HE2001 Microeconomics II

Tutorial 7

Academic Year 2025/2026, Semester 2

Quantitative Research Society @NTU

August 17, 2026

Overview

This tutorial covers:

- welfare analysis for price changes under perfect complements, quasilinear utility, and Cobb–Douglas utility;
- Hicksian and Marshallian demand, together with equivalent variation (EV) and compensating variation (CV);
- reservation prices and inverse demand under quasilinear preferences;
- indirect utility and expenditure-based welfare calculations;
- consumer surplus and its relation to EV and CV;
- past-year style applications involving price changes and discrete-choice quasilinear utility.

For academic reference only. This document is provided for educational purposes and does not constitute an official question paper, answer key, or any formally authorised academic material. Its content is not guaranteed to be complete or accurate.

Contents

Question 1: Bernice and Welfare Measures **3**
Problem . . . 3
Solution . . . 4
Method 1: ATQ-focused utility maximisation . . . 4
Method 2: Exam-style compression . . . 8
Method 3: Duality / expenditure and Hicksian demand . . . 9

Question 2: Steve, Reservation Prices, and Inverse Demand **11**
Problem . . . 11
Solution . . . 12
Method 1: Direct utility comparisons and optimisation . . . 12
Method 2: Utility function method . . . 15
Method 3: Indirect-utility / envelope viewpoint . . . 16

Question 3: Lolita, EV, CV, and Consumer Surplus **17**
Problem . . . 17
Solution . . . 18
Method 1: Direct optimisation and triangle-surplus method . . . 18
Method 2: Quasilinear welfare shortcut and integration method . . . 21
Method 3: Duality method . . . 23
Method 4: Legendre-transform method . . . 25

Question 4: Cobb Douglas Utility and Cost Minimisation **27**
Problem . . . 27
Solution . . . 27
Method 1: Direct optimisation and Hicksian reconstruction . . . 27
Method 2: Expenditure-share and expenditure-function method . . . 30
Method 3: Log-utility method . . . 31
Method 4: General Cobb Douglas theorem . . . 33

Sample Question 1: Fruitland, Cobb Douglas Utility, and Welfare Measures **34**
Problem . . . 34
Solution . . . 35
Method 1: Graphical interpretation and expenditure-function method . . . 35
(i) Compensating variation. . . . 35
(ii) Equivalent variation. . . . 36
Method 2: Hicksian minimisation method . . . 38

Sample Question 2: Bob, Quasi-linear Utility, and Reservation Prices **40**
Problem . . . 40
Solution . . . 40
Method 1: Definition and indifference-pricing method . . . 40
Method 2: Utility-value conversion method . . . 42

Method 3: Binary-choice simplification method 43

Question 1: Bernice and Welfare Measures

Bernice's preferences can be represented by $u(x, y) = \min\{x, y\}$, where x is pairs of earrings and y is dollars to spend on other things. She faces prices $(p_x, p_y) = (2, 1)$ and her income is 12.

Problem

(a) Draw in a graph, her budget constraint and several indifference curves, illustrating the optimal bundle.

(b) The price of a pair of earrings rises to \$3 and Bernice's income stays the same. On the same graph, draw her new budget constraint, illustrating the optimal point.

(c) What bundle would Bernice choose if she faced the original prices and had just enough income to reach the new indifference curve in (b)? Draw a budget line which passes through this bundle at the original prices. What is the level of income Bernice has at this new budget line?

(d) What is the equivalent variation (EV) of the price increase in earrings from \$2 to \$3? Remember that EV is the maximum income that at the original prices, when removed, just gives the consumer the same final utility level.

(e) What bundle would Bernice choose if she faced the new prices and had just enough income to reach the original indifference curve in (a)? Draw a budget line which passes through this bundle at the new prices. What is the level of income Bernice has at this budget line?

(f) What is the compensating variation (CV) of the price increase in earrings from \$2 to \$3 ? Remember that CV is the least income that at the new prices, when given, just restores the consumer's original utility level.

Solution

Method 1: ATQ-focused utility maximisation

(a) Since

$$u(x, y) = \min\{x, y\},$$

Bernice has Leontief preferences, so at the optimum she chooses the kink:

$$x = y.$$

Her budget constraint is

$$2x + y = 12.$$

Substituting $y = x$,

$$2x + x = 12 \implies 3x = 12 \implies x = 4.$$

Hence

$$y = 4.$$

Therefore the optimal bundle is

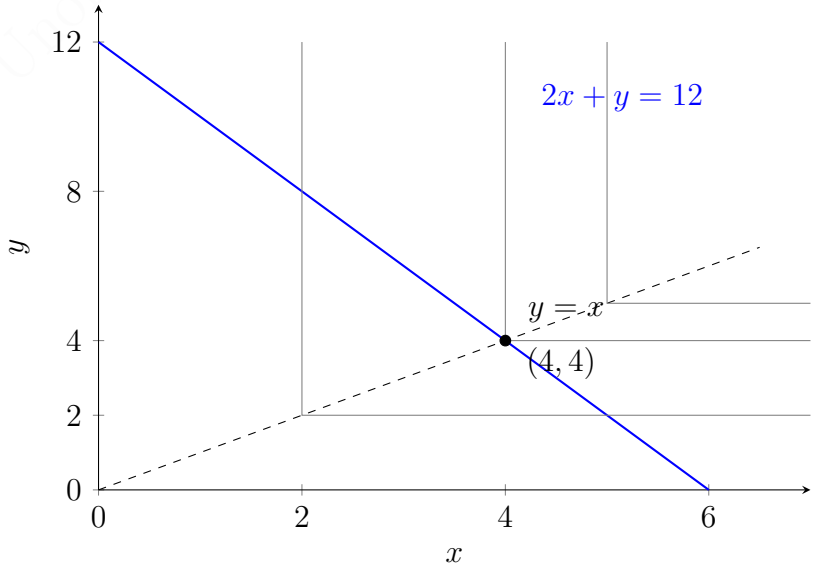
$$\boxed{(4, 4)}$$

and the original utility level is

$$\boxed{u_0 = 4.}$$

A correct graph should show:

- the original budget line $2x + y = 12$;
- several L-shaped indifference curves;
- the optimal point $(4, 4)$.



(b) When the price of earrings rises to 3, the new budget constraint is

$$3x + y = 12.$$

Again Bernice chooses the kink:

$$x = y.$$

Hence

$$3x + x = 12 \implies 4x = 12 \implies x = 3,$$

so

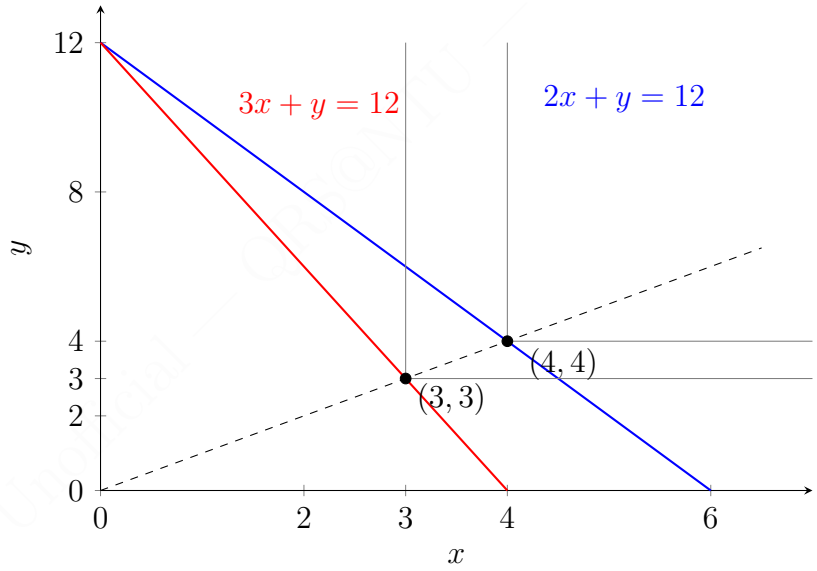
$$y = 3.$$

Therefore the new optimal bundle is

$$\boxed{(3, 3)}$$

and the new utility level is

$$\boxed{u_1 = 3.}$$



Unofficial – QRS@NTU – qrsntu.org

(c) From part (b),

$$u_1 = 3.$$

Since $u(x, y) = \min\{x, y\}$, attaining utility 3 at the original prices means choosing the kink bundle

