MH1201 Linear Algebra II

Midterm Exam (Solutions)

AY2024-25, Semester 2

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Overview

- **Q1 (Subspaces & spans):** subspace tests (recurrences, determinant counterexample, affine translates); span inclusion via definition; equality of spans under invertible change-of-generators.
- **Q2 (Linear operator on $\mathbb{R}^{2,2}$):** linearity from transpose/multiplication; standard matrix representation; explicit bases for $\mathrm{Null}(T)$ and $\mathrm{Range}(T)$; direct-sum verification via explicit decomposition (no blind rank–nullity).
- **Q3 (Polynomial identities & idempotents):** explicit construction of T with $T^4 = -I$ over any field; projection operators and internal direct sum $\mathrm{Null}(P) \oplus \mathrm{Range}(P)$ without rank–nullity.
- **Q4 (External direct sum):** verify all eight axioms for $V \times W$ with componentwise operations; construct basis and prove $\dim(V \oplus_{\text{ext}} W) = \dim V + \dim W$.
- **Method policy:** Each method is written as a standalone solution (no cross-references), with explicit theorem statements when invoked.

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Unofficial Solution

Question 1 (25 marks)

(a)(i) Linear recurrence subspace in $\mathbb{R}^\infty$

Problem

Let

$$V \coloneqq \mathbb{R}^\infty, \qquad W \coloneqq \{(x_1, x_2, x_3, \ldots) \in \mathbb{R}^\infty \mid \forall n \in \mathbb{N} : \ x_{n+2} = 4x_{n+1} - 2x_n\}.$$

Determine, with justification, whether or not W is a vector subspace of V.

Solution

Method 1: Direct subspace test

We first verify that $W \neq \varnothing$. The zero sequence

$$(0, 0, 0, \ldots)$$

belongs to W, since for every $n \in \mathbb{N}$,

$$0 = 4 \cdot 0 - 2 \cdot 0.$$

Now let $x = (x_1, x_2, \ldots) \in W$, $y = (y_1, y_2, \ldots) \in W$, and let $\alpha, \beta \in \mathbb{R}$. For each $n \in \mathbb{N}$,

$$x_{n+2} = 4x_{n+1} - 2x_n, \qquad y_{n+2} = 4y_{n+1} - 2y_n.$$

Hence

$$(\alpha x + \beta y)_{n+2} = \alpha x_{n+2} + \beta y_{n+2} = \alpha(4x_{n+1} - 2x_n) + \beta(4y_{n+1} - 2y_n),$$

so

$$(\alpha x + \beta y)_{n+2} = 4(\alpha x + \beta y)_{n+1} - 2(\alpha x + \beta y)_n.$$

Thus $\alpha x + \beta y \in W$.

Therefore W is nonempty and closed under linear combinations, so W is a vector subspace of $\mathbb{R}^\infty$.

$$\boxed{W \leq \mathbb{R}^\infty.}$$

Method 2: Kernel of a linear map

Again, $W \neq \varnothing$, since the zero sequence lies in W.

Define a map

$$L : \mathbb{R}^\infty \to \mathbb{R}^\infty$$

by

$$(Lx)_n \coloneqq x_{n+2} - 4x_{n+1} + 2x_n \qquad (n \in \mathbb{N}).$$

This map is linear, since for $x, y \in \mathbb{R}^\infty$ and $\alpha, \beta \in \mathbb{R}$,

$$L(\alpha x + \beta y)_n = (\alpha x + \beta y)_{n+2} - 4(\alpha x + \beta y)_{n+1} + 2(\alpha x + \beta y)_n = \alpha(Lx)_n + \beta(Ly)_n.$$

Now

$$x \in W \iff \forall n \in \mathbb{N} : \ x_{n+2} - 4x_{n+1} + 2x_n = 0 \iff Lx = 0.$$

Hence

$$W = \ker(L).$$

Since the kernel of a linear map is a subspace, W is a subspace of $\mathbb{R}^\infty$. Therefore

$$\boxed{W \leq \mathbb{R}^\infty.}$$

Method 3: Intersection of kernels of linear functionals

For each $n \in \mathbb{N}$, define

$$L_n : \mathbb{R}^\infty \to \mathbb{R}, \qquad L_n(x) \coloneqq x_{n+2} - 4x_{n+1} + 2x_n.$$

Each L_n is linear, being a linear combination of coordinate projections.

Then

$$W = \{x \in \mathbb{R}^\infty : \forall n \in \mathbb{N},\ L_n(x) = 0\} = \bigcap_{n\in\mathbb{N}} \ker(L_n).$$

Since each $\ker(L_n)$ is a subspace of $\mathbb{R}^\infty$, and an intersection of subspaces is again a subspace, it follows that W is a subspace of $\mathbb{R}^\infty$. Also $W \neq \varnothing$ because the zero sequence belongs to every $\ker(L_n)$. Thus

$$\boxed{W \leq \mathbb{R}^\infty.}$$

Method 4: The recurrence is determined by x_1 and x_2

Again, $W \neq \varnothing$ since the zero sequence belongs to W.

The defining relation

$$x_{n+2} = 4x_{n+1} - 2x_n \qquad (\forall n \in \mathbb{N})$$

shows that once x_1 and x_2 are chosen, all later terms are determined recursively:

$$x_3 = 4x_2 - 2x_1, \qquad x_4 = 4x_3 - 2x_2, \qquad \ldots$$

Thus every sequence in W is uniquely determined by its first two entries.

Define

$$T : W \to \mathbb{R}^2, \qquad T(x_1, x_2, x_3, \ldots) = (x_1, x_2).$$

This map is linear. Moreover, given any $(u, v) \in \mathbb{R}^2$, there is a unique sequence in W whose first two terms are u and v, obtained by recursively defining

$$x_1 = u, \qquad x_2 = v, \qquad x_{n+2} = 4x_{n+1} - 2x_n \ \ (n \in \mathbb{N}).$$

So T is bijective.

Hence W is naturally identified with $\mathbb{R}^2$, with addition and scalar multiplication corresponding to those of initial-value pairs. In particular, W is closed under linear combinations, and therefore W is a subspace of $\mathbb{R}^\infty$. Thus

$$\boxed{W \leq \mathbb{R}^\infty.}$$

(a)(ii) Singular 3×3 matrices

Problem

Let

$$V \coloneqq \mathbb{R}^{3,3}, \qquad W \coloneqq \{A \in \mathbb{R}^{3,3} \mid \det(A) = 0\}.$$

Determine, with justification, whether or not W is a vector subspace of V.

Solution

Method 1: Counterexample to closure under addition

We first verify that $W \neq \varnothing$. The zero matrix belongs to W, since

$$\det(0) = 0.$$

Now consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Both matrices are singular, so

$$\det(A) = 0, \qquad \det(B) = 0,$$

hence $A, B \in W$.

However,

$$A + B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3,$$

and

$$\det(I_3) = 1 \neq 0.$$

Thus $A + B \notin W$.

So W is not closed under addition, hence W is not a vector subspace of $\mathbb{R}^{3,3}$.

$$\boxed{W \not\leq \mathbb{R}^{3,3}.}$$

Method 2: Counterexample to closure under scalar combinations

Again, $W \neq \varnothing$ because $0 \in W$.

