Argue your answer.

(d) Today you buy an asset for price p_0. The asset pays out nothing in period 0. You sell the asset next period for p_1. Assuming there are no arbitrage opportunities, what can we say about the rate of return $\frac{p_1 - p_0}{p_0}$? Justify.

(e) Suppose now the asset is a house which provides benefit y_0 in period 0 (monetary value), with $y_0 < p_0$. The rate of return on buying the house for one period is $\frac{p_1 - (p_0 - y_0)}{p_0 - y_0}$. If there are no arbitrage opportunities, what can we say about this rate of return? Justify.

Solution

Method 1: Discounting and comparison to the risk-free asset

(a) Saving \$5 today yields next-period wealth

$$5(1 + r).$$

Borrowing \$5 today means repaying next period

$$5(1 + r).$$

$$\boxed{\text{Save: } 5(1 + r). \quad \text{Borrow: owe } 5(1 + r).}$$

(b) Present value of $a = (0, 1, 5)$:

$$PV(a) = 0 + \frac{1}{1 + r} + \frac{5}{(1 + r)^2}.$$

$$\boxed{PV(a) = \frac{1}{1 + r} + \frac{5}{(1 + r)^2}.}$$

(c) Asset paying \$5 each period starting in period 1 is a perpetuity:

$$PV = \sum_{t=1}^{\infty} \frac{5}{(1 + r)^t} = 5 \sum_{t=1}^{\infty} (1 + r)^{-t}.$$

Let $q = (1+r)^{-1} \in (0,1)$. Then

$$\sum_{t=1}^{\infty} q^t = \frac{q}{1-q}.$$

So

$$PV = 5 \cdot \frac{q}{1-q} = 5 \cdot \frac{(1+r)^{-1}}{1-(1+r)^{-1}} = 5 \cdot \frac{1}{r} = \frac{5}{r}.$$

$$\boxed{PV = \frac{5}{r}.}$$

(d) Asset bought at p_0, sold at p_1, no dividends/cash flows. Its gross return is p_1/p_0, hence rate of return is $(p_1 - p_0)/p_0$. If $\frac{p_1-p_0}{p_0} > r$, one could borrow at rate r, buy the asset, and repay next period, earning sure profit (arbitrage). If $\frac{p_1-p_0}{p_0} < r$, one could short the asset, invest proceeds at rate r, and buy back next period, earning sure profit. Therefore no-arbitrage implies equality:

$$\frac{p_1 - p_0}{p_0} = r \quad \Longleftrightarrow \quad p_1 = (1+r)p_0.$$

$$\boxed{\frac{p_1 - p_0}{p_0} = r.}$$

(e) With a house providing benefit y_0 in period 0, the effective net cost of owning the house over the period is $p_0 - y_0$. No-arbitrage requires the rate of return on the net investment equals the risk-free rate:

$$\frac{p_1 - (p_0 - y_0)}{p_0 - y_0} = r \quad \Longleftrightarrow \quad p_1 = (1+r)(p_0 - y_0).$$

$$\boxed{\frac{p_1 - (p_0 - y_0)}{p_0 - y_0} = r.}$$

Method 2: State the pricing rule and apply it

Pricing rule. With a risk-free gross return $1+r$ and no arbitrage, the time-0 price of a sure time-t payoff of 1 is $(1+r)^{-t}$. Equivalently, the present value of a deterministic cash-flow stream $(c_0, c_1, c_2, \dots)$ is

$$PV = \sum_{t=0}^{\infty} \frac{c_t}{(1+r)^t}.$$

(a) (a) is immediate from the definition of the gross return $1+r$.

(b) (b) Apply the PV formula:

$$PV(0,1,5) = \frac{1}{1+r} + \frac{5}{(1+r)^2}.$$

(c) (c) Apply the PV formula to $(0, 5, 5, 5, \dots)$ and evaluate the geometric series to get $5/r$.

(d) (d) The asset has cash flow $(-p_0, p_1)$. No-arbitrage implies $0 = -p_0 + \frac{p_1}{1+r}$, so $p_1 = (1+r)p_0$, i.e. return r.

(e) (e) The net cash flow is $\big(-(p_0 - y_0),\, p_1\big)$. No-arbitrage implies

$$0 = -(p_0 - y_0) + \frac{p_1}{1+r} \quad \Rightarrow \quad \frac{p_1 - (p_0 - y_0)}{p_0 - y_0} = r.$$

Final: $(a)\ 5(1+r)$; $(b)\ \dfrac{1}{1+r} + \dfrac{5}{(1+r)^2}$; $(c)\ \dfrac{5}{r}$; $(d)\ \dfrac{p_1 - p_0}{p_0} = r$; $(e)\ \dfrac{p_1 - (p_0 - y_0)}{p_0 - y_0} = r.$

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