$$(x, y) = (3, 3).$$

At prices $(2, 1)$, the required income is

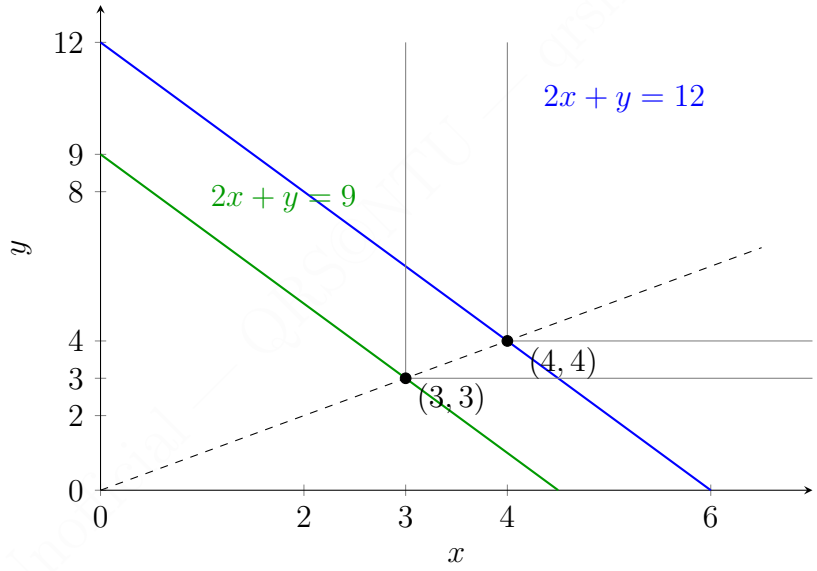
$$m = 2(3) + 1(3) = 9.$$

Thus the bundle is

$$\boxed{(3, 3)}$$

and the required budget line is

$$\boxed{2x + y = 9} \quad \text{or equivalently} \quad \boxed{y = 9 - 2x.}$$



(d) By definition, EV is the maximum income that can be taken away at the original prices while leaving Bernice at her final utility level. From part (c), to attain $u_1 = 3$ at the original prices, she needs income 9. Since her original income is 12,

$$EV = 12 - 9 = 3.$$

Therefore

$$\boxed{EV = 3.}$$

(e) From part (a),

$$u_0 = 4.$$

Since $u(x, y) = \min\{x, y\}$, attaining utility 4 at the new prices means choosing

$$(x, y) = (4, 4).$$

At prices $(3, 1)$, the required income is

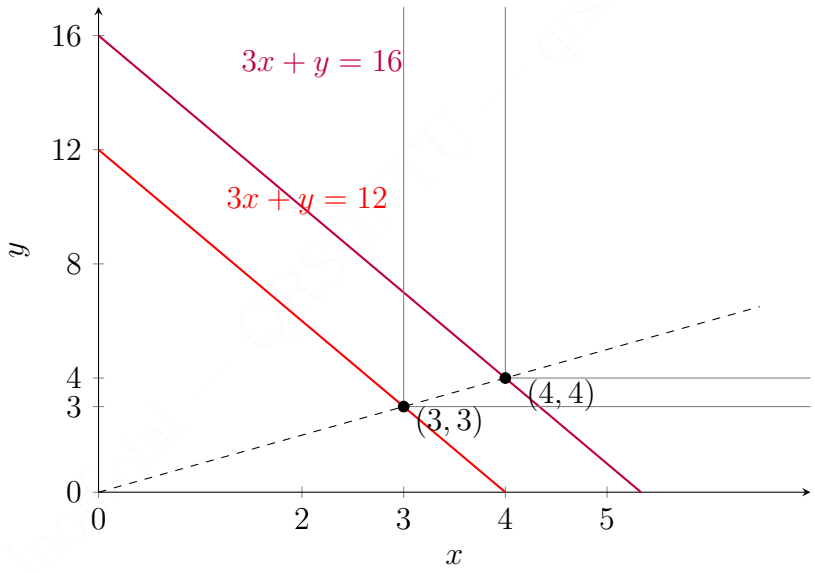
$$m = 3(4) + 1(4) = 16.$$

Thus the bundle is

$$\boxed{(4, 4)}$$

and the required budget line is

$$\boxed{3x + y = 16} \quad \text{or equivalently} \quad \boxed{y = 16 - 3x.}$$



(f) By definition, CV is the least income that must be given at the new prices to restore Bernice to her original utility. From part (e), she needs income 16 at the new prices, but currently has income 12. Hence

$$CV = 16 - 12 = 4.$$

Therefore

$$\boxed{CV = 4.}$$

Method 2: Exam-style compression

(a) Since $u(x, y) = \min\{x, y\}$, Bernice chooses the kink $x = y = u$. At the original prices,

$$2x + y = 12 \implies 2u + u = 12 \implies u_0 = 4.$$

Hence

$$\boxed{(x, y) = (4, 4).}$$

(b) At the new prices,

$$3x + y = 12 \implies 3u + u = 12 \implies u_1 = 3.$$

Hence

$$\boxed{(x, y) = (3, 3).}$$

(c) To attain the new utility level $u_1 = 3$ at the old prices, Bernice chooses

$$(3, 3).$$

Required income:

$$m = 2(3) + 3 = 9.$$

So the bundle is

$$\boxed{(3, 3)}$$

and the budget line is

$$\boxed{2x + y = 9.}$$

(d)

$$EV = 12 - 9 = 3.$$

Therefore

$$\boxed{EV = 3.}$$

(e) To attain the original utility level $u_0 = 4$ at the new prices, Bernice chooses

$$(4, 4).$$

Required income:

$$m = 3(4) + 4 = 16.$$

So the bundle is

$$\boxed{(4, 4)}$$

and the budget line is

$$\boxed{3x + y = 16.}$$

(f)

$$CV = 16 - 12 = 4.$$

Therefore

$$\boxed{CV = 4.}$$

Method 3: Duality / expenditure and Hicksian demand

(a) For $u(x, y) = \min\{x, y\}$, the Hicksian problem is

$$\min_{x,y} \; p_x x + p_y y \quad \text{subject to} \quad \min\{x, y\} \geq u.$$

The constraint is equivalent to

$$x \geq u, \qquad y \geq u,$$

so cost minimisation gives

$$h_x(p, u) = u, \qquad h_y(p, u) = u.$$

Hence the expenditure function is

$$e((p_x, p_y), u) = p_x u + p_y u = (p_x + p_y)u.$$

At prices $(2, 1)$,

$$e((2, 1), u) = 3u.$$

Since income is 12,

$$3u = 12 \implies u_0 = 4.$$

Therefore the optimal bundle is

$$\boxed{(4, 4)}.$$

(b) At the new prices $(3, 1)$,

$$e((3, 1), u) = 4u.$$

Since income is still 12,

$$4u = 12 \implies u_1 = 3.$$

Therefore the new optimal bundle is

$$\boxed{(3, 3)}.$$

(c) To attain $u_1 = 3$ at the old prices,

$$e((2, 1), 3) = 3 \cdot 3 = 9.$$

Thus the required bundle is

$$\boxed{(3, 3)}$$

and the budget line is

$$\boxed{2x + y = 9.}$$

Unofficial – QRS@NTU – Tutorial Solutions

(d) By definition,

$$EV = m - e(p_0, u_1).$$

Here $m = 12$, $p_0 = (2, 1)$, and $u_1 = 3$. Hence

$$EV = 12 - e((2, 1), 3) = 12 - 9 = 3.$$

Therefore

$$\boxed{EV = 3.}$$

(e) To attain $u_0 = 4$ at the new prices,

$$e((3, 1), 4) = 4 \cdot 4 = 16.$$

Thus the required bundle is

$$\boxed{(4, 4)}$$

and the budget line is

$$\boxed{3x + y = 16.}$$

(f) By definition,

$$CV = e(p_1, u_0) - m.$$

Here $p_1 = (3, 1)$, $u_0 = 4$, and $m = 12$. Hence

$$CV = e((3, 1), 4) - 12 = 16 - 12 = 4.$$

Therefore

$$\boxed{CV = 4.}$$

Unofficial QRS@NTU solutions

Question 2: Steve, Reservation Prices, and Inverse Demand

Steve consumes earplugs and other goods. His utility function $U(x, y)$ is increasing in both his consumption of earplugs x and other goods y. Suppose that Steve has a sufficiently large income I.

Problem

(a) Suppose $U(x, y) = v(x) + ay$ and that Steve can buy $n - 1$ units of earplugs at total cost c_{n-1}.

(i) Write out Steve's utility of consuming $n - 1$ units of earplugs and spending the rest on other goods.

(ii) Suppose that the nth earplug costs p_n, write out Steve's utility of consuming n units of earplugs and spending the rest on other goods.

(iii) Show that the reservation price for the nth earplug r_n is $\frac{p_y}{a} \times [v(n) - v(n-1)]$.
(Hint: look at the definition of the reservation price.)
Note that this implies that if earplugs were perfectly divisible, the reservation price function $r(n) = \frac{p_y}{a} v'(n)$.