To prove that W is not a subspace, it suffices to show that it is not closed under linear combinations. Consider the same matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $A, B \in W$, but taking the linear combination with coefficients 1 and 1 gives

$$1 \cdot A + 1 \cdot B = I_3 \notin W.$$

Therefore W is not closed under linear combinations, so it is not a subspace. Hence

$$\boxed{W \not\leq \mathbb{R}^{3,3}.}$$

Method 3: Determinant is not a linear condition

A vector subspace of $M_{3\times 3}(\mathbb{R})$ is typically defined by linear conditions on the entries. Here, however, the defining condition is

$$\det(A) = 0,$$

and the determinant is not a linear map on $\mathbb{R}^{3,3}$. Indeed,

$$\det(A+B) \neq \det(A) + \det(B)$$

in general. For example, with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we have

$$\det(A) = 0, \qquad \det(B) = 0, \qquad \det(A+B) = \det(I_3) = 1.$$

So matrices satisfying $\det(A) = 0$ are not preserved under addition. Therefore W is not a subspace:

$$\boxed{W \not\leq \mathbb{R}^{3,3}.}$$

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(a)(iii) Affine subset of $\mathbb{R}^{2,2}$

Problem

Let

$$V \coloneqq \mathbb{R}^{2,2},$$

and let

$$W \coloneqq \left\{ a\begin{bmatrix}2 & -3\\ 4 & 1\end{bmatrix} + b\begin{bmatrix}0 & 5\\ 1 & -2\end{bmatrix} + \begin{bmatrix}6 & -19\\ 10 & 7\end{bmatrix} \in \mathbb{R}^{2,2} \;\middle|\; a, b \in \mathbb{R} \right\}.$$

Determine, with justification, whether or not W is a vector subspace of V.

Solution

Method 1: Rewrite as a span

Let

$$M_1 = \begin{bmatrix}2 & -3\\ 4 & 1\end{bmatrix}, \qquad M_2 = \begin{bmatrix}0 & 5\\ 1 & -2\end{bmatrix}, \qquad C = \begin{bmatrix}6 & -19\\ 10 & 7\end{bmatrix}.$$

Then

$$W = \{aM_1 + bM_2 + C : \ a, b \in \mathbb{R}\}.$$

First, $W \neq \varnothing$, since taking $a = b = 0$ gives $C \in W$.

We now check whether $C \in \operatorname{span}\{M_1, M_2\}$. Suppose

$$aM_1 + bM_2 = C.$$

Comparing entries gives

$$2a = 6, \qquad 4a + b = 10.$$

From $2a = 6$, we get $a = 3$. Then $4a + b = 10$ gives

$$12 + b = 10 \quad \Rightarrow \quad b = -2.$$

Check the remaining entries:

$$-3a + 5b = -9 - 10 = -19, \qquad a - 2b = 3 - 2(-2) = 7.$$

Hence

$$C = 3M_1 - 2M_2 \in \operatorname{span}\{M_1, M_2\}.$$

Therefore every element of W can be rewritten as

$$aM_1 + bM_2 + C = aM_1 + bM_2 + 3M_1 - 2M_2 = (a+3)M_1 + (b-2)M_2,$$

so $W \subseteq \operatorname{span}\{M_1, M_2\}$.

Conversely, if $\alpha, \beta \in \mathbb{R}$, then

$$\alpha M_1 + \beta M_2 = (\alpha - 3)M_1 + (\beta + 2)M_2 + C \in W.$$

Hence

$$\operatorname{span}\{M_1, M_2\} \subseteq W.$$

Thus

$$W = \operatorname{span}\{M_1, M_2\},$$

which is a subspace of $\mathbb{R}^{2,2}$. Therefore

$$\boxed{W \leq \mathbb{R}^{2,2}.}$$

Method 2: Affine / translate criterion

With the same notation,

$$W = C + \operatorname{span}\{M_1, M_2\}.$$

Now $\operatorname{span}\{M_1, M_2\}$ is a subspace of $\mathbb{R}^{2,2}$. A translate $x_0 + S$ of a subspace S is itself a subspace if and only if $x_0 \in S$.

So it suffices to determine whether

$$C \in \operatorname{span}\{M_1, M_2\}.$$

As computed above,

$$C = 3M_1 - 2M_2,$$

so indeed $C \in \operatorname{span}\{M_1, M_2\}$. Hence

$$W = C + \operatorname{span}\{M_1, M_2\} = \operatorname{span}\{M_1, M_2\},$$

which is a subspace of $\mathbb{R}^{2,2}$. Therefore

$$\boxed{W \leq \mathbb{R}^{2,2}.}$$

Method 3: Zero-matrix test

Again let

$$M_1 = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}, \qquad C = \begin{bmatrix} 6 & -19 \\ 10 & 7 \end{bmatrix}.$$

Certainly $W \neq \varnothing$, since $C \in W$.

To check whether W is a subspace, a natural first test is whether $0 \in W$. We seek $a, b \in \mathbb{R}$ such that

$$aM_1 + bM_2 + C = 0.$$

Equivalently,

$$aM_1 + bM_2 = -C.$$

Comparing entries, we obtain

$$2a = -6, \qquad 4a + b = -10.$$

Thus

$$a = -3, \qquad b = 2.$$

Checking the remaining entries gives

$$-3a + 5b = 9 + 10 = 19, \qquad a - 2b = -3 - 4 = -7,$$

which agrees with $-C$. Hence

$$(-3)M_1 + 2M_2 + C = 0,$$

so $0 \in W$.

Now let $X, Y \in W$. Then for some $a_1, b_1, a_2, b_2 \in \mathbb{R}$,

$$X = a_1M_1 + b_1M_2 + C, \qquad Y = a_2M_1 + b_2M_2 + C.$$

Since $0 = (-3)M_1 + 2M_2 + C$, we may write

$$C = 3M_1 - 2M_2.$$

Substituting this into X and Y, we see that each element of W is actually a linear combination of M_1 and M_2, so $W = \operatorname{span}\{M_1, M_2\}$. Therefore W is a subspace:

$$\boxed{W \leq \mathbb{R}^{2,2}.}$$

(b)(i) Inclusion of spans

Problem

Let V be a vector space over $\mathbb{R}$, and let $a_1, \ldots, a_n, b_1, \ldots, b_n \in V$. Suppose that there is a matrix

$$M = [\lambda_{i,j}]_{i,j=1}^{n} \in \mathbb{R}^{n,n}$$

such that

$$b_i = \lambda_{i,1}a_1 + \cdots + \lambda_{i,n}a_n \qquad (i = 1, \ldots, n).$$

Explain — using the definition of span — why

$$\operatorname{span}(\{b_1, \ldots, b_n\}) \subseteq \operatorname{span}(\{a_1, \ldots, a_n\}).$$

Solution

Method 1: Directly from the definition of span

By definition,

$$\operatorname{span}(\{b_1, \ldots, b_n\}) = \left\{ \sum_{i=1}^{n} \alpha_i b_i \;\middle|\; \alpha_1, \ldots, \alpha_n \in \mathbb{R} \right\}.$$

Let

$$v \in \operatorname{span}(\{b_1, \ldots, b_n\}).$$

Then there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$v = \sum_{i=1}^{n} \alpha_i b_i.$$

Substituting

$$b_i = \sum_{j=1}^{n} \lambda_{i,j} a_j,$$

we obtain

$$v = \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{n} \lambda_{i,j} a_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \alpha_i \lambda_{i,j} \right) a_j.$$

Thus v is a linear combination of $a_1, \ldots, a_n$, so

$$v \in \operatorname{span}(\{a_1, \ldots, a_n\}).$$

Therefore

$$\boxed{\operatorname{span}(\{b_1, \ldots, b_n\}) \subseteq \operatorname{span}(\{a_1, \ldots, a_n\}).}$$

Method 2: Smallest-subspace property of span

For each $i \in \{1, \ldots, n\}$, we are given that

$$b_i = \lambda_{i,1}a_1 + \cdots + \lambda_{i,n}a_n.$$

Hence each b_i belongs to $\operatorname{span}(\{a_1, \ldots, a_n\})$. Therefore

$$\{b_1, \ldots, b_n\} \subseteq \operatorname{span}(\{a_1, \ldots, a_n\}).$$

Now $\operatorname{span}(\{a_1, \ldots, a_n\})$ is a subspace of V containing the set $\{b_1, \ldots, b_n\}$. By the smallest-subspace property of span, $\operatorname{span}(\{b_1, \ldots, b_n\})$ is contained in every subspace containing $\{b_1, \ldots, b_n\}$. In particular,

$$\operatorname{span}(\{b_1, \ldots, b_n\}) \subseteq \operatorname{span}(\{a_1, \ldots, a_n\}).$$

Thus

$$\boxed{\operatorname{span}(\{b_1, \ldots, b_n\}) \subseteq \operatorname{span}(\{a_1, \ldots, a_n\}).}$$

Method 3: Span as the range of a linear map

Define linear maps

$$A : \mathbb{R}^n \to V, \qquad A(e_j) = a_j,$$

and

$$B : \mathbb{R}^n \to V, \qquad B(e_i) = b_i,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$.

