MH1201 Linear Algebra II

Final Exam (Solutions)

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Overview

- Topics: linear maps on matrix spaces, diagonalisation, eigenspaces, Gram–Schmidt, QR decomposition, matrix-induced inner products, and dimension arguments.
- Strategy: the shortest exam-efficient method is placed first; a second method is included when it gives a genuinely different viewpoint.
- Notation: $\text{null}(T)$ and $\text{range}(T)$ denote the kernel and image of a linear map T.
- Mark schemes are unofficial but designed to reflect reasonable allocation of method, computation, justification, and final conclusion marks.

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Contents

Question 1 (20 Marks) 3
Problem . . . 3
Solution . . . 3
Method 1: Direct coordinate solution . . . 3
Method 2: Decomposition into symmetric and skew-symmetric parts . . . 5
Mark Scheme (20 marks) . . . 6

Question 2 (20 Marks) 7
Problem . . . 7
Solution . . . 7
Method 1: Eigenvalue and eigenspace computation . . . 7
Method 2: Minimal-polynomial viewpoint for the power . . . 9
Mark Scheme (20 marks) . . . 10

Question 3 (15 Marks) 11
Problem . . . 11
Solution . . . 11
Method 1: Dimension-count intersection argument . . . 11
Method 2: Geometric viewpoint . . . 12
Mark Scheme (15 marks) . . . 13

Question 4 (15 Marks) 14
Problem . . . 14
Solution . . . 14
Method 1: Gram–Schmidt on the columns . . . 14
Method 2: Reading R from inner products . . . 16
Mark Scheme (15 marks) . . . 16

Question 5 (15 Marks) 17
Problem . . . 17
Solution . . . 17
Method 1: Positive-definite matrix criterion . . . 17
Method 2: Completing the square . . . 18
Mark Scheme (15 marks) . . . 19

Question 6 (15 Marks) 20
Problem . . . 20
Solution . . . 20
Method 1: Use differences inside the span . . . 20
Method 2: Rank–nullity contradiction . . . 21
Mark Scheme (15 marks) . . . 22

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Question 1 (20 Marks)

Problem

Let n be a positive integer. Consider the function

$$T : \mathcal{M}_{n\times n}(\mathbb{R}) \to \mathcal{M}_{n\times n}(\mathbb{R}), \qquad T(M) = M + M^T.$$

(a) Show that T is a linear map.

(b) Fix $n = 2$.

(i) Determine the standard matrix for T.

(ii) Determine a basis for null(T).

(iii) Determine a basis for range(T).

(c) Show that

$$\dim \text{null}(T) = \frac{n(n-1)}{2}, \qquad \dim \text{range}(T) = \frac{n(n+1)}{2}$$

for all n.

Solution

Method 1: Direct coordinate solution

(a) For $M, N \in \mathcal{M}_{n\times n}(\mathbb{R})$ and $c \in \mathbb{R}$, we check the two linearity conditions separately. First, using $(M+N)^T = M^T + N^T$, we have

$$T(M+N) = (M+N) + (M+N)^T = M + N + M^T + N^T.$$

Rearranging the terms gives

$$T(M+N) = (M + M^T) + (N + N^T) = T(M) + T(N).$$

Next, using $(cM)^T = cM^T$, we get

$$T(cM) = cM + (cM)^T = cM + cM^T = c(M + M^T) = cT(M).$$

Hence T is additive and homogeneous, so

$$\boxed{T \text{ is a linear map.}}$$

(b)(i) Now fix $n = 2$ and use the ordered standard basis

$$(E_{11}, E_{12}, E_{21}, E_{22}).$$

For

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

we have

$$M^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \qquad T(M) = M + M^T = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}.$$

Thus, if M has coordinate vector $(a, b, c, d)^T$, then $T(M)$ has coordinate vector

$$(2a,\ b+c,\ b+c,\ 2d)^T.$$

Therefore the standard matrix of T is

$$\boxed{[T] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.}$$

Equivalently, this matrix may be read column-by-column from

$$T(E_{11}) = 2E_{11}, \quad T(E_{12}) = E_{12} + E_{21}, \quad T(E_{21}) = E_{12} + E_{21}, \quad T(E_{22}) = 2E_{22}.$$

(b)(ii) We need to solve $T(M) = 0$, i.e.

$$M + M^T = 0.$$

Using the expression above,

$$\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence

$$2a = 0, \qquad b + c = 0, \qquad 2d = 0.$$

So

$$a = 0, \qquad d = 0, \qquad c = -b.$$

Therefore every matrix in the null space has the form

$$M = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus

$$\boxed{\text{null}(T) = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.}$$