(b) Solve Steve's utility maximisation problem (subject to prices p_x, p_y and wealth w) to get his inverse demand function for earplugs x. Is it the same as the reservation price function?

(c) In the lecture's example, $a = 1$ and $p_y = 1$ such that the reservation price curve is $p_x = v'(x)$. What do you think the additional term $\frac{p_y}{a}$ captures?

Solution

Method 1: Direct utility comparisons and optimisation

(a) (i) Since

$$U(x, y) = v(x) + ay,$$

if Steve consumes $n - 1$ earplugs and total spending on those earplugs is c_{n-1}, then he has

$$I - c_{n-1}$$

dollars left for the other good. Since the other good costs p_y per unit, he buys

$$y = \frac{I - c_{n-1}}{p_y}$$

units of it. Therefore his utility is

$$U_{n-1} = v(n-1) + a\left(\frac{I - c_{n-1}}{p_y}\right).$$

Hence

$$\boxed{U_{n-1} = v(n-1) + \frac{a}{p_y}(I - c_{n-1})\,.}$$

(ii) If the nth earplug costs p_n, then total spending on earplugs becomes

$$c_{n-1} + p_n,$$

so the remaining money is

$$I - c_{n-1} - p_n.$$

Hence Steve buys

$$y = \frac{I - c_{n-1} - p_n}{p_y}$$

units of the other good, and his utility becomes

$$U_n = v(n) + a\left(\frac{I - c_{n-1} - p_n}{p_y}\right).$$

Therefore

$$\boxed{U_n = v(n) + \frac{a}{p_y}(I - c_{n-1} - p_n)\,.}$$

(iii) By definition, the reservation price r_n is the price that makes Steve indifferent between consuming $n-1$ earplugs and consuming n earplugs. Therefore

$$v(n-1) + \frac{a}{p_y}(I - c_{n-1}) = v(n) + \frac{a}{p_y}(I - c_{n-1} - r_n).$$

Cancelling the common term $\frac{a}{p_y}(I - c_{n-1})$,

$$v(n-1) = v(n) - \frac{a}{p_y} r_n.$$

Rearranging gives

$$\frac{a}{p_y} r_n = v(n) - v(n-1),$$

so

$$\boxed{r_n = \frac{p_y}{a}[v(n) - v(n-1)]\,.}$$

In the perfectly divisible case, the finite difference becomes a derivative, so

$$\boxed{r(x) = \frac{p_y}{a} v'(x)\,.}$$

(b) Steve solves

$$\max_{x,y} \; v(x) + ay \quad \text{subject to} \quad p_x x + p_y y = w.$$

Substitute

$$y = \frac{w - p_x x}{p_y}$$

into utility:

$$U(x) = v(x) + a\left(\frac{w - p_x x}{p_y}\right) = v(x) + \frac{aw}{p_y} - \frac{a p_x}{p_y} x.$$

Since $\frac{aw}{p_y}$ is constant, Steve equivalently solves

$$\max_{x} \; v(x) - \frac{a p_x}{p_y} x.$$

The first-order condition is

$$v'(x) - \frac{a p_x}{p_y} = 0.$$

Hence

$$p_x = \frac{p_y}{a} v'(x).$$

Therefore Steve's inverse demand function is

$$\boxed{p_x = \frac{p_y}{a} v'(x)\,.}$$

It is not literally the same as the discrete reservation-price formula

$$r_n = \frac{p_y}{a}[v(n) - v(n-1)],$$

because that uses a finite difference, whereas inverse demand uses a derivative. However, in the perfectly divisible case the reservation-price function becomes

$$r(x) = \frac{p_y}{a}v'(x),$$

so the two coincide in the continuous formulation.

(c) The factor

$$\frac{p_y}{a}$$

converts utility units into money units. One unit of the other good costs p_y dollars and contributes a units of utility, so one util corresponds to

$$\frac{p_y}{a}$$

dollars' worth of the other good. Thus $\frac{p_y}{a}$ measures the money value of one util, using the outside good as the benchmark.

Therefore

$\boxed{\frac{p_y}{a} \text{ is the utility-to-money conversion factor.}}$

Unofficial_QRS@NTU_consumer_notes

Method 2: Utility function method

(a) (i)

$$U_{n-1} = v(n-1) + a\left(\frac{I - c_{n-1}}{p_y}\right) = \boxed{v(n-1) + \frac{a}{p_y}(I - c_{n-1})}.$$

(ii)

$$U_n = v(n) + a\left(\frac{I - c_{n-1} - p_n}{p_y}\right) = \boxed{v(n) + \frac{a}{p_y}(I - c_{n-1} - p_n)}.$$

(iii) Reservation price means indifference:

$$v(n-1) + \frac{a}{p_y}(I - c_{n-1}) = v(n) + \frac{a}{p_y}(I - c_{n-1} - r_n).$$

Hence

$$\boxed{r_n = \frac{p_y}{a}[v(n) - v(n-1)]}.$$

If earplugs are perfectly divisible,

$$\boxed{r(x) = \frac{p_y}{a}v'(x)}.$$

(b)

$$\max_{x,y} \; v(x) + ay \quad \text{s.t.} \quad p_x x + p_y y = w.$$

Substitute $y = (w - p_x x)/p_y$:

$$\max_{x} \; v(x) - \frac{ap_x}{p_y}x.$$

FOC:

$$v'(x) = \frac{ap_x}{p_y}.$$

Therefore

$$\boxed{p_x = \frac{p_y}{a}v'(x)}.$$

This matches the reservation-price function only in the perfectly divisible case.

(c)

$$\boxed{\frac{p_y}{a} \text{ converts utils into dollars via the outside good.}}$$

Unofficial – Original solutions

Method 3: Indirect-utility / envelope viewpoint

(a) (i)

$$\boxed{U_{n-1} = v(n-1) + \frac{a}{p_y}(I - c_{n-1})}.$$

(ii)

$$\boxed{U_n = v(n) + \frac{a}{p_y}(I - c_{n-1} - p_n)}.$$

(iii) Reservation price is the price that leaves utility unchanged, so

$$\boxed{r_n = \frac{p_y}{a}[v(n) - v(n-1)]}.$$

(b) Define the indirect objective

$$\Phi(x; p_x, p_y, w) = v(x) + a\frac{w - p_x x}{p_y}.$$

Then indirect utility is

$$V(p_x, p_y, w) = \max_x \Phi(x; p_x, p_y, w).$$

The optimiser x^* satisfies

$$\frac{\partial \Phi}{\partial x} = v'(x) - \frac{a p_x}{p_y} = 0.$$

Hence

$$\boxed{p_x = \frac{p_y}{a} v'(x)}.$$

Because wealth enters only through the additive term aw/p_y, earplug demand has no income effect in the interior solution.

(c) Since y is quasi-linear,

$$U(x, y) = v(x) + ay,$$

the outside good acts as the numeraire benchmark for utility. Therefore

$$\boxed{\frac{p_y}{a} \text{ captures the dollar value of one unit of utility relative to the outside good.}}$$

Question 3: Lolita, EV, CV, and Consumer Surplus

Lolita, an intelligent and charming Holstein cow, consumes only two goods, cow feed (made of ground corn and oats) and hay. Her preferences are represented by the utility function

$$U(x, y) = x - \frac{x^2}{2} + y,$$

where x is her consumption of cow feed and y is her consumption of hay. Lolita has been instructed in the mysteries of budgets and optimization and always maximizes her utility subject to her budget constraint. Lolita has an income of $\$I$ that she is allowed to spend as she wishes on cow feed and hay. The price of hay is always \$1, and the price of cow feed will be denoted by p, where $0 < p \leq 1$.

Problem

(a) Write Lolita's demand function for cow feed and hay as a function of I and p.

(b) Write out the indirect utility function of Lolita at income I and prices p. (Remember that indirect utility is the maximum level of utility that she enjoys at this price and income.)

Suppose that Lolita's daily income is \$3 and that the price of feed is \$0.50. There is a price increase from \$0.50 to \$1 for cow feed.

(c) Consider the largest fee Lolita is willing to pay to avoid having the price of cow feed rise to \$1. Is this EV or CV?

(d) Consider the extra money would you have to pay Lolita to make her as well-off as she was at the old prices. Is this EV or CV?

(e) Use the indirect utility function to calculate Lolita's CV and EV.

(f) What is the change in consumer surplus of the price increase of cow feed from \$0.5 to \$1?