Then

$$\text{range}(A) = \text{span}(\{a_1, \ldots, a_n\}), \qquad \text{range}(B) = \text{span}(\{b_1, \ldots, b_n\}).$$

From

$$b_i = \sum_{j=1}^{n} \lambda_{i,j} a_j,$$

we get

$$B(e_i) = \sum_{j=1}^{n} \lambda_{i,j} A(e_j) = A\left(\sum_{j=1}^{n} \lambda_{i,j} e_j\right) = A(Me_i).$$

Since this holds for each basis vector e_i, we have

$$B = A \circ M.$$

Therefore

$$\text{range}(B) = \text{range}(A \circ M) \subseteq \text{range}(A),$$

that is,

$$\text{span}(\{b_1, \ldots, b_n\}) \subseteq \text{span}(\{a_1, \ldots, a_n\}).$$

Hence

$$\boxed{\text{span}(\{b_1, \ldots, b_n\}) \subseteq \text{span}(\{a_1, \ldots, a_n\}).}$$

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(b)(ii) Equality of spans when M is invertible

Problem

Under the assumptions of part (b)(i), show that if M is invertible, then

$$\operatorname{span}(\{b_1, \ldots, b_n\}) = \operatorname{span}(\{a_1, \ldots, a_n\}).$$

Solution

Method 1: Express each a_i as a linear combination of the b_j

From part (b)(i), we already know that

$$\operatorname{span}(\{b_1, \ldots, b_n\}) \subseteq \operatorname{span}(\{a_1, \ldots, a_n\}).$$

So it remains to prove the reverse inclusion.

Assume M is invertible, and write

$$M^{-1} = [\mu_{i,j}]_{i,j=1}^{n}.$$

The relations

$$b_i = \sum_{j=1}^{n} \lambda_{i,j} a_j \qquad (i = 1, \ldots, n)$$

can be inverted to solve for the a_i's in terms of the b_j's:

$$a_i = \sum_{j=1}^{n} \mu_{i,j} b_j \qquad (i = 1, \ldots, n).$$

Hence each $a_i \in \operatorname{span}(\{b_1, \ldots, b_n\})$. Therefore

$$\operatorname{span}(\{a_1, \ldots, a_n\}) \subseteq \operatorname{span}(\{b_1, \ldots, b_n\}).$$

Combining this with the inclusion from part (b)(i), we obtain

$$\operatorname{span}(\{a_1, \ldots, a_n\}) = \operatorname{span}(\{b_1, \ldots, b_n\}).$$

Thus

$$\boxed{\operatorname{span}(\{b_1, \ldots, b_n\}) = \operatorname{span}(\{a_1, \ldots, a_n\}).}$$

Method 2: Smallest-subspace argument in both directions

By part (b)(i),

$$\operatorname{span}(\{b_1, \ldots, b_n\}) \subseteq \operatorname{span}(\{a_1, \ldots, a_n\}).$$

Now assume M is invertible. Then

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = M^{-1} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix},$$

so each a_i is a linear combination of $b_1, \ldots, b_n$. Hence

$$\{a_1, \ldots, a_n\} \subseteq \operatorname{span}(\{b_1, \ldots, b_n\}).$$

Since $\operatorname{span}(\{b_1, \ldots, b_n\})$ is a subspace containing $\{a_1, \ldots, a_n\}$, the smallest-subspace property yields

$$\operatorname{span}(\{a_1, \ldots, a_n\}) \subseteq \operatorname{span}(\{b_1, \ldots, b_n\}).$$

Together with the inclusion from part (b)(i), this gives

$$\operatorname{span}(\{a_1, \ldots, a_n\}) = \operatorname{span}(\{b_1, \ldots, b_n\}).$$

Therefore

$$\boxed{\operatorname{span}(\{b_1, \ldots, b_n\}) = \operatorname{span}(\{a_1, \ldots, a_n\}).}$$

Method 3: Range equality via an invertible change of coordinates

Retain the linear maps

$$A : \mathbb{R}^n \to V, \qquad A(e_j) = a_j,$$

and

$$B : \mathbb{R}^n \to V, \qquad B(e_i) = b_i.$$

As in part (b)(i), we have

$$B = A \circ M.$$

Thus

$$\operatorname{span}(\{b_1, \ldots, b_n\}) = \operatorname{range}(B) = \operatorname{range}(A \circ M).$$

If M is invertible, then $M : \mathbb{R}^n \to \mathbb{R}^n$ is surjective. Therefore every vector $u \in \mathbb{R}^n$ can be written as $u = Mv$ for some $v \in \mathbb{R}^n$. Hence every element of $\operatorname{range}(A)$ has the form

$$A(u) = A(Mv) = (A \circ M)(v),$$

so

$$\operatorname{range}(A) \subseteq \operatorname{range}(A \circ M).$$

The reverse inclusion

$$\operatorname{range}(A \circ M) \subseteq \operatorname{range}(A)$$

always holds. Therefore

$$\operatorname{range}(A \circ M) = \operatorname{range}(A).$$

That is,

$$\operatorname{span}(\{b_1, \ldots, b_n\}) = \operatorname{span}(\{a_1, \ldots, a_n\}).$$

Thus

$$\boxed{\operatorname{span}(\{b_1, \ldots, b_n\}) = \operatorname{span}(\{a_1, \ldots, a_n\}).}$$

Mark Scheme (25 marks)

- **(a)(i) (5 marks)**
  - (1) Verifies $W \neq \varnothing$ (e.g. the zero sequence satisfies the recurrence).
  - (2) Correctly proves closure under linear combinations, or identifies W as a kernel / intersection of kernels of linear maps.
  - (1) States the relevant subspace criterion clearly.
  - (1) Concludes that W is a subspace of $\mathbb{R}^{\infty}$.
- **(a)(ii) (5 marks)**
  - (1) Verifies $W \neq \varnothing$ (e.g. $0 \in W$).
  - (1) Identifies a suitable subspace axiom to test, typically closure under addition.
  - (2) Gives explicit $A, B \in W$ with $\det(A) = \det(B) = 0$ but $\det(A + B) \neq 0$.
  - (1) Concludes that W is not a subspace.
- **(a)(iii) (5 marks)**
  - (1) Verifies $W \neq \varnothing$ (e.g. by taking $a = b = 0$).
  - (2) Correctly checks that $C \in \operatorname{span}\{M_1, M_2\}$ (or equivalently that $0 \in W$) by solving for coefficients.
  - (1) Deduces that $W = \operatorname{span}\{M_1, M_2\}$, or equivalently that the translate equals the underlying subspace.
  - (1) Concludes that W is a subspace of $\mathbb{R}^{2,2}$.
- **(b)(i) (4 marks)**
  - (1) Writes the definition of span as the set of all linear combinations.
  - (2) Substitutes each $b_i = \sum_j \lambda_{i,j} a_j$ into a general element of $\operatorname{span}(\{b_1, \ldots, b_n\})$, or uses the smallest-subspace / range argument correctly.
  - (1) Concludes that $\operatorname{span}(\{b_1, \ldots, b_n\}) \subseteq \operatorname{span}(\{a_1, \ldots, a_n\})$.
- **(b)(ii) (6 marks)**
  - (2) Uses invertibility of M correctly to express each a_i as a linear combination of the b_j, or proves range equality via surjectivity.
  - (2) Establishes the reverse inclusion $\operatorname{span}(\{a_1, \ldots, a_n\}) \subseteq \operatorname{span}(\{b_1, \ldots, b_n\})$.
  - (1) Combines with part (b)(i) to obtain equality.
  - (1) States the final equality clearly.