(b)(iii) For any matrix M, the matrix $T(M) = M + M^T$ is symmetric, because

$$T(M)^T = (M + M^T)^T = M^T + M = T(M).$$

So

$$\text{range}(T) \subseteq \{\text{symmetric } 2 \times 2 \text{ matrices}\}.$$

Conversely, if S is symmetric, then $S^T = S$, and

$$T\left(\frac{1}{2}S\right) = \frac{1}{2}S + \left(\frac{1}{2}S\right)^T = \frac{1}{2}S + \frac{1}{2}S = S.$$

Hence every symmetric 2×2 matrix lies in the range. Therefore

$$\text{range}(T) = \{\text{symmetric } 2 \times 2 \text{ matrices}\}.$$

A general symmetric 2×2 matrix has the form

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence

$$\boxed{\text{range}(T) = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.}$$

(c) For general n, the null space is the set of all matrices satisfying

$$M + M^T = 0.$$

This means $M^T = -M$, so M is skew-symmetric. A skew-symmetric matrix has zero diagonal entries, because

$$m_{ii} = -m_{ii} \quad \Longrightarrow \quad m_{ii} = 0.$$

For each pair $i < j$, the entry m_{ij} may be chosen freely, while the entry below the diagonal is then forced to be

$$m_{ji} = -m_{ij}.$$

There are

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

such pairs. Therefore

$$\boxed{\dim \operatorname{null}(T) = \frac{n(n-1)}{2}.}$$

Similarly, the range is the space of symmetric matrices. A symmetric matrix is determined by its n diagonal entries and its entries above the diagonal. There are n diagonal entries and $\binom{n}{2}$ entries above the diagonal, so

$$\dim \operatorname{range}(T) = n + \binom{n}{2} = n + \frac{n(n-1)}{2} = \boxed{\frac{n(n+1)}{2}}.$$

Method 2: Decomposition into symmetric and skew-symmetric parts

(a) Linearity also follows from the fact that transposition is linear and the identity map is linear. Since

$$T = I + S,$$

where $S(M) = M^T$, the map T is a sum of linear maps and is therefore linear.

(b) and (c) Every matrix $M \in \mathcal{M}_{n\times n}(\mathbb{R})$ has the unique decomposition

$$M = \frac{M + M^T}{2} + \frac{M - M^T}{2},$$

where

$$\frac{M + M^T}{2} \text{ is symmetric,} \qquad \frac{M - M^T}{2} \text{ is skew-symmetric.}$$

Under T,

$$T(M) = M + M^T = 2\left(\frac{M + M^T}{2}\right).$$

Thus T doubles the symmetric part and kills the skew-symmetric part. Therefore

$$\operatorname{null}(T) = \{\text{skew-symmetric matrices}\}, \qquad \operatorname{range}(T) = \{\text{symmetric matrices}\}.$$

For $n = 2$, this gives exactly the bases found in Method 1. For general n, counting free entries gives

$$\boxed{\dim \operatorname{null}(T) = \frac{n(n-1)}{2}, \qquad \dim \operatorname{range}(T) = \frac{n(n+1)}{2}.}$$

Mark Scheme (20 marks)

- **(a) Linearity (4 marks)**
  - (2) Correctly proves additivity using $(M+N)^T = M^T + N^T$.
  - (1) Correctly proves homogeneity using $(cM)^T = cM^T$.
  - (1) States the conclusion that T is linear.
- **(b) Case $n = 2$ (9 marks)**
  - (3) Correct standard matrix in the ordered basis $(E_{11}, E_{12}, E_{21}, E_{22})$.
  - (3) Correctly solves $M + M^T = 0$ and gives a valid basis for null(T).
  - (3) Identifies the range as the symmetric 2×2 matrices and gives a valid basis.
- **(c) General dimension formulae (7 marks)**
  - (3) Correctly identifies null(T) as the space of skew-symmetric matrices and counts $\binom{n}{2}$ free entries.
  - (3) Correctly identifies range(T) as the space of symmetric matrices and counts $n + \binom{n}{2}$ free entries.
  - (1) Presents both final dimension formulae clearly.

