Definite integrals & FTC (Ch1)

Riemann sum / net area

$\int_a^b f(x)\,dx := \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$, $\Delta x = \frac{b-a}{n}$, $x_i^* \in [x_{i-1}, x_i]$ (if limit exists, independent of sample points). Net area: $\int_a^b f$ = (area above) − (area below).

Core properties (integrable on $[a,b]$)

Linearity: $\int_a^b(\alpha f+\beta g) = \alpha\int_a^b f + \beta\int_a^b g$; $\int_a^a f = 0$; $\int_a^b f = -\int_b^a f$.
Additivity: $\int_a^b f = \int_a^c f + \int_c^b f$.
Order/Bounds: $f \ge g \Rightarrow \int_a^b f \ge \int_a^b g$; $m \le f \le M \Rightarrow m(b-a) \le \int_a^b f \le M(b-a)$.
Triangle ineq.: $\left|\int_a^b f\right| \le \int_a^b |f|$; if $|f| \le M$ then $\left|\int_a^b f\right| \le M(b-a)$.
Symmetry: f odd $\Rightarrow \int_{-a}^a f = 0$; f even $\Rightarrow \int_{-a}^a f = 2\int_0^a f$.

FTC (conditions matter)

FTC I: f cont on $[a,b]$, $F(x) = \int_a^x f(t)dt \Rightarrow F'(x) = f(x)$ on (a,b).
Leibniz (moving limit): $\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))g'(x)$ (f cont, g diff).
More general: $\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v(x))v'(x) - f(u(x))u'(x)$.
FTC II: f cont on $[a,b]$, $F' = f \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$.
Average value: $f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$ (cont).
MVT (integrals): f cont $\Rightarrow \exists c \in (a,b)$ s.t. $\int_a^b f = f(c)(b-a)$, so $f_{\text{avg}} = f(c)$.

u-substitution

Indef: $u = g(x)$ diff $\Rightarrow \int f(g(x))g'(x)\,dx = \int f(u)\,du$.
Def: $g \in C^1[a,b]$, f cont on range(g): $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ (change limits).
Heuristic: spot inner $g(x)$ + missing $g'(x)$ (up to constant).

Improper integrals

Type 1 (infinite): $\int_a^\infty f := \lim_{t\to\infty}\int_a^t f$; $\int_{-\infty}^b f := \lim_{t\to-\infty}\int_t^b f$.
Type 2 (blow-up): if $f \to \infty$ at a^+, then $\int_a^b f := \lim_{t\to a^+}\int_t^b f$ (split at interior singularities).
Convergence is required for the value to be defined; never cancel divergences across a split.
p-tests: $\int_1^\infty x^{-p}\,dx$ conv $\Leftrightarrow p > 1$; $\int_0^1 x^{-p}\,dx$ conv $\Leftrightarrow p < 1$.
Comparison: $0 \le f \le g$, $\int g$ conv $\Rightarrow \int f$ conv; $0 \le g \le f$, $\int g$ div $\Rightarrow \int f$ div.
Limit comparison: $f, g > 0$, $\lim_{x\to\cdot}\frac{f}{g} = c \in (0,\infty) \Rightarrow$ same conv/div.
Alternating/absolute: if $\int |f|$ conv $\Rightarrow \int f$ conv (absolute conv).

Techniques of integration (Ch3)

Integration by parts (IBP)

IBP: $\int u\,dv = uv - \int v\,du$; Def IBP: $\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$.
Choose u: **L**og, **I**nv trig, **A**lgebraic, **T**rig, **E**xp (LIATE).
Tabular (poly×trig/exp): alternate signs; diff poly to 0; integrate other factor repeatedly.
Repeated-cycle: set $I = \int e^{ax}\sin bx\,dx$ or $\int e^{ax}\cos bx\,dx$, IBP twice $\Rightarrow$ solve linear equation for I.
Definite IBP pitfall: evaluate uv at endpoints *before* simplifying $\int v\,du$.

High-yield closed forms (from repeated IBP)

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\big(a\sin bx - b\cos bx\big) + C \ (a,b \in \mathbb{R}).$$
$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\big(a\cos bx + b\sin bx\big) + C.$$

Trig integrals: pattern selection (no identities here)

$\int \sin^m x\cos^n x\,dx$: n odd $\Rightarrow$ save $\cos x\,dx$, convert rest via *Formula Sheet*, $u = \sin x$; m odd $\Rightarrow$ save $\sin x\,dx$, $u = \cos x$; both even $\Rightarrow$ power-reduction (*Formula Sheet*).
$\int \tan^m x\sec^n x\,dx$: n even $\Rightarrow$ save $\sec^2 x\,dx$, $u = \tan x$; m odd $\Rightarrow$ save $\sec x\tan x\,dx$, $u = \sec x$.
Mixed arguments (e.g. $\sin kx\cos mx$): product-to-sum (*Formula Sheet*).

Trig substitution: radicals $\sqrt{\cdot}$

$\sqrt{a^2-x^2}$: $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$, $\sqrt{a^2-x^2} = a\cos\theta$.
$\sqrt{a^2+x^2}$: $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$, $\sqrt{a^2+x^2} = a\sec\theta$.
$\sqrt{x^2-a^2}$: $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\,d\theta$, $\sqrt{x^2-a^2} = a\tan\theta$.
Back-sub: triangle method or $\theta = \arcsin(x/a)$ etc (*Formula Sheet*).
Complete square first for $\sqrt{ax^2+bx+c}$.