Solution

Method 1: Direct optimisation and triangle-surplus method

(a) Lolita solves

$$\max_{x,y} \; x - \frac{x^2}{2} + y \qquad \text{subject to} \qquad px + y = I,$$

with $x \geq 0$, $y \geq 0$. Substituting

$$y = I - px$$

into utility gives the one-variable problem

$$\max_{0 \leq x \leq I/p} \; I + (1-p)x - \frac{x^2}{2}.$$

Since

$$\frac{d^2}{dx^2}\left(I + (1-p)x - \frac{x^2}{2}\right) = -1 < 0,$$

the objective is strictly concave, so the unconstrained maximiser is given by

$$(1-p) - x = 0 \quad \Longrightarrow \quad x = 1 - p.$$

Hence the demand for cow feed is

$$x^*(I,p) = \min\left\{1-p, \frac{I}{p}\right\},$$

and the demand for hay is

$$y^*(I,p) = I - p\,x^*(I,p).$$

Therefore

$$\boxed{x^*(I,p) = \min\left\{1-p, \frac{I}{p}\right\}, \qquad y^*(I,p) = I - p\min\left\{1-p, \frac{I}{p}\right\}.}$$

(b) Using the demand found in part (a), the indirect utility is

$$V(I,p) = \max_{0 \leq x \leq I/p} \left\{I + (1-p)x - \frac{x^2}{2}\right\}.$$

There are two cases.

If $1 - p \leq I/p$, then the interior optimum is feasible and

$$x^* = 1 - p, \qquad y^* = I - p(1-p).$$

Hence

$$V(I,p) = (1-p) - \frac{(1-p)^2}{2} + I - p(1-p) = I + \frac{(1-p)^2}{2}.$$

If $1 - p > I/p$, then the optimum is at the boundary $x^* = I/p$, $y^* = 0$, so

$$V(I,p) = \frac{I}{p} - \frac{1}{2}\left(\frac{I}{p}\right)^2.$$

Therefore

$$\boxed{V(I,p) = \begin{cases} I + \dfrac{(1-p)^2}{2}, & \text{if } 1 - p \le \dfrac{I}{p}, \\ \dfrac{I}{p} - \dfrac{1}{2}\left(\dfrac{I}{p}\right)^2, & \text{if } 1 - p > \dfrac{I}{p}. \end{cases}}$$

(c) The largest fee Lolita is willing to pay *to avoid* the price rise is the amount that can be taken away *at the old prices* while leaving her at the same utility as after the price increase. That is exactly the *equivalent variation*.

Hence

$$\boxed{\text{This is EV.}}$$

(d) The extra money that must be given to Lolita *at the new prices* to make her as well-off as she was at the old prices is, by definition, the *compensating variation*.

Hence

$$\boxed{\text{This is CV.}}$$

(e) Here $I = 3$, initially $p_0 = 0.5$, and finally $p_1 = 1$. Since

$$1 - p_0 = 0.5 \le \frac{3}{0.5} = 6, \qquad 1 - p_1 = 0 \le \frac{3}{1} = 3,$$

both situations lie in the interior branch of V. Thus

$$V(3, 0.5) = 3 + \frac{(1-0.5)^2}{2} = 3 + \frac{(0.5)^2}{2} = 3 + \frac{1}{8} = \frac{25}{8},$$

and

$$V(3,1) = 3 + \frac{(1-1)^2}{2} = 3.$$

For EV, let m^{EV} be the income at old prices $p_0 = 0.5$ that gives final utility 3:

$$V(m^{EV}, 0.5) = 3.$$

Since we remain on the interior branch,

$$m^{EV} + \frac{1}{8} = 3 \quad \Longrightarrow \quad m^{EV} = \frac{23}{8}.$$

Hence

$$EV = 3 - \frac{23}{8} = \frac{1}{8}.$$

For CV, let m^{CV} be the income at new prices $p_1 = 1$ that gives initial utility 25/8:

$$V(m^{CV}, 1) = \frac{25}{8}.$$

Since $V(m, 1) = m$,

$$m^{CV} = \frac{25}{8}.$$

Hence

$$CV = \frac{25}{8} - 3 = \frac{1}{8}.$$

Therefore

$$\boxed{EV = \frac{1}{8}, \qquad CV = \frac{1}{8}.}$$

(f) From part (a), for $0 < p \leq 1$ and with income large enough for the interior solution,

$$x = 1 - p.$$

So the inverse demand curve is

$$p = 1 - x.$$

When price rises from 0.5 to 1, quantity falls from

$$x_0 = 1 - 0.5 = 0.5 \qquad \text{to} \qquad x_1 = 1 - 1 = 0.$$

The lost consumer surplus is the area of the triangle under inverse demand and above the new price line between $x = 0$ and $x = 0.5$:

$$\frac{1}{2} \times 0.5 \times 0.5 = \frac{1}{8}.$$

Since the question asks for the *change* in consumer surplus,

$$\Delta CS = CS_{\text{new}} - CS_{\text{old}} = -\frac{1}{8}.$$

Therefore

$$\boxed{\Delta CS = -\frac{1}{8}.}$$

Method 2: Quasilinear welfare shortcut and integration method

(a) With

$$U(x,y) = \phi(x) + y, \qquad \phi(x) = x - \frac{x^2}{2},$$

the problem is quasilinear in y. Substituting $y = I - px$, Lolita solves

$$\max_{0 \le x \le I/p} \phi(x) - px + I.$$

Since

$$\phi'(x) = 1 - x,$$

the interior first-order condition is

$$\phi'(x) = p \quad \Longleftrightarrow \quad 1 - x = p \quad \Longleftrightarrow \quad x = 1 - p.$$

Hence

$$\boxed{x^*(I,p) = \min\left\{1 - p, \frac{I}{p}\right\}, \qquad y^*(I,p) = I - p\,x^*(I,p).}$$

(b) The indirect utility is

$$V(I,p) = I + \max_{0 \le x \le I/p} \{\phi(x) - px\}.$$

If $1 - p \le I/p$, then the interior maximiser is feasible and

$$V(I,p) = I + \phi(1-p) - p(1-p).$$

Since

$$\phi(1-p) = (1-p) - \frac{(1-p)^2}{2},$$

this becomes

$$V(I,p) = I + \frac{(1-p)^2}{2}.$$

If $1 - p > I/p$, then $x^* = I/p$, so

$$V(I,p) = I + \phi\left(\frac{I}{p}\right) - p\left(\frac{I}{p}\right) = \frac{I}{p} - \frac{1}{2}\left(\frac{I}{p}\right)^2.$$

Therefore

$$\boxed{V(I,p) = \begin{cases} I + \dfrac{(1-p)^2}{2}, & \text{if } 1 - p \le \dfrac{I}{p}, \\ \dfrac{I}{p} - \dfrac{1}{2}\left(\dfrac{I}{p}\right)^2, & \text{if } 1 - p > \dfrac{I}{p}. \end{cases}}$$

Unofficial solutions

(c) The largest fee Lolita is willing to pay *at the old prices* to avoid the price increase is

$$\boxed{\text{EV.}}$$

(d) The extra money needed *at the new prices* to restore old utility is

$$\boxed{\text{CV.}}$$

(e) Since $I = 3$ is sufficiently large here, demand remains on the quasilinear interior branch:

$$x(p) = 1 - p, \qquad 0 < p \leq 1.$$

Therefore the welfare loss from the price increase from $p_0 = 0.5$ to $p_1 = 1$ is

$$\int_{p_0}^{p_1} x(p)\, dp = \int_{0.5}^{1} (1 - p)\, dp = \left[p - \frac{p^2}{2}\right]_{0.5}^{1} = \frac{1}{8}.$$

For quasilinear preferences, the welfare loss equals both EV and CV. Hence

$$\boxed{EV = \frac{1}{8}, \qquad CV = \frac{1}{8}.}$$

(f) The change in consumer surplus is

$$\Delta CS = -\int_{0.5}^{1} x(p)\, dp = -\int_{0.5}^{1} (1 - p)\, dp = -\frac{1}{8}.$$

Therefore

$$\boxed{\Delta CS = -\frac{1}{8}.}$$

Unofficial Lesson Notes

Method 3: Duality method

(a) For fixed I and p, the Marshallian problem is

$$\max_{x,y} \; x - \frac{x^2}{2} + y \qquad \text{s.t.} \qquad px + y = I.$$

Since utility is quasilinear, it is convenient to isolate the non-numeraire part. Define

$$\phi(x) = x - \frac{x^2}{2}.$$

Then

$$U(x, y) = \phi(x) + y.$$

The optimal x solves

$$\max_{0 \le x \le I/p} \{\phi(x) - px\}.$$

The first-order condition for an interior maximiser is

$$\phi'(x) = p \quad \Longleftrightarrow \quad 1 - x = p \quad \Longleftrightarrow \quad x = 1 - p.$$

Hence

$$\boxed{x^*(I, p) = \min\left\{1 - p, \frac{I}{p}\right\}, \qquad y^*(I, p) = I - p\, x^*(I, p).}$$