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Question 2 (20 Marks)

Problem

Let a and b be real numbers such that $b \neq 0$. Consider

$$A = \begin{bmatrix} 0 & 1 & b \\ 4 & 0 & 2b \\ 0 & 0 & a \end{bmatrix}.$$

(a) Show that the characteristic polynomial for A is $-(a-\lambda)(4-\lambda^2)$. Hence determine the eigenvalues of A in terms of a and b.

(b) Let b be nonzero. Show that A is diagonalizable if and only if $a \neq \beta$. Determine β.

(c) Fix $a = -2$, with $b \neq 0$.

(i) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

(ii) Determine A^{20}.

Solution

Method 1: Eigenvalue and eigenspace computation

(a) We compute the characteristic polynomial using $\det(A - \lambda I)$. We have

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & b \\ 4 & -\lambda & 2b \\ 0 & 0 & a-\lambda \end{bmatrix}.$$

This matrix is block upper triangular, so its determinant is the product of the determinant of the top-left 2×2 block and the bottom-right entry:

$$\det(A - \lambda I) = (a-\lambda) \det \begin{bmatrix} -\lambda & 1 \\ 4 & -\lambda \end{bmatrix}.$$

Now

$$\det \begin{bmatrix} -\lambda & 1 \\ 4 & -\lambda \end{bmatrix} = (-\lambda)(-\lambda) - 4 = \lambda^2 - 4.$$

Hence

$$\det(A - \lambda I) = (a-\lambda)(\lambda^2 - 4) = -(a-\lambda)(4-\lambda^2).$$

Therefore the eigenvalues are the roots of this polynomial, namely

$$\boxed{a,\ 2,\ -2.}$$

Notice that the eigenvalues do not depend on b, although the eigenspaces may depend on b.

(b) We are given $b \neq 0$. If $a \notin \{2, -2\}$, then the eigenvalues $a, 2, -2$ are three distinct real eigenvalues. A 3×3 matrix with three distinct eigenvalues has three linearly independent eigenvectors, so A is diagonalizable.

It remains to examine the repeated-eigenvalue cases $a = 2$ and $a = -2$.

First suppose $a = 2$. Then $\lambda = 2$ has algebraic multiplicity 2. We check the dimension of its eigenspace:

$$A - 2I = \begin{bmatrix} -2 & 1 & b \\ 4 & -2 & 2b \\ 0 & 0 & 0 \end{bmatrix}.$$

The first two rows are independent when $b \neq 0$. Indeed, if the second row were a multiple of the first row, then comparing the first entry gives the multiplier -2, but then the third entry would have to be $-2b$, not $2b$. Since $b \neq 0$, this is impossible. Hence

$$\operatorname{rank}(A - 2I) = 2, \qquad \dim E_2 = 3 - 2 = 1.$$

But $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1, so A is not diagonalizable when $a = 2$.

Next suppose $a = -2$. Then $\lambda = -2$ has algebraic multiplicity 2. We compute

$$A + 2I = \begin{bmatrix} 2 & 1 & b \\ 4 & 2 & 2b \\ 0 & 0 & 0 \end{bmatrix}.$$

Here the second row is exactly twice the first row, so

$$\operatorname{rank}(A + 2I) = 1, \qquad \dim E_{-2} = 3 - 1 = 2.$$

Thus the repeated eigenvalue -2 has a two-dimensional eigenspace. The remaining eigenvalue 2 contributes one more independent eigenvector. Therefore A has three linearly independent eigenvectors and is diagonalizable when $a = -2$.

Combining the cases, the only value of a for which diagonalizability fails is $a = 2$. Therefore

$$\boxed{\beta = 2, \qquad A \text{ is diagonalizable if and only if } a \neq 2.}$$

(c)(i) Now fix $a = -2$ with $b \neq 0$. Then

$$A = \begin{bmatrix} 0 & 1 & b \\ 4 & 0 & 2b \\ 0 & 0 & -2 \end{bmatrix}.$$

We find eigenvectors for the eigenvalues 2 and -2.