Question 2 (25 marks)

Problem

Define a function $T : \mathbb{R}^{2,2} \to \mathbb{R}^{2,2}$ as follows:

$$\forall A \in \mathbb{R}^{2,2} : \quad T(A) \text{ df} = A \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} A^{\mathsf{T}}.$$

Note: The symbol "T" denotes matrix transposition.

(a) **[5 points]** Show that $T \in L(\mathbb{R}^{2,2})$.

(b) **[5 points]** Compute the standard matrix representation of T.

(c) **[5 points]** Derive a basis for Null(T).

(d) **[5 points]** Derive a basis for Range(T).

(e) **[5 points]** Determine (with justification) whether or not $\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T)$. CAUTION: Your solution cannot be a blind application of the Rank-Nullity Theorem!

Solution

Method 1: Full coordinate solution + hyperplane characterisation

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Compute

$$AM = \begin{bmatrix} a+2b & 2a+b \\ c+2d & 2c+d \end{bmatrix}, \qquad NA^{\mathsf{T}} = \begin{bmatrix} 2a+b & 2c+d \\ a+2b & c+2d \end{bmatrix}.$$

Hence

$$T(A) = AM - NA^{\mathsf{T}} = \begin{bmatrix} b-a & 2a+b-2c-d \\ -a-2b+c+2d & c-d \end{bmatrix}.$$

Write $T(A) = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, so

$$x = b-a, \quad y = 2a+b-2c-d, \quad z = -a-2b+c+2d, \quad w = c-d.$$

(a) **Linearity. Theorem.** The maps $A \mapsto AM$, $A \mapsto NA$, and $A \mapsto A^{\mathsf{T}}$ are linear (fixed-matrix multiplication and transposition are linear). Sums/differences of linear maps are linear. Thus $T(A) = AM - N(A^{\mathsf{T}})$ is linear, so $T \in L(\mathbb{R}^{2,2})$.

$$\boxed{T \in L(\mathbb{R}^{2,2}).}$$

(b) **Standard matrix representation.** Use the standard basis $E_{11}, E_{12}, E_{21}, E_{22}$ and identify $\mathbb{R}^{2,2} \cong \mathbb{R}^4$ via

$$\begin{bmatrix} u & v \\ s & t \end{bmatrix} \leftrightarrow (u, v, s, t).$$

Compute (by substituting $(a, b, c, d) = (1, 0, 0, 0)$, etc.) the columns:

$$T(E_{11}) = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix},\ T(E_{12}) = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix},\ T(E_{21}) = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix},\ T(E_{22}) = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}.$$

Hence

$$[T] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 1 & -2 & -1 \\ -1 & -2 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

$$\boxed{[T] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 1 & -2 & -1 \\ -1 & -2 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}}$$

(c) **Basis for** $\mathrm{Null}(T)$**.** Solve $T(A) = 0$:

$$b - a = 0, \quad c - d = 0, \quad 2a + b - 2c - d = 0, \quad -a - 2b + c + 2d = 0.$$

From $b = a$ and $c = d$, the third gives $2a + a - 2c - c = 0 \Rightarrow 3a - 3c = 0 \Rightarrow a = c$. Thus $a = b = c = d = t$, so

$$\mathrm{Null}(T) = \left\{ t \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \mathrm{span} \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

$$\boxed{\mathrm{Null}(T) = \mathrm{span} \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.}$$

(d) **Basis for** $\mathrm{Range}(T)$**.**

Step 1: Range is contained in a hyperplane. Let $\ell : \mathbb{R}^{2,2} \to \mathbb{R}$ be the linear functional "sum of all entries":

$$\ell\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) = x + y + z + w.$$

For $T(A) = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ as above,

$$x + y + z + w = (b - a) + (2a + b - 2c - d) + (-a - 2b + c + 2d) + (c - d) = 0.$$

Hence $\mathrm{Range}(T) \subseteq H := \ker(\ell) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : x + y + z + w = 0 \right\}$.

Step 2: Hyperplane is contained in the range (explicit preimage). Take $Y = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in H$, so $x + y + z + w = 0$. Choose $d = 0$, set $c = w$, and define

$$a = \frac{y - x + 2w}{3}, \qquad b = x + a = \frac{2x + y + 2w}{3}.$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$b - a = x, \qquad c - d = w, \qquad 2a + b - 2c - d = y,$$

and

$$-a - 2b + c + 2d = -a - 2b + w = -(x + y + w) = z$$

(using $x + y + z + w = 0$). Thus $T(A) = Y$ and $H \subseteq \text{Range}(T)$. Therefore $\text{Range}(T) = H$. A basis of H is, for example,

$$R_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \quad R_3 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix},$$

which are linearly independent by entrywise coefficient comparison. Hence they form a basis of $\text{Range}(T)$.

$$\boxed{\text{A basis for Range}(T) \text{ is } \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}.}$$

(e) **Direct sum: $\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T)$. Theorem (Internal direct sum criterion).** $V = U \oplus W$ iff $V = U + W$ and $U \cap W = \{0\}$.

Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. From (c), $\text{Null}(T) = \text{span}\{J\}$ and from (d), $\text{Range}(T) = H = \ker(\ell)$.

Intersection: If $X \in \text{Null}(T) \cap \text{Range}(T)$ then $X = tJ$ and also $\ell(X) = 0$. But $\ell(tJ) = 4t$, so $4t = 0 \Rightarrow t = 0 \Rightarrow X = 0$. Hence the intersection is $\{0\}$.

Sum: For any $B \in \mathbb{R}^{2,2}$, let $s = \ell(B)$ and set $t = \frac{s}{4}$. Then $\ell(B - tJ) = \ell(B) - 4t = 0$, so $B - tJ \in \text{Range}(T)$, and $tJ \in \text{Null}(T)$. Thus $B = (B - tJ) + tJ \in \text{Range}(T) + \text{Null}(T)$, so $\mathbb{R}^{2,2} = \text{Range}(T) + \text{Null}(T)$. Therefore $\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T)$.

$$\boxed{\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T).}$$

Method 2: Standard matrix method + rank via constraint

Theorem 1 (Basic matrix operations). For fixed matrices M, N, the maps $A \mapsto AM$, $A \mapsto NA$, and $A \mapsto A^{\mathsf{T}}$ are linear. Sums, differences, and compositions of linear maps are linear.

Theorem 2 (Images span the range). If $V = \text{span}\{v_1, \ldots, v_k\}$, then

$$\text{range}(T) = \text{span}\{T(v_1), \ldots, T(v_k)\}.$$

Theorem 3 (Internal direct sum criterion). For subspaces $U, W \leq V$,

$$V = U \oplus W \iff (V = U + W) \text{ and } (U \cap W = \{0\}).$$

Let $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

(a) **Linearity.** For $A, B \in \mathbb{R}^{2,2}$ and $\lambda \in \mathbb{R}$,

$$T(A+\lambda B) = (A+\lambda B)M - N(A+\lambda B)^{\mathsf{T}} = AM + \lambda BM - NA^{\mathsf{T}} - \lambda NB^{\mathsf{T}} = T(A) + \lambda T(B),$$

using Theorem 1. Hence $T \in L(\mathbb{R}^{2,2})$.

$$\boxed{T \in L(\mathbb{R}^{2,2}).}$$

(b) **Standard matrix representation.** Use the standard basis $E_{11}, E_{12}, E_{21}, E_{22}$ of $\mathbb{R}^{2,2}$ and the identification

$$\begin{bmatrix} u & v \\ s & t \end{bmatrix} \leftrightarrow (u, v, s, t) \in \mathbb{R}^4.$$

Compute columns (each by direct substitution into $T(A) = AM - NA^{\mathsf{T}}$):

$$T(E_{11}) = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}, \quad T(E_{12}) = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}, \quad T(E_{21}) = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}, \quad T(E_{22}) = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}.$$