(b) The expenditure function at utility level u is

$$e(p, u) = \min_x \{px + u - \phi(x)\} = u + \min_x \{px - \phi(x)\}.$$

Equivalently,

$$e(p, u) = u - \max_x \{\phi(x) - px\}.$$

For $0 < p \le 1$, the maximiser of $\phi(x) - px$ is $x = 1 - p$, so

$$\max_x \{\phi(x) - px\} = \phi(1 - p) - p(1 - p) = \frac{(1 - p)^2}{2}.$$

Therefore

$$e(p, u) = u - \frac{(1 - p)^2}{2}.$$

Using the dual identity

$$V(I, p) = \max\{u : e(p, u) \le I\},$$

we obtain

$$I = e(p, u) = u - \frac{(1 - p)^2}{2} \quad \Longrightarrow \quad u = I + \frac{(1 - p)^2}{2},$$

whenever the interior solution applies. Thus

$$\boxed{V(I,p) = I + \frac{(1-p)^2}{2}}$$

on the relevant branch, and together with the boundary case this yields

$$\boxed{V(I,p) = \begin{cases} I + \dfrac{(1-p)^2}{2}, & \text{if } 1-p \le \dfrac{I}{p}, \\ \dfrac{I}{p} - \dfrac{1}{2}\left(\dfrac{I}{p}\right)^2, & \text{if } 1-p > \dfrac{I}{p}. \end{cases}}$$

(c) The largest fee paid at old prices to avoid the price increase is

$$\boxed{\text{EV.}}$$

(d) The extra money required at new prices to restore the old utility is

$$\boxed{\text{CV.}}$$

(e) At $I = 3$, both $p_0 = 0.5$ and $p_1 = 1$ lie on the interior branch. Hence

$$V(3, 0.5) = 3 + \frac{(0.5)^2}{2} = \frac{25}{8}, \qquad V(3,1) = 3.$$

From the expenditure function,

$$e(p,u) = u - \frac{(1-p)^2}{2}.$$

Therefore

$$EV = I - e(p_0, V(3,p_1)) = 3 - e(0.5, 3) = 3 - \left(3 - \frac{1}{8}\right) = \frac{1}{8},$$

and

$$CV = e(p_1, V(3,p_0)) - I = e\left(1, \frac{25}{8}\right) - 3 = \frac{25}{8} - 3 = \frac{1}{8}.$$

Thus

$$\boxed{EV = \frac{1}{8}, \qquad CV = \frac{1}{8}.}$$

(f) Since quasilinearity implies

$$\max_x \{\phi(x) - px\} = \frac{(1-p)^2}{2},$$

the surplus loss from $p_0 = 0.5$ to $p_1 = 1$ is

$$\frac{(1-0.5)^2}{2} - \frac{(1-1)^2}{2} = \frac{1}{8}.$$

Hence the change in consumer surplus is

$$\boxed{\Delta CS = -\frac{1}{8}.}$$

Method 4: Legendre-transform method

(a) Let

$$\phi(x) = x - \frac{x^2}{2}.$$

The indirect problem is

$$V(I,p) = I + \sup_{0 \le x \le I/p} \{\phi(x) - px\}.$$

On the interior region, the supremum is attained where

$$\phi'(x) = p \quad \Longleftrightarrow \quad 1 - x = p \quad \Longleftrightarrow \quad x = 1 - p.$$

Hence

$$\boxed{x^*(I,p) = \min\left\{1 - p, \frac{I}{p}\right\}, \qquad y^*(I,p) = I - p\,x^*(I,p).}$$

(b) The Legendre transform of ϕ at slope p is

$$\phi^*(p) := \sup_x \{\phi(x) - px\}.$$

For $0 < p \le 1$, the maximiser is $x = 1 - p$, so

$$\phi^*(p) = (1-p) - \frac{(1-p)^2}{2} - p(1-p) = \frac{(1-p)^2}{2}.$$

Therefore, on the interior branch,

$$V(I,p) = I + \phi^*(p) = I + \frac{(1-p)^2}{2}.$$

Including the boundary branch,

$$\boxed{V(I,p) = \begin{cases} I + \dfrac{(1-p)^2}{2}, & \text{if } 1 - p \le \dfrac{I}{p}, \\ \dfrac{I}{p} - \dfrac{1}{2}\left(\dfrac{I}{p}\right)^2, & \text{if } 1 - p > \dfrac{I}{p}. \end{cases}}$$

(c) The largest fee paid at old prices to avoid the price increase is

$$\boxed{\text{EV.}}$$

(d) The extra money needed at new prices to restore old utility is

$$\boxed{\text{CV.}}$$

(e) Since $I = 3$ is large relative to both prices, the relevant formula is

$$V(I,p) = I + \phi^*(p) = I + \frac{(1-p)^2}{2}.$$

Thus

$$V(3,0.5) = 3 + \phi^*(0.5) = 3 + \frac{1}{8} = \frac{25}{8}, \qquad V(3,1) = 3 + \phi^*(1) = 3.$$

Hence

$$EV = \phi^*(0.5) - \phi^*(1) = \frac{1}{8}, \qquad CV = \phi^*(0.5) - \phi^*(1) = \frac{1}{8}.$$

Therefore

$$\boxed{EV = \frac{1}{8}, \qquad CV = \frac{1}{8}.}$$

(f) Consumer surplus change equals the change in the conjugate surplus term:

$$\Delta CS = \phi^*(1) - \phi^*(0.5) = 0 - \frac{1}{8} = -\frac{1}{8}.$$

Therefore

$$\boxed{\Delta CS = -\frac{1}{8}.}$$

Unofficial_QRS@NTU_Tutorials

Question 4: Cobb Douglas Utility and Cost Minimisation

Consider a consumer with cobb douglas utility

$$U(x_1, x_2) = x_1^{0.4} x_2^{0.6}.$$

Problem

(a) Derive the demand functions for x_1 and x_2 for prices p_1, p_2 and income I.

(b) Derive the consumer's indirect utility function.

Consider initial prices $p_1 = 10, p_2 = 10$. The price of good 2 increases from 10 to 20. Income remains fixed at $I = 100$.

(c) What is the consumer's maximum utility at the initial prices.

(d) * What is the lowest cost consumption bundle (x_1, x_2) which at the new prices, gives the consumer the utility in (c)?

Solution

Method 1: Direct optimisation and Hicksian reconstruction

(a) The consumer solves

$$\max_{x_1, x_2 > 0} \; x_1^{0.4} x_2^{0.6} \qquad \text{subject to} \qquad p_1 x_1 + p_2 x_2 = I.$$

Form the Lagrangian

$$\mathcal{L} = x_1^{0.4} x_2^{0.6} + \lambda(I - p_1 x_1 - p_2 x_2).$$

The first-order conditions are

$$0.4 x_1^{-0.6} x_2^{0.6} = \lambda p_1, \qquad 0.6 x_1^{0.4} x_2^{-0.4} = \lambda p_2.$$

Dividing the first equation by the second gives

$$\frac{0.4}{0.6} \frac{x_2}{x_1} = \frac{p_1}{p_2},$$

so

$$\frac{2}{3} \frac{x_2}{x_1} = \frac{p_1}{p_2} \quad \Longrightarrow \quad x_2 = \frac{3p_1}{2p_2} x_1.$$

Substitute into the budget constraint:

$$p_1 x_1 + p_2 \left(\frac{3p_1}{2p_2} x_1 \right) = I \quad \Longrightarrow \quad \frac{5}{2} p_1 x_1 = I.$$

Unofficial QRS@NTU Solutions

Hence

$$x_1^*(p_1, p_2, I) = \frac{2I}{5p_1} = \frac{0.4I}{p_1}.$$

Then

$$x_2^*(p_1, p_2, I) = \frac{I - p_1 x_1^*}{p_2} = \frac{I - 0.4I}{p_2} = \frac{0.6I}{p_2}.$$

Therefore

$$\boxed{x_1^*(p_1, p_2, I) = \frac{0.4I}{p_1}, \qquad x_2^*(p_1, p_2, I) = \frac{0.6I}{p_2}.}$$

(b) Substitute the Marshallian demands into utility:

$$V(p_1, p_2, I) = \left(\frac{0.4I}{p_1}\right)^{0.4} \left(\frac{0.6I}{p_2}\right)^{0.6}.$$

Hence

$$\boxed{V(p_1, p_2, I) = 0.4^{0.4} 0.6^{0.6}\, I\, p_1^{-0.4} p_2^{-0.6}.}$$

(c) At the initial prices $p_1 = p_2 = 10$ and income $I = 100$,

$$x_1^* = \frac{0.4(100)}{10} = 4, \qquad x_2^* = \frac{0.6(100)}{10} = 6.$$

Therefore the maximum utility is

$$u_0 = 4^{0.4} 6^{0.6}.$$

Hence

$$\boxed{u_0 = 4^{0.4} 6^{0.6} \approx 5.101698.}$$

(d) Now prices are $p_1 = 10$, $p_2 = 20$. We seek the expenditure-minimising bundle that attains utility u_0. Thus solve

$$\min_{x_1, x_2 > 0} 10x_1 + 20x_2 \qquad \text{subject to} \qquad x_1^{0.4} x_2^{0.6} = u_0.$$