For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} -2 & 1 & b \\ 4 & -2 & 2b \\ 0 & 0 & -4 \end{bmatrix}.$$

Solving $(A - 2I)x = 0$ gives from the third row

$$-4x_3 = 0 \quad \Longrightarrow \quad x_3 = 0.$$

Then the first row gives

$$-2x_1 + x_2 = 0 \quad \Longrightarrow \quad x_2 = 2x_1.$$

Taking $x_1 = 1$, we get the eigenvector

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

For $\lambda = -2$,

$$A + 2I = \begin{bmatrix} 2 & 1 & b \\ 4 & 2 & 2b \\ 0 & 0 & 0 \end{bmatrix}.$$

The second row is twice the first, so the eigenspace is determined by the single equation

$$2x_1 + x_2 + bx_3 = 0.$$

Two convenient independent choices are:

	x_1	x_2	x_3
v_2	1	-2	0
v_3	0	$-b$	1

Thus

$$v_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ -b \\ 1 \end{bmatrix}.$$

Taking these eigenvectors as the columns of P, in the order corresponding to eigenvalues $2, -2, -2$, gives

$$\boxed{P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & -b \\ 0 & 0 & 1 \end{bmatrix}, \qquad D = \operatorname{diag}(2, -2, -2).}$$

Since the columns of P are linearly independent eigenvectors, P is invertible. Therefore

$$\boxed{A = PDP^{-1}.}$$

(c)(ii) Using the diagonalisation from part (c)(i),

$$A^{20} = (PDP^{-1})^{20} = PD^{20}P^{-1}.$$

Now

$$D^{20} = \operatorname{diag}(2^{20}, (-2)^{20}, (-2)^{20}).$$

Since 20 is even,

$$(-2)^{20} = 2^{20}.$$

Therefore

$$D^{20} = 2^{20} I_3.$$

Hence

$$A^{20} = P(2^{20} I_3)P^{-1} = 2^{20} PP^{-1} = 2^{20} I_3.$$

Thus

$$\boxed{A^{20} = 2^{20} I_3 = 1{,}048{,}576 I_3.}$$

Method 2: Minimal-polynomial viewpoint for the power

(c)(ii) For $a = -2$, part (b) shows that A is diagonalizable and its only eigenvalues are 2 and -2. Hence its minimal polynomial has no repeated factors and divides

$$(x - 2)(x + 2) = x^2 - 4.$$

Therefore

$$A^2 - 4I_3 = 0, \qquad \text{so} \qquad A^2 = 4I_3.$$

Raising both sides to the tenth power gives

$$A^{20} = (A^2)^{10} = (4I_3)^{10} = 4^{10} I_3 = 2^{20} I_3.$$

This gives the same final answer without explicitly multiplying P, D, and P^{-1}.

Mark Scheme (20 marks)

- **(a) Characteristic polynomial and eigenvalues (5 marks)**
  - (2) Recognises or uses block upper-triangular determinant structure.
  - (2) Computes $\det(A - \lambda I) = -(a - \lambda)(4 - \lambda^2)$ correctly.
  - (1) States eigenvalues $a, 2, -2$.
- **(b) Diagonalizability condition (6 marks)**
  - (1) Notes that distinct eigenvalues imply diagonalizability when $a \neq \pm 2$.
  - (2) Correctly shows the case $a = 2$ is defective using $\dim E_2 = 1$.
  - (2) Correctly shows the case $a = -2$ has enough eigenvectors using $\dim E_{-2} = 2$.
  - (1) Concludes $\beta = 2$ and states the iff condition.
- **(c) Diagonalisation and power when $a = -2$ (9 marks)**
  - (3) Finds correct eigenvectors for $\lambda = 2$ and $\lambda = -2$.
  - (2) Forms valid P and D with $A = PDP^{-1}$.
  - (2) Uses $A^{20} = PD^{20}P^{-1}$ or an equivalent minimal-polynomial argument.
  - (2) Gives $A^{20} = 2^{20}I_3 = 1{,}048{,}576I_3$.

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Question 3 (15 Marks)

Problem

Consider

$$A = \begin{bmatrix} 8 & -2 & -3 \\ -6 & 4 & 3 \\ -6 & 2 & 5 \end{bmatrix}.$$

(a) Show that both $(1,3,0)^T$ and $(1,0,2)^T$ are eigenvectors of A. Furthermore, show that the eigenvalues corresponding to these eigenvectors are the same.

(b) Let B be another 3×3 matrix with eigenvalue λ. Suppose that the eigenspace of B corresponding to λ has dimension two. Show that $A + B$ has an eigenvalue $2 + \lambda$.

Solution

Method 1: Dimension-count intersection argument

(a) Let

$$v_1 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

We verify the eigenvector condition directly.

For v_1,

$$Av_1 = \begin{bmatrix} 8 & -2 & -3 \\ -6 & 4 & 3 \\ -6 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 8-6+0 \\ -6+12+0 \\ -6+6+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = 2v_1,$$

v_1 is an eigenvector of A with eigenvalue 2.