Thus

$$[T] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 1 & -2 & -1 \\ -1 & -2 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

$$\boxed{[T] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 1 & -2 & -1 \\ -1 & -2 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}}$$

(c) **Basis for** Null(T)**.** Solve $T(A) = 0$ for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by using the coordinate formula derived from $AM - NA^{\mathsf{T}}$:

$$T(A) = \begin{bmatrix} b - a & 2a + b - 2c - d \\ -a - 2b + c + 2d & c - d \end{bmatrix}.$$

Setting this equal to 0 gives

$$b - a = 0, \qquad c - d = 0, \qquad 2a + b - 2c - d = 0, \qquad -a - 2b + c + 2d = 0.$$

From $b = a$ and $c = d$, the third equation becomes $3a - 3c = 0$, so $a = c$. Hence $a = b = c = d = t$. Therefore

$$\text{Null}(T) = \left\{ t \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

$$\boxed{\text{Null}(T) = \text{span} \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.}$$

(d) **Basis for** Range(T)**.** By Theorem 2,

$$\text{Range}(T) = \text{span}\{T(E_{11}), T(E_{12}), T(E_{21}), T(E_{22})\}.$$

Define the linear functional $\ell : \mathbb{R}^{2,2} \to \mathbb{R}$ by $\ell\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) = x + y + z + w$. From the coordinate expression for $T(A)$, one checks (by direct cancellation) that $\ell(T(A)) = 0$ for all A; hence

$$\text{Range}(T) \subseteq \ker(\ell) =: H = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : x + y + z + w = 0 \right\},$$

and $\dim(H) = 3$ (one independent linear constraint).

Now exhibit three linearly independent elements of Range(T):

$$R_1 := T(E_{11}) = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}, \quad R_2 := T(E_{12}) = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}, \quad R_3 := T(E_{21}) = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}.$$

If $\alpha R_1 + \beta R_2 + \gamma R_3 = 0$, then comparing the $(2,2)$ entry gives $\gamma = 0$; then comparing $(1,1)$ and $(2,1)$ gives $-\alpha + \beta = 0$ and $-\alpha - 2\beta = 0$, hence $\alpha = \beta = 0$. Thus R_1, R_2, R_3 are linearly independent, so $\dim \text{Range}(T) \geq 3$. Since Range$(T) \subseteq H$ and $\dim(H) = 3$, it follows that Range$(T) = H$ and $\{R_1, R_2, R_3\}$ is a basis of Range(T).

$$\boxed{\text{A basis for Range}(T) \text{ is } \left\{ \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \right\}.}$$

(e) **Determine whether** $\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T)$**.** Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. From (c), Null$(T) =$ span$\{J\}$, and from (d), Range$(T) = H = \ker(\ell)$.

Intersection: If $X \in \text{Null}(T) \cap \text{Range}(T)$, then $X = tJ$ and also $\ell(X) = 0$. But $\ell(tJ) = 4t$, so $t = 0$ and $X = 0$. Thus Null$(T) \cap$ Range$(T) = \{0\}$.

Sum: Let $B \in \mathbb{R}^{2,2}$ be arbitrary and set $t = \frac{\ell(B)}{4}$. Then $\ell(B - tJ) = 0$, so $B - tJ \in H =$ Range(T), and $tJ \in$ Null(T). Hence

$$B = (B - tJ) + tJ \in \text{Range}(T) + \text{Null}(T).$$

Therefore $\mathbb{R}^{2,2} = \text{Null}(T) + \text{Range}(T)$.

By Theorem 3, $\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T)$.

$$\boxed{\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T).}$$

Method 3: Operator factorisation + invariant hyperplane

Theorem 1 (Linearity of basic matrix operations). For fixed matrices M, N, the maps $A \mapsto AM$, $A \mapsto NA$, and $A \mapsto A^{\mathsf{T}}$ are linear.

Theorem 2 (Internal direct sum criterion). $V = U \oplus W$ iff $V = U + W$ and $U \cap W = \{0\}$.

Let $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $N = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and define $\ell : \mathbb{R}^{2,2} \to \mathbb{R}$ by

$$\ell\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) = x + y + z + w.$$

(a) **Linearity.** By Theorem 1, $A \mapsto AM$ is linear, $A \mapsto A^{\mathsf{T}}$ is linear, and $A \mapsto N(A^{\mathsf{T}})$ is linear as a composition. Hence

$$T(A) = AM - N(A^{\mathsf{T}})$$

is linear and $T \in L(\mathbb{R}^{2,2})$.

$$\boxed{T \in L(\mathbb{R}^{2,2}).}$$

(b) **Standard matrix representation.** Using the standard basis $E_{11}, E_{12}, E_{21}, E_{22}$ and the identification $\mathbb{R}^{2,2} \cong \mathbb{R}^4$ via $(x_{11}, x_{12}, x_{21}, x_{22})$, compute:

$$T(E_{11}) = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix},\ T(E_{12}) = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix},\ T(E_{21}) = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix},\ T(E_{22}) = \begin{bmatrix} 0 & -1 \\ 2 & -1 \end{bmatrix}.$$

Thus

$$\boxed{[T] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 1 & -2 & -1 \\ -1 & -2 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}}.$$

(c) **Basis for** Null(T)**.** Compute $T(A)$ explicitly for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$T(A) = \begin{bmatrix} b - a & 2a + b - 2c - d \\ -a - 2b + c + 2d & c - d \end{bmatrix}.$$

Setting $T(A) = 0$ yields $b = a$, $c = d$, and then $3a - 3c = 0$, so $a = b = c = d = t$. Hence

$$\text{Null}(T) = \text{span}\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}.$$

$$\boxed{\text{Null}(T) = \text{span}\left\{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right\}.}$$

(d) **Basis for** Range(T)**.** First show Range$(T) \subseteq \ker(\ell)$. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $T(A) = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ as above,

$$\ell(T(A)) = x + y + z + w = (b - a) + (2a + b - 2c - d) + (-a - 2b + c + 2d) + (c - d) = 0.$$

Thus $\text{Range}(T) \subseteq H := \ker(\ell)$, where

$$H = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : x + y + z + w = 0 \right\}, \qquad \dim(H) = 3.$$

Next show $H \subseteq \text{Range}(T)$ by explicitly solving $T(A) = Y$ for any $Y = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in H$. Set $d = 0$, $c = w$, and define

$$a = \frac{y - x + 2w}{3}, \qquad b = x + a.$$

Then $b - a = x$, $c - d = w$, $2a + b - 2c - d = y$, and the remaining entry becomes

$$-a - 2b + c + 2d = -a - 2b + w = -(x + y + w) = z$$

using $x + y + z + w = 0$. Hence $T(A) = Y$ and $H \subseteq \text{Range}(T)$. Therefore $\text{Range}(T) = H$. A basis for H is for example

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\},$$

which is easily checked linearly independent by comparing entries.

$$\boxed{\text{Range}(T) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : x + y + z + w = 0 \right\}, \ \dim \text{Range}(T) = 3.}$$

(e) **Direct sum.** Let $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. We have $\text{Null}(T) = \text{span}\{J\}$ and $\text{Range}(T) = H = \ker(\ell)$.

Intersection: if $tJ \in H$, then $0 = \ell(tJ) = 4t$, so $t = 0$. Hence $\text{Null}(T) \cap \text{Range}(T) = \{0\}$.

Sum: for any B, set $t = \frac{\ell(B)}{4}$. Then $\ell(B - tJ) = 0$, so $B - tJ \in H = \text{Range}(T)$ and $tJ \in \text{Null}(T)$, giving $B \in \text{Range}(T) + \text{Null}(T)$. Thus $\mathbb{R}^{2,2} = \text{Range}(T) + \text{Null}(T)$.