At an interior Hicksian optimum,

$$\frac{MU_1}{MU_2} = \frac{p_1}{p_2}.$$

Since

$$MU_1 = 0.4 x_1^{-0.6} x_2^{0.6}, \qquad MU_2 = 0.6 x_1^{0.4} x_2^{-0.4},$$

this gives

$$\frac{0.4}{0.6} \frac{x_2}{x_1} = \frac{10}{20} = \frac{1}{2}.$$

Hence

$$\frac{2}{3} \frac{x_2}{x_1} = \frac{1}{2} \quad \Longrightarrow \quad \frac{x_2}{x_1} = \frac{3}{4} \quad \Longrightarrow \quad x_2 = \frac{3}{4} x_1.$$

Impose the utility constraint:

$$x_1^{0.4}\left(\frac{3}{4}x_1\right)^{0.6} = u_0 \quad \Longrightarrow \quad x_1\left(\frac{3}{4}\right)^{0.6} = u_0.$$

Therefore

$$x_1 = u_0\left(\frac{4}{3}\right)^{0.6}.$$

Since

$$u_0 = 4^{0.4}6^{0.6},$$

we get

$$x_1 = 4^{0.4}6^{0.6}\left(\frac{4}{3}\right)^{0.6} = 4^{0.4}8^{0.6} = 4\cdot 2^{0.6} = 4\cdot 2^{3/5}.$$

Thus

$$x_2 = \frac{3}{4}x_1 = 3\cdot 2^{3/5}.$$

Numerically,

$$x_1 \approx 6.062866, \qquad x_2 \approx 4.547150.$$

Therefore the lowest-cost bundle is

$$\boxed{(x_1, x_2) = \left(4\cdot 2^{3/5},\, 3\cdot 2^{3/5}\right) \approx (6.062866,\, 4.547150).}$$

Unofficial_QRS@NTU_Solutions

Method 2: Expenditure-share and expenditure-function method

(a) For Cobb Douglas utility $x_1^{0.4}x_2^{0.6}$, the Marshallian demands allocate fixed expenditure shares:

$$40\% \text{ of income to good 1}, \qquad 60\% \text{ of income to good 2}.$$

Hence

$$\boxed{x_1^*(p_1,p_2,I) = \frac{0.4I}{p_1}, \qquad x_2^*(p_1,p_2,I) = \frac{0.6I}{p_2}.}$$

(b) Substituting these into utility,

$$V(p_1,p_2,I) = \left(\frac{0.4I}{p_1}\right)^{0.4}\left(\frac{0.6I}{p_2}\right)^{0.6}.$$

Therefore

$$\boxed{V(p_1,p_2,I) = 0.4^{0.4}0.6^{0.6}\, I\, p_1^{-0.4}p_2^{-0.6}.}$$

(c) At $(p_1,p_2,I) = (10,10,100)$,

$$x_1^* = 4, \qquad x_2^* = 6.$$

Therefore

$$\boxed{u_0 = 4^{0.4}6^{0.6} \approx 5.101698.}$$

(d) For Cobb Douglas utility $x_1^{0.4}x_2^{0.6}$, the expenditure function is

$$e(p_1,p_2,u) = \frac{u\, p_1^{0.4}p_2^{0.6}}{0.4^{0.4}0.6^{0.6}}.$$

At the new prices $(10,20)$, the minimum expenditure to attain u_0 is

$$e(10,20,u_0) = \frac{4^{0.4}6^{0.6}\,10^{0.4}20^{0.6}}{0.4^{0.4}0.6^{0.6}}.$$

Since $u_0 = 0.4^{0.4}0.6^{0.6}\cdot 100\cdot 10^{-0.4}10^{-0.6}$, this simplifies to

$$e(10,20,u_0) = 100\left(\frac{20}{10}\right)^{0.6} = 100\cdot 2^{0.6} = 100\cdot 2^{3/5}.$$

Numerically,

$$e(10,20,u_0) \approx 151.571657.$$

Hicksian demands allocate the same expenditure shares, so

$$h_1(10,20,u_0) = \frac{0.4\,e(10,20,u_0)}{10} = 4\cdot 2^{3/5} \approx 6.062866,$$

$$h_2(10,20,u_0) = \frac{0.6\,e(10,20,u_0)}{20} = 3\cdot 2^{3/5} \approx 4.547150.$$

Therefore

$$\boxed{(x_1,x_2) = \left(4\cdot 2^{3/5},\, 3\cdot 2^{3/5}\right) \approx (6.062866,\, 4.547150).}$$

Method 3: Log-utility method

(a) Since log is strictly increasing, maximising

$$U(x_1, x_2) = x_1^{0.4} x_2^{0.6}$$

is equivalent to maximising

$$0.4 \ln x_1 + 0.6 \ln x_2$$

subject to

$$p_1 x_1 + p_2 x_2 = I.$$

Form the Lagrangian

$$\mathcal{L} = 0.4 \ln x_1 + 0.6 \ln x_2 + \lambda(I - p_1 x_1 - p_2 x_2).$$

The first-order conditions are

$$\frac{0.4}{x_1} = \lambda p_1, \qquad \frac{0.6}{x_2} = \lambda p_2.$$

Multiplying by x_1 and x_2 respectively,

$$x_1 = \frac{0.4}{\lambda p_1}, \qquad x_2 = \frac{0.6}{\lambda p_2}.$$

Using the budget constraint,

$$p_1 x_1 + p_2 x_2 = \frac{0.4}{\lambda} + \frac{0.6}{\lambda} = \frac{1}{\lambda} = I.$$

Hence $\lambda = 1/I$, so

$$\boxed{x_1^*(p_1, p_2, I) = \frac{0.4I}{p_1}, \qquad x_2^*(p_1, p_2, I) = \frac{0.6I}{p_2}.}$$

(b) Substituting back,

$$V(p_1, p_2, I) = \left(\frac{0.4I}{p_1}\right)^{0.4} \left(\frac{0.6I}{p_2}\right)^{0.6}.$$

Therefore

$$\boxed{V(p_1, p_2, I) = 0.4^{0.4} 0.6^{0.6} \, I \, p_1^{-0.4} p_2^{-0.6}.}$$

(c) At $(10, 10, 100)$,

$$x_1^* = 4, \qquad x_2^* = 6,$$

Therefore

$$\boxed{u_0 = 4^{0.4} 6^{0.6} \approx 5.101698.}$$

Unofficial – use at your own risk

(d) At the new prices $(10, 20)$, the Hicksian problem is

$$\min_{x_1,x_2>0} \; 10x_1 + 20x_2 \qquad \text{s.t.} \qquad 0.4\ln x_1 + 0.6\ln x_2 = \ln u_0.$$

Form the Lagrangian

$$\mathcal{L} = 10x_1 + 20x_2 + \mu\big(\ln u_0 - 0.4\ln x_1 - 0.6\ln x_2\big).$$

The first-order conditions are

$$10 = \mu\frac{0.4}{x_1}, \qquad 20 = \mu\frac{0.6}{x_2}.$$

Hence

$$x_1 = \frac{0.4\mu}{10}, \qquad x_2 = \frac{0.6\mu}{20},$$

so

$$\frac{x_2}{x_1} = \frac{0.6/20}{0.4/10} = \frac{3}{4}.$$

Thus $x_2 = \frac{3}{4}x_1$. Imposing the utility constraint,

$$x_1^{0.4}\left(\frac{3}{4}x_1\right)^{0.6} = u_0$$

gives

$$x_1 = 4\cdot 2^{3/5} \approx 6.062866, \qquad x_2 = 3\cdot 2^{3/5} \approx 4.547150.$$

Therefore

$$\boxed{(x_1, x_2) = \big(4\cdot 2^{3/5},\, 3\cdot 2^{3/5}\big) \approx (6.062866,\, 4.547150).}$$

Unofficial Solutions

Method 4: General Cobb Douglas theorem

(a) For the general Cobb Douglas utility

$$U(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha},$$

the Marshallian demands are

$$x_1^* = \frac{\alpha I}{p_1}, \qquad x_2^* = \frac{(1-\alpha)I}{p_2}.$$

Here $\alpha = 0.4$, so

$$\boxed{x_1^*(p_1, p_2, I) = \frac{0.4I}{p_1}, \qquad x_2^*(p_1, p_2, I) = \frac{0.6I}{p_2}.}$$