For v_2,

$$Av_2 = \begin{bmatrix} 8 & -2 & -3 \\ -6 & 4 & 3 \\ -6 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8+0-6 \\ -6+0+6 \\ -6+0+10 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}.$$

Since

$$\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 2v_2,$$

v_2 is also an eigenvector of A with eigenvalue 2. Therefore both given vectors are eigenvectors corresponding to the same eigenvalue

$$\boxed{2.}$$

(b) From part (a), both v_1 and v_2 lie in the eigenspace of A corresponding to the eigenvalue 2. They are linearly independent because neither is a scalar multiple of the other. Hence

$$E_A := \text{span}\{v_1, v_2\}$$

is a two-dimensional subspace of $\mathbb{R}^3$, and every nonzero vector $x \in E_A$ satisfies

$$Ax = 2x.$$

Let $E_B(\lambda)$ be the eigenspace of B corresponding to λ. By assumption,

$$\dim E_B(\lambda) = 2.$$

Both E_A and $E_B(\lambda)$ are subspaces of $\mathbb{R}^3$. We use the dimension formula

$$\dim(E_A + E_B(\lambda)) = \dim E_A + \dim E_B(\lambda) - \dim(E_A \cap E_B(\lambda)).$$

Since $E_A + E_B(\lambda)$ is a subspace of $\mathbb{R}^3$,

$$\dim(E_A + E_B(\lambda)) \leq 3.$$

Therefore

$$3 \geq 2 + 2 - \dim(E_A \cap E_B(\lambda)).$$

Rearranging gives

$$\dim(E_A \cap E_B(\lambda)) \geq 1.$$

Thus there exists a nonzero vector

$$0 \neq x \in E_A \cap E_B(\lambda).$$

Because $x \in E_A$,

$$Ax = 2x.$$

Because $x \in E_B(\lambda)$,

$$Bx = \lambda x.$$

Hence

$$(A + B)x = Ax + Bx = 2x + \lambda x = (2 + \lambda)x.$$

Since $x \neq 0$, this shows that $2 + \lambda$ is an eigenvalue of $A + B$. Therefore

$$\boxed{2 + \lambda \text{ is an eigenvalue of } A + B.}$$

Method 2: Geometric viewpoint

(b) The eigenspace

$$E_A = \text{span}\{v_1, v_2\}$$

is a plane through the origin in $\mathbb{R}^3$. The eigenspace $E_B(\lambda)$ is also a plane through the origin, because it has dimension two. Two planes through the origin in $\mathbb{R}^3$ must intersect in at least a line, so they share a nonzero vector x.

For this common nonzero vector,

$$Ax = 2x \qquad \text{and} \qquad Bx = \lambda x.$$

Therefore

$$(A + B)x = (2 + \lambda)x,$$

which proves that $2 + \lambda$ is an eigenvalue of $A + B$.

Mark Scheme (15 marks)

- **(a) Eigenvector verification (5 marks)**
  - (2) Correctly computes $A(1,3,0)^T = 2(1,3,0)^T$.
  - (2) Correctly computes $A(1,0,2)^T = 2(1,0,2)^T$.
  - (1) States that the common eigenvalue is 2.
- **(b) Existence of eigenvalue $2+\lambda$ (10 marks)**
  - (2) Defines or identifies the two-dimensional eigenspace $E_A = \text{span}\{v_1, v_2\}$.
  - (3) Correctly argues that E_A and $E_B(\lambda)$ have a nonzero intersection.
  - (2) Chooses a nonzero vector x in the intersection and states $Ax = 2x$, $Bx = \lambda x$.
  - (2) Computes $(A+B)x = (2+\lambda)x$.
  - (1) Concludes that $2+\lambda$ is an eigenvalue.

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Question 4 (15 Marks)

Problem

(a) Consider the following vectors in $\mathbb{R}^3$ with the usual Euclidean inner product:

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}.$$

Apply the Gram–Schmidt procedure to $\{u_1, u_2, u_3\}$ to obtain an orthonormal basis for $\mathbb{R}^3$.

(b) Let

$$A = \begin{bmatrix} 0 & 0 & 5 \\ 1 & 4 & 7 \\ 0 & 8 & 6 \end{bmatrix}.$$

Find the QR-decomposition for A, i.e. find an orthogonal matrix Q and an upper-triangular matrix R such that $A = QR$.