By Theorem 2,

$$\boxed{\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T).}$$

Method 4: Codimension-1 decomposition theorem

Theorem (Codimension-1 decomposition). Let V be a vector space and $\ell : V \to \mathbb{R}$ be a nonzero linear functional. Let $H = \ker(\ell)$. If $v_0 \in V$ satisfies $\ell(v_0) = 1$, then

$$V = H \oplus \operatorname{span}\{v_0\}.$$

Proof: For any $v \in V$, write $v = (v - \ell(v)v_0) + \ell(v)v_0$ with $v - \ell(v)v_0 \in H$ and $\ell(v)v_0 \in \operatorname{span}\{v_0\}$. If $h \in H \cap \operatorname{span}\{v_0\}$ then $h = tv_0$ and $0 = \ell(h) = t$, so $h = 0$.

Let $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and ℓ be sum-of-entries.

(a) **Linearity.** As before,

$$T(A) = AM - N(A^{\mathsf{T}})$$

is a difference of linear maps, hence linear.

$$\boxed{T \in L(\mathbb{R}^{2,2}).}$$

(b) **Standard matrix representation.** Compute $T(E_{11}), T(E_{12}), T(E_{21}), T(E_{22})$ to obtain

$$\boxed{[T] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 1 & -2 & -1 \\ -1 & -2 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}}.$$

(c) **Basis for** $\operatorname{Null}(T)$**.** Solve $T(A) = 0$ to get $A = tJ$ where $J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Hence

$$\boxed{\operatorname{Null}(T) = \operatorname{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.}$$

(d) **Basis for** $\operatorname{Range}(T)$**.** Show $\operatorname{Range}(T) \subseteq \ker(\ell)$ by verifying $\ell(T(A)) = 0$ from the explicit formula for $T(A)$. Conversely, for any $Y \in \ker(\ell)$, the explicit construction (set $d = 0$, $c = w$, $a = \frac{y-x+2w}{3}$, $b = x + a$) yields A with $T(A) = Y$, so $\ker(\ell) \subseteq \operatorname{Range}(T)$. Thus $\operatorname{Range}(T) = \ker(\ell)$ and a basis is any basis of that hyperplane, e.g. the three matrices in Method 1.

$$\boxed{\operatorname{Range}(T) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : x + y + z + w = 0 \right\}.}$$

(e) **Direct sum.** Let $H = \ker(\ell) = \operatorname{Range}(T)$ and let J be all-ones. Since $\ell(J) = 4 \neq 0$, set $v_0 = \frac{1}{4}J$, so $\ell(v_0) = 1$. By the codimension-1 decomposition theorem,

$$\mathbb{R}^{2,2} = H \oplus \operatorname{span}\{v_0\} = \operatorname{Range}(T) \oplus \operatorname{Null}(T).$$

$$\boxed{\mathbb{R}^{2,2} = \operatorname{Null}(T) \oplus \operatorname{Range}(T).}$$

Method 5: Duality/annihilator dimension squeeze

Theorem (Dimension squeeze via annihilator). Let V be finite-dimensional, $\ell \neq 0$ a linear functional, and $H = \ker(\ell)$ so $\dim(H) = \dim(V) - 1$. If $\text{Range}(T) \subseteq H$ and $\dim \text{Range}(T) \geq \dim(H)$, then $\text{Range}(T) = H$.

Let ℓ be sum-of-entries and $H = \ker(\ell)$.

(a) **Linearity.** $T(A) = AM - N(A^{\mathsf{T}})$ is a difference of linear maps, hence linear.

$$\boxed{T \in L(\mathbb{R}^{2,2}).}$$

(b) **Standard matrix representation.** Compute $T(E_{ij})$ to obtain

$$\boxed{[T] = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 1 & -2 & -1 \\ -1 & -2 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}.}$$

(c) **Basis for** $\text{Null}(T)$**.** Solve $T(A) = 0$ to get $A = tJ$ with J all-ones, so

$$\boxed{\text{Null}(T) = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.}$$

(d) **Basis for** $\text{Range}(T)$**.** First, $\ell(T(A)) = 0$ for all A (direct cancellation), hence $\text{Range}(T) \subseteq H = \ker(\ell)$. Since $\dim(\mathbb{R}^{2,2}) = 4$ and $\ell \neq 0$, we have $\dim(H) = 3$.

Now show $\dim \text{Range}(T) \geq 3$ by producing three independent outputs:

$$R_1 = T(E_{11}), \quad R_2 = T(E_{12}), \quad R_3 = T(E_{21}),$$

and check independence exactly as in Method 2(d) by comparing entries, yielding only the trivial combination. Thus $\dim \text{Range}(T) \geq 3 = \dim(H)$. By the squeeze theorem, $\text{Range}(T) = H$.

A basis of H (hence of $\text{Range}(T)$) can be taken as any three independent vectors in H, e.g. $\{R_1, R_2, R_3\}$.

$$\boxed{\text{Range}(T) = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : x + y + z + w = 0 \right\}.}$$

(e) **Direct sum.** With $\text{Null}(T) = \text{span}\{J\}$ and $\text{Range}(T) = H = \ker(\ell)$:

$$\text{Null}(T) \cap \text{Range}(T) = \{0\} \quad \text{since} \quad tJ \in H \Rightarrow 0 = \ell(tJ) = 4t \Rightarrow t = 0,$$

and every B decomposes as $B = (B - \frac{\ell(B)}{4}J) + \frac{\ell(B)}{4}J$ with the first term in H and second in $\text{span}\{J\}$. Thus $\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T)$.

$$\boxed{\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T).}$$

Mark Scheme (25 marks)

- **(a) (5 marks)**
  - (2) States and uses linearity of transpose and multiplication by fixed matrices.
  - (2) Shows $T(A+B) = T(A) + T(B)$ and $T(\lambda A) = \lambda T(A)$.
  - (1) Concludes $T \in L(\mathbb{R}^{2,2})$.
- **(b) (5 marks)**
  - (2) Correctly computes $T(E_{11}), T(E_{12}), T(E_{21}), T(E_{22})$.
  - (2) Converts each to $(x_{11}, x_{12}, x_{21}, x_{22})$ coordinates (consistent ordering).
  - (1) Assembles the 4×4 matrix $[T]$ with correct column order.
- **(c) (5 marks)**
  - (2) Sets up $T(A) = 0$ and derives the correct linear system in a, b, c, d.
  - (2) Solves to obtain $a = b = c = d$ and describes $\text{Null}(T)$.
  - (1) States a correct basis for $\text{Null}(T)$.
- **(d) (5 marks)**
  - (2) Identifies a necessary constraint for outputs (e.g. sum of entries is 0) so $\text{Range}(T) \subseteq H$.
  - (2) Proves $H \subseteq \text{Range}(T)$ (explicit preimage / solvability argument) or equivalently proves $\dim \text{Range}(T) = 3$ with basis.
  - (1) States a correct basis for $\text{Range}(T)$.
- **(e) (5 marks)**
  - (2) Shows $\text{Null}(T) \cap \text{Range}(T) = \{0\}$ (e.g. via entry-sum invariant).
  - (2) Shows $\text{Null}(T) + \text{Range}(T) = \mathbb{R}^{2,2}$ by constructing a decomposition for arbitrary B.
  - (1) Concludes $\mathbb{R}^{2,2} = \text{Null}(T) \oplus \text{Range}(T)$ with the internal direct sum criterion.

Unofficial Solutions by QRS@NTU

Question 3 (25 marks)

(a) Existence of $T \in L(V)$ such that $T^4 = -I_V$

Problem

Let V be a 4-dimensional vector space over an arbitrary field.
[10 points] Show that there is a $T \in L(V)$ such that $T^4 = -I_V$.
Note: T^4 df $= T \circ T \circ T \circ T$.

Solution

Method 1: Explicit basis construction

Since $\dim V = 4$, we may choose a basis

$$(v_1, v_2, v_3, v_4)$$

of V.

Define $T \in L(V)$ by prescribing its values on this basis:

$$T(v_1) = v_2, \qquad T(v_2) = v_3, \qquad T(v_3) = v_4, \qquad T(v_4) = -v_1.$$

By the theorem that a linear map is uniquely determined by its values on a basis, this defines a linear transformation $T \in L(V)$.