(b) The general indirect utility formula is

$$V(p_1, p_2, I) = \alpha^{\alpha}(1-\alpha)^{1-\alpha} I\, p_1^{-\alpha} p_2^{-(1-\alpha)}.$$

Substituting $\alpha = 0.4$ yields

$$\boxed{V(p_1, p_2, I) = 0.4^{0.4} 0.6^{0.6}\, I\, p_1^{-0.4} p_2^{-0.6}.}$$

(c) At $(p_1, p_2, I) = (10, 10, 100)$,

$$x_1^* = 4, \qquad x_2^* = 6,$$

so

$$\boxed{u_0 = 4^{0.4} 6^{0.6} \approx 5.101698.}$$

(d) The general expenditure function for Cobb Douglas utility is

$$e(p_1, p_2, u) = \frac{u\, p_1^{\alpha} p_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}.$$

With $\alpha = 0.4$,

$$e(p_1, p_2, u) = \frac{u\, p_1^{0.4} p_2^{0.6}}{0.4^{0.4} 0.6^{0.6}}.$$

Hence at the new prices,

$$e(10, 20, u_0) = 100 \cdot 2^{0.6} = 100 \cdot 2^{3/5} \approx 151.571657.$$

The Hicksian demands are

$$h_1(p_1, p_2, u) = \frac{0.4\, e(p_1, p_2, u)}{p_1}, \qquad h_2(p_1, p_2, u) = \frac{0.6\, e(p_1, p_2, u)}{p_2}.$$

Therefore

$$h_1(10, 20, u_0) = \frac{0.4(100 \cdot 2^{3/5})}{10} = 4 \cdot 2^{3/5} \approx 6.062866,$$

$$h_2(10, 20, u_0) = \frac{0.6(100 \cdot 2^{3/5})}{20} = 3 \cdot 2^{3/5} \approx 4.547150.$$

Hence

$$\boxed{(x_1, x_2) = \left(4 \cdot 2^{3/5},\, 3 \cdot 2^{3/5}\right) \approx (6.062866,\, 4.547150).}$$

Sample Question 1: Fruitland, Cobb Douglas Utility, and Welfare Measures

There are two types of fruits grown for consumption in Fruitland, A and B, with initial prices p_A and p_B respectively. Due to climate change, a recent drought caused the price of A to increase to $p'_A > p_A$. The price of B, p_B remains the same.

Assume that a typical consumer in Fruitland has cobb douglas utility

$$U(C_A, C_B) = C_A^{0.5} C_B^{0.5},$$

and an income of M, where C_A and C_B is the consumption of A and B respectively.

Problem

(a) Illustrate graphically in separate diagrams, (i) the compensating variation and (ii) the equivalent variation of the price increase.

(b) Suppose $M = 100$, $p_A = 10$, $p_B = 10$ and $p'_A = 20$. Calculate the exact value of the compensating variation.

(c) Will equivalent variation be equal to compensating variation in this case? Explain. (Hint: you don't need to calculate EV)

Unofficial – QRS@NTU – Use at your own risk

Solution

Method 1: Graphical interpretation and expenditure-function method

(a) Let the initial prices be (p_A, p_B) and the new prices be (p'_A, p_B), with $p'_A > p_A$. Since only the price of fruit A rises, the budget line becomes steeper after the price change. The consumer moves from the initial utility level u_0 to the lower utility level u_1, where

$$u_1 < u_0.$$

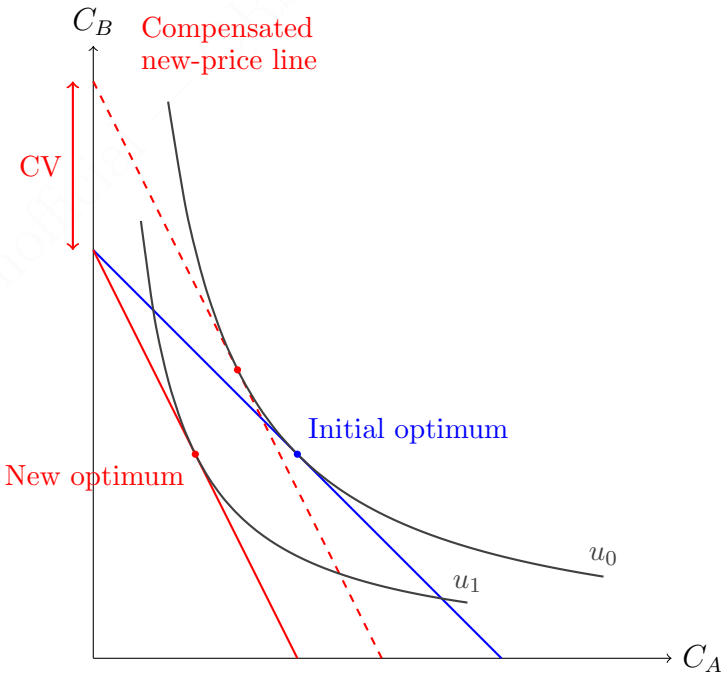
To illustrate the welfare measures clearly, it is helpful to draw two separate diagrams.

(i) Compensating variation.

CV is the extra income needed *at the new prices* to restore the consumer to the original utility level u_0. Hence:

- the original budget line is tangent to u_0;
- the new budget line is tangent to u_1;
- a compensated budget line, parallel to the new budget line, is shifted outward until it is tangent to u_0.

The monetary gap between the new budget line and this compensated budget line is the compensating variation.

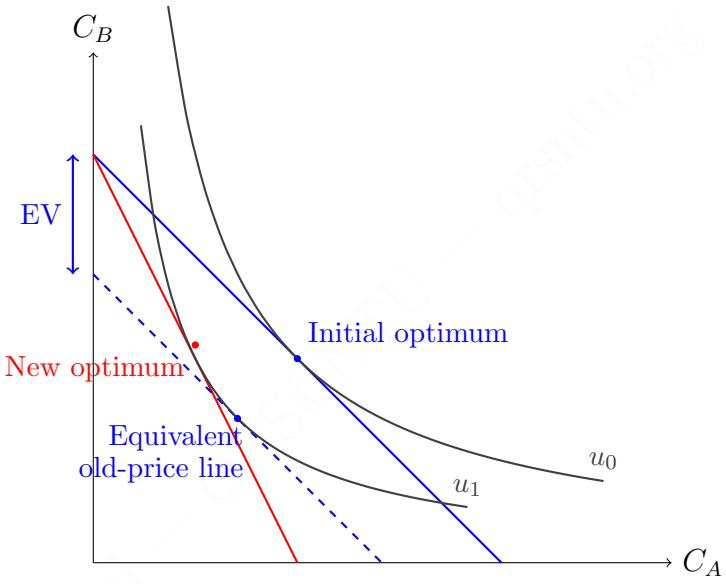


(ii) Equivalent variation.

EV is the amount of income that could be taken away *at the old prices* so that the consumer is just as badly off as after the price increase. Hence:

- the original budget line is tangent to u_0;
- the new budget line is tangent to u_1;
- an equivalent budget line, parallel to the original budget line, is shifted inward until it is tangent to u_1.

The monetary gap between the original budget line and this inward-shifted old-price line is the equivalent variation.



These diagrams show the standard Hicksian welfare interpretation:

$$\text{CV} = e(p'_A, p_B, u_0) - e(p_A, p_B, u_0), \qquad \text{EV} = e(p_A, p_B, u_0) - e(p_A, p_B, u_1).$$

(b) For Cobb Douglas utility

$$U(C_A, C_B) = C_A^{0.5} C_B^{0.5},$$

the expenditure function is

$$e(p_A, p_B, u) = 2u\sqrt{p_A p_B}.$$

At the initial prices $(10, 10)$ and income $M = 100$, the consumer spends half the income on each good, so

$$C_A^* = \frac{50}{10} = 5, \qquad C_B^* = \frac{50}{10} = 5.$$

Hence the initial utility is

$$u_0 = \sqrt{5 \cdot 5} = 5.$$

CV is the extra income needed at the new prices $(20, 10)$ to attain the original utility $u_0 = 5$. Thus

$$CV = e(20, 10, 5) - 100.$$

Compute:

$$e(20, 10, 5) = 2 \cdot 5\sqrt{20 \cdot 10} = 10\sqrt{200} = 10 \cdot 10\sqrt{2} = 100\sqrt{2}.$$

Therefore

$$CV = 100\sqrt{2} - 100 = 100(\sqrt{2} - 1).$$

Hence

$$\boxed{CV = 100(\sqrt{2} - 1).}$$

(c) No, EV will not be equal to CV in this case.

The reason is that Cobb Douglas preferences are *not* quasilinear, so demand exhibits income effects. Fruit A is a normal good under Cobb Douglas preferences, and the price increase is harmful. For a price increase of a normal good, restoring the consumer to the original utility at the new prices is more expensive than measuring the equivalent utility loss at the old prices. Hence

$$CV > EV.$$

Therefore

$$\boxed{EV \neq CV, \qquad CV > EV.}$$

Method 2: Hicksian minimisation method

(a) The graphical interpretation is the same as in Method 1.