Solution

Method 1: Gram–Schmidt on the columns

(a) We apply Gram–Schmidt to

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}.$$

First,

$$\|u_1\| = \sqrt{0^2 + 1^2 + 0^2} = 1.$$

Hence

$$e_1 = \frac{u_1}{\|u_1\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Next, subtract the projection of u_2 onto e_1:

$$u_2 \cdot e_1 = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 4.$$

Thus

$$\mathrm{proj}_{e_1}(u_2) = (u_2 \cdot e_1)e_1 = 4e_1 = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}.$$

So

$$w_2 = u_2 - 4e_1 = \begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 8 \end{bmatrix}.$$

Since

$$\|w_2\| = 8,$$

we get

$$e_2 = \frac{w_2}{\|w_2\|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, subtract the projections of u_3 onto both e_1 and e_2:

$$u_3 \cdot e_1 = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 7, \qquad u_3 \cdot e_2 = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 6.$$

Therefore

$$w_3 = u_3 - 7e_1 - 6e_2 = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 7 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

Since

$$\|w_3\| = 5,$$

we obtain

$$e_3 = \frac{w_3}{\|w_3\|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Thus an orthonormal basis for $\mathbb{R}^3$ is

$$\boxed{\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.}$$

(b) The columns of A are precisely

$$u_1,\ u_2,\ u_3.$$

From part (a), the orthonormal vectors are

$$e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$Q = [e_1\ e_2\ e_3] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Since the columns of Q are orthonormal, Q is an orthogonal matrix, i.e.

$$Q^T Q = I.$$

To form R, write each original column u_j in terms of the orthonormal basis e_1, e_2, e_3:

$$u_1 = e_1,$$

$$u_2 = 4e_1 + 8e_2,$$

$$u_3 = 7e_1 + 6e_2 + 5e_3.$$

Therefore the coordinate columns form

$$R = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 8 & 6 \\ 0 & 0 & 5 \end{bmatrix}.$$

Equivalently, since Q is orthogonal,

$$R = Q^T A.$$

Thus

$$\boxed{A = QR, \qquad Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 8 & 6 \\ 0 & 0 & 5 \end{bmatrix}.}$$

Method 2: Reading R from inner products

(b) After obtaining e_1, e_2, e_3 in part (a), the entries of R may be found from

$$R_{ij} = e_i \cdot u_j \qquad (1 \le i \le j \le 3),$$

with $R_{ij} = 0$ for $i > j$ because R is upper triangular. Computing these inner products gives

$$R_{11} = e_1 \cdot u_1 = 1, \qquad R_{12} = e_1 \cdot u_2 = 4, \qquad R_{13} = e_1 \cdot u_3 = 7,$$

$$R_{22} = e_2 \cdot u_2 = 8, \qquad R_{23} = e_2 \cdot u_3 = 6, \qquad R_{33} = e_3 \cdot u_3 = 5.$$

Hence

$$R = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 8 & 6 \\ 0 & 0 & 5 \end{bmatrix},$$

which is the same result as in Method 1.

Mark Scheme (15 marks)

- **(a) Gram–Schmidt (8 marks)**
  - (2) Correctly obtains $e_1 = (0, 1, 0)^T$.
  - (2) Correctly subtracts the projection of u_2 and obtains $e_2 = (0, 0, 1)^T$.
  - (3) Correctly subtracts projections of u_3 onto e_1, e_2 and obtains $e_3 = (1, 0, 0)^T$.
  - (1) Presents the final orthonormal basis clearly.
- **(b) QR decomposition (7 marks)**
  - (2) Forms an orthogonal matrix Q using the orthonormal vectors as columns.
  - (3) Computes the correct upper-triangular matrix R.
  - (1) Checks or states that $A = QR$.
  - (1) Ensures R is upper triangular and the dimensions are correct.

Question 5 (15 Marks)

Problem

Consider the vector space $\mathbb{R}^2$. Fix c and consider

$$A = \begin{bmatrix} 25 & 5 \\ 5 & c \end{bmatrix}.$$

Define

$$\langle u, v \rangle = u^T A v, \qquad u, v \in \mathbb{R}^2.$$

(a) Show that the product is both additive and homogeneous in the first slot.

(b) Show that the product is symmetric.

(c) Determine the values of c for which the product is an inner product.

Solution

Method 1: Positive-definite matrix criterion

(a) Let $u_1, u_2, v \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$. We prove additivity and homogeneity in the first slot. For additivity,

$$\langle u_1 + u_2, v \rangle = (u_1 + u_2)^T A v.$$

Using $(u_1 + u_2)^T = u_1^T + u_2^T$, we get

$$\langle u_1 + u_2, v \rangle = (u_1^T + u_2^T) A v = u_1^T A v + u_2^T A v = \langle u_1, v \rangle + \langle u_2, v \rangle .$$

For homogeneity,

$$\langle \alpha u, v \rangle = (\alpha u)^T A v = \alpha u^T A v = \alpha \langle u, v \rangle .$$

Hence the product is additive and homogeneous in the first slot.