Partial fractions (PF)

Pre-step: if $\deg P \ge \deg Q$ do long division. Factor Q over $\mathbb{R}$.
Templates: (i) $(ax+b)^r$: $\sum_{k=1}^r \frac{A_k}{(ax+b)^k}$; (ii) irreducible $(ax^2+bx+c)^r$: $\sum_{k=1}^r \frac{A_kx+B_k}{(ax^2+bx+c)^k}$.
Fast solve: distinct linear factors $\Rightarrow$ cover-up (at roots) + solve leftovers.
Quadratic: write $Ax+B = \lambda(2ax+b)+\mu \Rightarrow \lambda\ln|ax^2+bx+c|$ + complete square for arctan(·) (see *Formula Sheet*).

Applications of integration (Ch2)

Area between curves

Vertical slices: $A = \int_{x=a}^b(\text{top}-\text{bottom})\,dx$ (split where curves cross / switch order).
Horizontal slices: $A = \int_{y=c}^d(\text{right}-\text{left})\,dy$ (often avoids splitting).
Bounds: solve intersections; ensure integrand ≥ 0 on each piece.

Volumes: general slicing

General: $V = \int A(x)\,dx$ (slices $\perp$ x-axis) or $V = \int A(y)\,dy$ (slices $\perp$ y-axis).
Revolution: build $A(\cdot)$ from radii; check units: A (length)^2 $\Rightarrow$ V (length)^3.

Volumes: disks / washers (revolution)

About $y = c$ using dx: $R_{out}(x) = |y_{top}(x)-c|$, $R_{in}(x) = |y_{bot}(x)-c|$,
$V = \pi\int_a^b\left(R_{out}^2 - R_{in}^2\right)dx$.
About $x = c$ using dy: $R_{out}(y) = |x_{right}(y)-c|$, $R_{in}(y) = |x_{left}(y)-c|$,
$V = \pi\int_c^d\left(R_{out}^2 - R_{in}^2\right)dy$.
Rule: washers slice **perpendicular** to axis; shells slice **parallel**.

Volumes: cylindrical shells (revolution)

About $x = c$ using dx (vertical strips): $r(x) = |x-c|$, $h(x) = y_{top} - y_{bot}$, $V = 2\pi\int_a^b r(x)h(x)\,dx$.
About $y = c$ using dy (horizontal strips): $r(y) = |y-c|$, $h(y) = x_{right} - x_{left}$, $V = 2\pi\int_c^d r(y)h(y)\,dy$.

Arc length / surface area

If $y = f(x)$, $f' \in C[a,b]$: $L = \int_a^b\sqrt{1+(f')^2}\,dx$; S about $y = c$: $S = 2\pi\int_a^b|f(x)-c|\sqrt{1+(f')^2}\,dx$.
If $x = g(y)$, $g' \in C[c,d]$: $L = \int_c^d\sqrt{1+(g')^2}\,dy$; S about $x = c$: $S = 2\pi\int_c^d|g(y)-c|\sqrt{1+(g')^2}\,dy$.
(Parametric, if used): $L = \int_\alpha^\beta\sqrt{(x')^2+(y')^2}\,dt$.

Average value

$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$ (cont on $[a,b]$).

Work / pumping / hydrostatic force (templates)

Work (1D): $W = \int_a^b F(x)\,dx$ (force along line).
Hooke spring: $F = kx$ (displacement from natural length) $\Rightarrow W = \int_{x=a}^b kx\,dx$ (see *Formula Sheet*).
Pumping: slice at height y: $dW = \rho g\,A(y)\,D(y)\,dy$, so $W = \int dW$ (use weight density ρg).
Hydrostatic force: pressure $p(y) = \rho g\,h(y)$ (depth), $dF = p(y)\,w(y)\,dy$, $F = \int_{y=c}^d \rho g\,h(y)w(y)\,dy$.

Numerical integration (Ch3)

Rules

Partition: $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$.
Left: $L_n = \sum_{i=1}^n f(x_{i-1})\Delta x$; Right: $R_n = \sum_{i=1}^n f(x_i)\Delta x$.
Midpoint: $M_n = \sum_{i=1}^n f\big(\frac{x_{i-1}+x_i}{2}\big)\Delta x$.
Trapezoid: $T_n = \frac{\Delta x}{2}\big(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\big) = \frac{L_n+R_n}{2}$.
Simpson (n even): $S_n = \frac{\Delta x}{3}\big(f(x_0) + 4\sum_{\text{odd } i} f(x_i) + 2\sum_{\text{even } i\ne 0,n} f(x_i) + f(x_n)\big)$.

Error bounds (need smoothness)

If f'' cont on $[a,b]$, $|f''| \le K$: $|E_T| \le \frac{K(b-a)^3}{12n^2}$, $|E_M| \le \frac{K(b-a)^3}{24n^2}$.
If $f^{(4)}$ cont on $[a,b]$, $|f^{(4)}| \le K$ (n even): $|E_S| \le \frac{K(b-a)^5}{180n^4}$.
Quick monotonicity check (common in tutorials): if $f'' \ge 0$ then T_n overestimates and M_n underestimates (reverse if $f'' \le 0$).