For CV, use the *new* price slope and restore the *old* indifference curve. For EV, use the *old* price slope and reduce income until the *new* indifference curve is just reached.

More precisely:

$$\text{CV uses } (p'_A, p_B) \text{ and } u_0, \qquad \text{EV uses } (p_A, p_B) \text{ and } u_1.$$

(b) At the initial prices $(10, 10)$ and income 100, Cobb Douglas expenditure shares imply

$$C_A^* = 5, \qquad C_B^* = 5,$$

so

$$u_0 = \sqrt{5 \cdot 5} = 5.$$

To compute CV, solve the Hicksian problem at the new prices:

$$\min_{C_A, C_B > 0} 20C_A + 10C_B \qquad \text{subject to} \qquad C_A^{0.5} C_B^{0.5} = 5.$$

Equivalently,

$$C_A C_B = 25.$$

Substitute

$$C_B = \frac{25}{C_A}$$

into expenditure:

$$E(C_A) = 20C_A + 10\left(\frac{25}{C_A}\right) = 20C_A + \frac{250}{C_A}.$$

Differentiate:

$$E'(C_A) = 20 - \frac{250}{C_A^2}.$$

Setting $E'(C_A) = 0$,

$$20 = \frac{250}{C_A^2} \quad \Longrightarrow \quad C_A^2 = \frac{25}{2} \quad \Longrightarrow \quad C_A = \frac{5}{\sqrt{2}}.$$

Then

$$C_B = \frac{25}{5/\sqrt{2}} = 5\sqrt{2}.$$

Thus minimum expenditure is

$$20\left(\frac{5}{\sqrt{2}}\right) + 10(5\sqrt{2}) = 50\sqrt{2} + 50\sqrt{2} = 100\sqrt{2}.$$

Hence

$$CV = 100\sqrt{2} - 100 = 100(\sqrt{2} - 1).$$

Therefore

$$\boxed{CV = 100(\sqrt{2} - 1).}$$

Tutorial_QRS@NTU_answers

(c) At the new prices $(20, 10)$ and income 100, the Marshallian bundle is

$$C'_A = \frac{50}{20} = 2.5, \qquad C'_B = \frac{50}{10} = 5.$$

Hence the new utility is

$$u_1 = \sqrt{2.5 \cdot 5} = \frac{5}{\sqrt{2}}.$$

Since preferences are Cobb Douglas, there are income effects. Therefore EV and CV are not equal in general. Moreover, because fruit A is a normal good and its price rises,

$$CV > EV.$$

Hence

$$\boxed{EV \neq CV, \qquad CV > EV.}$$

Unofficial_QRS@NTU_orsntu.org

Sample Question 2: Bob, Quasi-linear Utility, and Reservation Prices

Bob, who has wealth of w is deciding where to go for tea-time. There are 3 possible options, i) X, ii) Y, iii) Z, and he can choose only 1 of the options.

If he chooses an option, its value is 1. If he does not choose the option, its value is 0. For example, if he chooses Y, then $X = 0$, $Y = 1$, $Z = 0$ and so on.

His utility depends on which item he chooses and his leftover holdings of cash m as follows:

$$u(X, Y, Z, m) = a\sqrt{X} + b\sqrt{Y} + c\sqrt{Z} + dm,$$

where $a, b, c, d > 0$.

Problem

(a) State the definition of quasi-linear utility. Is Bob's utility quasi-linear?

(b) Solve for Bob's reservation prices for option X, option Y and option Z.

Solution

Method 1: Definition and indifference-pricing method

(a) A utility function is *quasi-linear* if it can be written in the form

$$u(q, m) = \phi(q) + m$$

up to a positive affine rescaling of the money variable, where utility is linear in one good, usually money.

Here,

$$u(X, Y, Z, m) = a\sqrt{X} + b\sqrt{Y} + c\sqrt{Z} + dm.$$

Since $d > 0$, this is linear in money m. Therefore Bob's utility is quasi-linear. Equivalently, one may write

$$u(X, Y, Z, m) = \phi(X, Y, Z) + dm,$$

where

$$\phi(X, Y, Z) = a\sqrt{X} + b\sqrt{Y} + c\sqrt{Z}.$$

Hence

$$\boxed{\text{Bob's utility is quasi-linear.}}$$

Tutorial_QRS@NTU_Solutions

(b) Since $X, Y, Z \in \{0, 1\}$, we have

$$\sqrt{X} = X, \qquad \sqrt{Y} = Y, \qquad \sqrt{Z} = Z.$$

So the utility may be simplified to

$$u(X, Y, Z, m) = aX + bY + cZ + dm.$$

The reservation price for option X is the maximum amount Bob is willing to pay to choose X rather than choose no option. If he chooses no option, his utility is

$$u(0, 0, 0, w) = dw.$$

If he chooses option X and pays price r_X, then his leftover cash is $w - r_X$, so his utility is

$$u(1, 0, 0, w - r_X) = a + d(w - r_X).$$

At the reservation price he is indifferent:

$$a + d(w - r_X) = dw.$$

Hence

$$dr_X = a \quad \Longrightarrow \quad \boxed{r_X = \frac{a}{d}.}$$

Similarly, for option Y,

$$u(0, 1, 0, w - r_Y) = b + d(w - r_Y),$$

and indifference with choosing no option gives

$$b + d(w - r_Y) = dw \quad \Longrightarrow \quad \boxed{r_Y = \frac{b}{d}.}$$

Likewise, for option Z,

$$u(0, 0, 1, w - r_Z) = c + d(w - r_Z),$$

so

$$c + d(w - r_Z) = dw \quad \Longrightarrow \quad \boxed{r_Z = \frac{c}{d}.}$$

Therefore

$$\boxed{r_X = \frac{a}{d}, \qquad r_Y = \frac{b}{d}, \qquad r_Z = \frac{c}{d}.}$$

Unofficial_ORSOC_HE2001_Tutorial_Solutions.pdf

Method 2: Utility-value conversion method

(a) A utility function is quasi-linear if utility is linear in one argument, typically money. Here

$$u(X, Y, Z, m) = a\sqrt{X} + b\sqrt{Y} + c\sqrt{Z} + dm$$

is linear in m, so Bob's utility is quasi-linear.

Hence

$$\boxed{\text{Bob's utility is quasi-linear.}}$$

(b) Since one extra dollar of leftover cash raises utility by

$$d,$$

the marginal utility of money is d. Therefore a utility increment of size Δu has money equivalent

$$\frac{\Delta u}{d}.$$

Now the utility bonus from choosing option X rather than no option is

$$a\sqrt{1} = a.$$

So its reservation price is

$$\boxed{r_X = \frac{a}{d}.}$$

Similarly, option Y gives utility bonus

$$b\sqrt{1} = b,$$

so

$$\boxed{r_Y = \frac{b}{d}.}$$

Option Z gives utility bonus

$$c\sqrt{1} = c,$$

so

$$\boxed{r_Z = \frac{c}{d}.}$$

Therefore

$$\boxed{r_X = \frac{a}{d}, \qquad r_Y = \frac{b}{d}, \qquad r_Z = \frac{c}{d}.}$$

Unofficial — QRS@NTU — qrs.ntu.edu.sg

Method 3: Binary-choice simplification method

(a) Because each of X, Y, Z is binary,

$$X, Y, Z \in \{0, 1\},$$

the square-root terms simplify exactly:

$$\sqrt{X} = X, \qquad \sqrt{Y} = Y, \qquad \sqrt{Z} = Z.$$

Hence Bob's utility can be rewritten as

$$u(X, Y, Z, m) = aX + bY + cZ + dm.$$

This is of quasi-linear form: an additive term depending on the discrete choice variables plus a linear money term.

Therefore

$$\boxed{\text{Bob's utility is quasi-linear.}}$$

(b) In the simplified form

$$u(X, Y, Z, m) = aX + bY + cZ + dm,$$

choosing no option yields utility

$$dw.$$

Choosing option X and paying price p yields

$$a + d(w - p).$$

Bob is indifferent when

$$a + d(w - p) = dw \quad \Longrightarrow \quad p = \frac{a}{d}.$$

So

$$\boxed{r_X = \frac{a}{d}.}$$

Repeating the same argument for Y and Z,

$$\boxed{r_Y = \frac{b}{d}, \qquad r_Z = \frac{c}{d}.}$$

Therefore

$$\boxed{r_X = \frac{a}{d}, \qquad r_Y = \frac{b}{d}, \qquad r_Z = \frac{c}{d}.}$$

Unofficial – Use at your own discretion.