(b) We show that the product is symmetric. Since

$$A = \begin{bmatrix} 25 & 5 \\ 5 & c \end{bmatrix},$$

we have

$$A^T = A.$$

Also, $u^T A v$ is a 1×1 scalar, so it is equal to its transpose. Therefore

$$\langle u, v \rangle = u^T A v = (u^T A v)^T.$$

Using $(XYZ)^T = Z^T Y^T X^T$, we get

$$(u^T A v)^T = v^T A^T u.$$

Since $A^T = A$,

$$v^T A^T u = v^T A u = \langle v, u \rangle .$$

Thus

$$\boxed{\langle u, v \rangle = \langle v, u \rangle .}$$

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(c) For the product to be an inner product on $\mathbb{R}^2$, after parts (a) and (b), the remaining essential condition is positive definiteness:

$$\langle u, u\rangle > 0 \quad \text{for all } u \neq 0.$$

Because A is real symmetric, the product $u^T Av$ defines an inner product exactly when A is positive definite.

For a real symmetric 2×2 matrix, Sylvester's criterion says that A is positive definite iff its leading principal minors are positive. Here the first leading principal minor is

$$25 > 0.$$

The determinant is

$$\det A = 25c - 5 \cdot 5 = 25c - 25 = 25(c-1).$$

Thus A is positive definite iff

$$25(c-1) > 0.$$

This is equivalent to

$$c > 1.$$

Therefore the product is an inner product exactly when

$$\boxed{c > 1.}$$

Method 2: Completing the square

(c) Let $u = (x, y)^T$. Then

$$\langle u, u\rangle = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 25 & 5 \\ 5 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

First compute

$$\begin{bmatrix} 25 & 5 \\ 5 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 25x + 5y \\ 5x + cy \end{bmatrix}.$$

Hence

$$\langle u, u\rangle = x(25x + 5y) + y(5x + cy) = 25x^2 + 10xy + cy^2.$$

Completing the square,

$$25x^2 + 10xy + cy^2 = (5x + y)^2 + (c-1)y^2.$$

If $c > 1$, then for any nonzero (x, y):

- if $y \neq 0$, then $(c-1)y^2 > 0$;
- if $y = 0$ and $x \neq 0$, then $(5x + y)^2 = 25x^2 > 0$.

Thus $\langle u, u\rangle > 0$ for all nonzero u.

If $c = 1$, choose $u = (1, -5)^T$. Then

$$\langle u, u\rangle = (5-5)^2 + 0 = 0$$

even though $u \neq 0$, so positive definiteness fails.

If $c < 1$, choose again $u = (1, -5)^T$. Then

$$\langle u, u\rangle = (5-5)^2 + (c-1)(-5)^2 = 25(c-1) < 0,$$

so positive definiteness also fails. Hence the necessary and sufficient condition is

$$\boxed{c > 1.}$$

Mark Scheme (15 marks)

- **(a) Linearity in the first slot (4 marks)**
  - (2) Correct proof of additivity.
  - (2) Correct proof of homogeneity.
- **(b) Symmetry (4 marks)**
  - (2) Notes that A is symmetric or uses $A^T = A$.
  - (2) Correctly proves $u^T Av = v^T Au$.
- **(c) Inner product condition (7 marks)**
  - (2) Recognises that positive definiteness is the remaining condition.
  - (3) Correctly applies the 2×2 positive-definite criterion or completes the square.
  - (1) Handles the failure when $c \leq 1$ or explains necessity.
  - (1) States the final answer $c > 1$.

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Question 6 (15 Marks)

Problem

Let V be a vector space, and let $\{v_1, v_2, \ldots, v_{2024}\}$ be a set of linearly independent vectors in V. Pick any $w \in V$ and set

$$U = \text{span}\{v_1 + w, v_2 + w, \ldots, v_{2024} + w\}.$$

Show that

$$\dim(U) \geq 2023.$$

Solution

Method 1: Use differences inside the span

This question has one main part: prove the lower bound $\dim(U) \geq 2023$.