We now compute T^4 on the basis vectors:

$$T^4(v_1) = T^3(v_2) = T^2(v_3) = T(v_4) = -v_1,$$

$$T^4(v_2) = T^3(v_3) = T^2(v_4) = T(-v_1) = -v_2,$$
$$T^4(v_3) = T^3(v_4) = T^2(-v_1) = T(-v_2) = -v_3,$$
$$T^4(v_4) = T^3(-v_1) = -T^3(v_1) = -v_4.$$

Thus

$$T^4(v_i) = -v_i \qquad (i = 1, 2, 3, 4).$$

Since (v_1, v_2, v_3, v_4) is a basis, two linear maps that agree on all basis vectors are equal. Therefore

$$T^4 = -I_V.$$

Hence there exists $T \in L(V)$ such that $T^4 = -I_V$, namely the one defined above.

$$\boxed{\exists T \in L(V) \text{ such that } T^4 = -I_V.}$$

Method 2: Polynomial-module / quotient-space construction

Let $\mathbb{F}$ denote the underlying field. Consider the quotient vector space

$$U := \mathbb{F}[t]/(t^4+1).$$

Every class in U has a unique representative of degree at most 3, so

$$\{[1],[t],[t^2],[t^3]\}$$

is a basis of U. In particular,

$$\dim U = 4.$$

Define a linear map $\widetilde{T} : U \to U$ by

$$\widetilde{T}([p(t)]) = [t\,p(t)].$$

This is linear because multiplication by t distributes over addition and scalar multiplication.

Now compute $\widetilde{T}^4$:

$$\widetilde{T}^4([p(t)]) = [t^4 p(t)].$$

But in the quotient $U = \mathbb{F}[t]/(t^4+1)$, we have

$$t^4 \equiv -1 \pmod{t^4+1},$$

so

$$[t^4 p(t)] = [-p(t)] = -[p(t)].$$

Hence

$$\widetilde{T}^4 = -I_U.$$

Since $\dim V = 4 = \dim U$, there exists a vector space isomorphism

$$\Phi : V \to U.$$

Define

$$T := \Phi^{-1} \circ \widetilde{T} \circ \Phi \in L(V).$$

Then

$$T^4 = \Phi^{-1} \circ \widetilde{T}^4 \circ \Phi = \Phi^{-1} \circ (-I_U) \circ \Phi = -I_V.$$

Therefore there exists $T \in L(V)$ such that $T^4 = -I_V$.

$$\boxed{\exists\, T \in L(V) \text{ such that } T^4 = -I_V.}$$

(b) Idempotent maps and the decomposition $V = \mathrm{Null}(P) \oplus \mathrm{Range}(P)$

Problem

Let V be a vector space over an arbitrary field, and let $P \in L(V)$ be idempotent, i.e.

$$P^2 = P.$$

(i) **[4 points]** Show, for all $v \in V$, that $v - P(v) \in \mathrm{Null}(P)$.

(ii) **[7 points]** Show that $\mathrm{Null}(P) \cap \mathrm{Range}(P) = \{0_V\}$.

(iii) **[4 points]** Hence, show that $V = \mathrm{Null}(P) \oplus \mathrm{Range}(P)$.

CAUTION: The Rank-Nullity Theorem cannot be applied here!

Solution

Method 1: Direct computation and internal direct sum criterion

(i) Let $v \in V$. Then

$$P(v - P(v)) = P(v) - P^2(v).$$

Since $P^2 = P$, this becomes

$$P(v - P(v)) = P(v) - P(v) = 0_V.$$

Therefore $v - P(v) \in \text{Null}(P)$. Hence

$$\boxed{\forall v \in V : \ v - P(v) \in \text{Null}(P).}$$

(ii) Let

$$x \in \text{Null}(P) \cap \text{Range}(P).$$

Then $x \in \text{Range}(P)$, so there exists $y \in V$ such that

$$x = P(y).$$

Also $x \in \text{Null}(P)$, so

$$P(x) = 0_V.$$

But

$$P(x) = P(P(y)) = P^2(y) = P(y) = x,$$

since $P^2 = P$. Therefore

$$x = 0_V.$$

Thus every element of $\text{Null}(P) \cap \text{Range}(P)$ is 0_V, so

$$\text{Null}(P) \cap \text{Range}(P) = \{0_V\}.$$

Hence

$$\boxed{\text{Null}(P) \cap \text{Range}(P) = \{0_V\}.}$$

(iii) Let $v \in V$. Then

$$v = (v - P(v)) + P(v).$$

By part (i),

$$v - P(v) \in \text{Null}(P),$$

and by definition,

$$P(v) \in \text{Range}(P).$$

Hence

$$V = \text{Null}(P) + \text{Range}(P).$$

Together with part (ii), we obtain

$$V = \text{Null}(P) \oplus \text{Range}(P).$$

Therefore

$$\boxed{V = \text{Null}(P) \oplus \text{Range}(P).}$$

Method 2: Complementary idempotent $Q = I_V - P$

Define

$$Q := I_V - P.$$

We first compute

$$Q^2 = (I_V - P)^2 = I_V - 2P + P^2 = I_V - P = Q,$$

since $P^2 = P$. Thus Q is also idempotent.

(i) For any $v \in V$,

$$Q(v) = v - P(v).$$

Now

$$P(Q(v)) = P(v - P(v)) = P(v) - P^2(v) = P(v) - P(v) = 0_V,$$

so

$$Q(v) = v - P(v) \in \text{Null}(P).$$

Hence

$$\boxed{\forall v \in V : \ v - P(v) \in \text{Null}(P).}$$

(ii) Let

$$x \in \text{Null}(P) \cap \text{Range}(P).$$

Then $x = P(y)$ for some $y \in V$, and also $P(x) = 0_V$. But

$$P(x) = P(P(y)) = P^2(y) = P(y) = x.$$

So $x = 0_V$. Therefore

$$\text{Null}(P) \cap \text{Range}(P) = \{0_V\}.$$

Hence

$$\boxed{\text{Null}(P) \cap \text{Range}(P) = \{0_V\}.}$$

(iii) For every $v \in V$,

$$v = Q(v) + P(v) = (v - P(v)) + P(v).$$

By part (i),

$$Q(v) = v - P(v) \in \text{Null}(P),$$

and clearly

$$P(v) \in \text{Range}(P).$$

Thus

$$V = \text{Null}(P) + \text{Range}(P).$$

Together with part (ii), we obtain

$$V = \text{Null}(P) \oplus \text{Range}(P).$$

Therefore

$$\boxed{V = \text{Null}(P) \oplus \text{Range}(P).}$$

Method 3: Mutual inclusions $\text{Range}(I_V - P) = \text{Null}(P)$ **and** $\text{Range}(P) = \text{Null}(I_V - P)$
Let

$$Q := I_V - P.$$

We claim that

$$\text{Range}(Q) = \text{Null}(P) \qquad \text{and} \qquad \text{Range}(P) = \text{Null}(Q).$$