For each $i = 1, 2, \ldots, 2023$, consider the difference

$$(v_i + w) - (v_{2024} + w).$$

The vector $v_i + w$ is one of the spanning vectors of U, and $v_{2024} + w$ is also one of the spanning vectors of U. Since U is a subspace, it is closed under subtraction. Therefore

$$(v_i + w) - (v_{2024} + w) \in U.$$

But

$$(v_i + w) - (v_{2024} + w) = v_i - v_{2024}.$$

Hence

$$v_1 - v_{2024},\ v_2 - v_{2024},\ \ldots,\ v_{2023} - v_{2024} \in U.$$

We now show that these 2023 vectors are linearly independent. Suppose

$$\sum_{i=1}^{2023} c_i(v_i - v_{2024}) = 0.$$

Expanding the left-hand side gives

$$\sum_{i=1}^{2023} c_i v_i - \sum_{i=1}^{2023} c_i v_{2024} = 0.$$

Since v_{2024} is fixed, this can be written as

$$\sum_{i=1}^{2023} c_i v_i - \left(\sum_{i=1}^{2023} c_i\right) v_{2024} = 0.$$

This is a linear relation among the linearly independent vectors

$$v_1, v_2, \ldots, v_{2024}.$$

Therefore every coefficient in this relation must be zero. In particular, the coefficients of $v_1, v_2, \ldots, v_{2023}$ give

$$c_1 = c_2 = \cdots = c_{2023} = 0.$$

Hence

$$v_1 - v_{2024},\ v_2 - v_{2024},\ \ldots,\ v_{2023} - v_{2024}$$

are linearly independent.

Since U contains these 2023 linearly independent vectors, its dimension must be at least 2023. Therefore

$$\boxed{\dim(U) \geq 2023.}$$

Method 2: Rank–nullity contradiction

Define

$$F : \mathbb{R}^{2024} \to U, \qquad F(c_1, \ldots, c_{2024}) = \sum_{i=1}^{2024} c_i(v_i + w).$$

By construction, the range of F is exactly U, so

$$\operatorname{rank}(F) = \dim(U).$$

Suppose for contradiction that

$$\dim(U) \leq 2022.$$

Then by rank–nullity,

$$\dim \operatorname{null}(F) = 2024 - ank(F) = 2024 - \dim(U) \geq 2.$$

Now consider the hyperplane

$$H = \left\{ c = (c_1, \ldots, c_{2024}) \in \mathbb{R}^{2024} : \sum_{i=1}^{2024} c_i = 0 \right\}.$$

This is a subspace of $\mathbb{R}^{2024}$ of dimension 2023. Since

$$\dim \operatorname{null}(F) \geq 2 \qquad \text{and} \qquad \dim H = 2023,$$

the dimension formula gives

$$\dim(\operatorname{null}(F) \cap H) \geq \dim \operatorname{null}(F) + \dim H - 2024 \geq 2 + 2023 - 2024 = 1.$$

So there exists a nonzero vector

$$0 \neq c = (c_1, \ldots, c_{2024}) \in \operatorname{null}(F) \cap H.$$

Because $c \in \operatorname{null}(F)$,

$$0 = F(c) = \sum_{i=1}^{2024} c_i(v_i + w).$$

Expanding this gives

$$0 = \sum_{i=1}^{2024} c_i v_i + \left(\sum_{i=1}^{2024} c_i \right) w.$$

Because $c \in H$,

$$\sum_{i=1}^{2024} c_i = 0.$$

Thus

$$\sum_{i=1}^{2024} c_i v_i = 0.$$

Since $v_1, \ldots, v_{2024}$ are linearly independent, all coefficients must be zero:

$$c_1 = c_2 = \cdots = c_{2024} = 0.$$

This contradicts $c \neq 0$. Hence our assumption was false, and therefore

$$\boxed{\dim(U) \geq 2023.}$$

Mark Scheme (15 marks)

- **Proof using differences (15 marks)**
  - (3) Correctly observes that $v_i - v_{2024} = (v_i + w) - (v_{2024} + w)$ lies in U for $1 \leq i \leq 2023$.
  - (4) Sets up a linear relation among the vectors $v_i - v_{2024}$.
  - (4) Converts it into a linear relation among $v_1, \ldots, v_{2024}$.
  - (3) Uses the linear independence of $v_1, \ldots, v_{2024}$ to show all coefficients are zero.
  - (1) Concludes U contains 2023 linearly independent vectors and hence $\dim(U) \geq 2023$.
- Alternative rank–nullity/intersection arguments should receive full credit if they establish the same lower bound rigorously.

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