First, if $x \in \text{Range}(Q)$, say $x = Q(v) = v - P(v)$, then

$$P(x) = P(v - P(v)) = P(v) - P^2(v) = 0_V,$$

so $x \in \text{Null}(P)$. Hence

$$\text{Range}(Q) \subseteq \text{Null}(P).$$

Conversely, if $x \in \text{Null}(P)$, then $P(x) = 0_V$, so

$$Q(x) = x - P(x) = x.$$

Thus $x \in \text{Range}(Q)$. Therefore

$$\text{Range}(Q) = \text{Null}(P).$$

Similarly, if $x \in \text{Range}(P)$, say $x = P(v)$, then

$$Q(x) = x - P(x) = P(v) - P^2(v) = 0_V,$$

so $x \in \text{Null}(Q)$. Hence

$$\text{Range}(P) \subseteq \text{Null}(Q).$$

Conversely, if $x \in \text{Null}(Q)$, then

$$Q(x) = x - P(x) = 0_V,$$

so $x = P(x) \in \text{Range}(P)$. Therefore

$$\text{Range}(P) = \text{Null}(Q).$$

Now we prove the required parts.
(i) For any $v \in V$,

$$v - P(v) = Q(v) \in \text{Range}(Q) = \text{Null}(P).$$

Thus

$$\boxed{\forall v \in V : \ v - P(v) \in \text{Null}(P).}$$

(ii) Let

$$x \in \text{Null}(P) \cap \text{Range}(P).$$

Then

$$x \in \text{Range}(Q) \cap \text{Range}(P).$$

Also, from $x \in \text{Null}(P)$, we get $P(x) = 0_V$, while from $x \in \text{Range}(P)$, we may write $x = P(y)$, hence

$$x = P(y) = P(x) = 0_V.$$

Therefore

$$\text{Null}(P) \cap \text{Range}(P) = \{0_V\}.$$

So

$$\boxed{\text{Null}(P) \cap \text{Range}(P) = \{0_V\}.}$$

(iii) For every $v \in V$,

$$v = (I_V - P)(v) + P(v) = Q(v) + P(v).$$

Now

$$Q(v) \in \text{Range}(Q) = \text{Null}(P), \qquad P(v) \in \text{Range}(P).$$

Hence

$$V = \text{Null}(P) + \text{Range}(P).$$

Together with part (ii), this yields

$$V = \text{Null}(P) \oplus \text{Range}(P).$$

Therefore

$$\boxed{V = \text{Null}(P) \oplus \text{Range}(P).}$$

Mark Scheme (25 marks)

- **(a) (10 marks)**
  - (2) Uses $\dim V = 4$ to choose a basis of V (or to identify a 4-dimensional model space).
  - (3) Correctly defines a linear map T with the intended cyclic action.
  - (3) Verifies that $T^4(v_i) = -v_i$ for basis vectors (or proves the equivalent identity in a model space).
  - (2) Concludes that $T^4 = -I_V$.
- **(b)(i) (4 marks)**
  - (2) Computes $P(v - P(v)) = P(v) - P^2(v)$.
  - (1) Uses $P^2 = P$ to deduce $P(v - P(v)) = 0_V$.
  - (1) Concludes that $v - P(v) \in \text{Null}(P)$.
- **(b)(ii) (7 marks)**
  - (2) Lets $x \in \text{Null}(P) \cap \text{Range}(P)$ and writes $x = P(y)$.
  - (2) Uses $x \in \text{Null}(P)$ to obtain $P(x) = 0_V$.
  - (2) Computes $P(x) = P(P(y)) = P^2(y) = P(y) = x$.
  - (1) Concludes $x = 0_V$, and hence the intersection is $\{0_V\}$.
- **(b)(iii) (4 marks)**
  - (2) Shows $v = (v - P(v)) + P(v)$ with summands in $\text{Null}(P)$ and $\text{Range}(P)$.
  - (1) Uses part (ii) to establish trivial intersection.
  - (1) Concludes that $V = \text{Null}(P) \oplus \text{Range}(P)$.

Question 4 (25 marks)

(a) The external direct sum $V \oplus_{\text{ext}} W$ is a vector space

Problem

Let V and W be vector spaces over a field $\mathbb{F}$ (not necessarily $\mathbb{R}$ or $\mathbb{C}$).

Define a binary operation $+ : (V \times W) \times (V \times W) \to V \times W$ by

$$(v, w) + (\tilde{v}, \tilde{w}) \coloneqq (v +_V \tilde{v},\ w +_W \tilde{w}),$$

for all $v, \tilde{v} \in V$ and $w, \tilde{w} \in W$.

Define a scalar multiplication $\cdot : \mathbb{F} \times (V \times W) \to V \times W$ by

$$\lambda \cdot (v, w) \coloneqq (\lambda \cdot_V v,\ \lambda \cdot_W w),$$

for all $v \in V$, $w \in W$, and $\lambda \in \mathbb{F}$.

Show that $V \times W$ is a vector space over $\mathbb{F}$ when equipped with these operations.

Note: All eight axioms of a vector space must be verified.

Solution

Method 1: Full verification of all eight axioms componentwise

Let

$$x = (v, w), \qquad y = (\tilde{v}, \tilde{w}), \qquad z = (\hat{v}, \hat{w}) \in V \times W,$$

and let $\lambda, \mu \in \mathbb{F}$. We verify the eight vector space axioms one by one.

A1. Associativity of addition. We compute

$$(x + y) + z = (v +_V \tilde{v},\ w +_W \tilde{w}) + (\hat{v}, \hat{w}) = ((v +_V \tilde{v}) +_V \hat{v},\ (w +_W \tilde{w}) +_W \hat{w}).$$

Since addition in V and in W is associative,

$$((v +_V \tilde{v}) +_V \hat{v},\ (w +_W \tilde{w}) +_W \hat{w}) = (v +_V (\tilde{v} +_V \hat{v}),\ w +_W (\tilde{w} +_W \hat{w})).$$

But this is exactly

$$(v, w) + (\tilde{v} +_V \hat{v},\ \tilde{w} +_W \hat{w}) = (v, w) + \big((\tilde{v}, \tilde{w}) + (\hat{v}, \hat{w})\big) = x + (y + z).$$

Thus

$$(x + y) + z = x + (y + z).$$

A2. Commutativity of addition. We have

$$x + y = (v +_V \tilde{v},\ w +_W \tilde{w}).$$

Since addition in V and in W is commutative,

$$(v +_V \tilde{v},\ w +_W \tilde{w}) = (\tilde{v} +_V v,\ \tilde{w} +_W w) = (\tilde{v}, \tilde{w}) + (v, w) = y + x.$$

Hence

$$x + y = y + x.$$

A3. Existence of an additive identity. Let 0_V and 0_W denote the zero vectors of V and W, respectively. Consider $(0_V, 0_W) \in V \times W$. Then

$$x + (0_V, 0_W) = (v +_V 0_V,\ w +_W 0_W) = (v, w) = x,$$

since 0_V is the additive identity in V and 0_W is the additive identity in W. Similarly,

$$(0_V, 0_W) + x = x.$$

Therefore $(0_V, 0_W)$ is an additive identity in $V \times W$.

A4. Existence of additive inverses. Let

$$-x \coloneqq (-_V v,\ -_W w),$$

where $-_V v$ is the additive inverse of v in V, and $-_W w$ is the additive inverse of w in W. Then

$$x + (-x) = (v +_V (-_V v),\ w +_W (-_W w)) = (0_V, 0_W).$$

Similarly,

$$(-x) + x = (0_V, 0_W).$$

Thus every element of $V \times W$ has an additive inverse.

A5. Associativity of scalar multiplication. We compute

$$(\lambda\mu) \cdot x = ((\lambda\mu) \cdot_V v,\ (\lambda\mu) \cdot_W w).$$

Since scalar multiplication is associative in V and in W,

$$((\lambda\mu) \cdot_V v,\ (\lambda\mu) \cdot_W w) = \big(\lambda \cdot_V (\mu \cdot_V v),\ \lambda \cdot_W (\mu \cdot_W w)\big).$$

By definition of scalar multiplication on $V \times W$, this is

$$\lambda \cdot (\mu \cdot v,\ \mu \cdot w) = \lambda \cdot (\mu \cdot x).$$

Hence

$$(\lambda\mu) \cdot x = \lambda \cdot (\mu \cdot x).$$

A6. Identity element of scalar multiplication. Let $1 \in \mathbb{F}$ be the multiplicative identity. Then

$$1 \cdot x = (1 \cdot_V v,\ 1 \cdot_W w) = (v, w) = x,$$

since $1 \cdot_V v = v$ in V and $1 \cdot_W w = w$ in W. Thus

$$1 \cdot x = x.$$

A7. Distributivity of scalar multiplication over vector addition. We have

$$x + y = (v +_V \tilde{v},\ w +_W \tilde{w}).$$

Hence

$$\lambda \cdot (x + y) = \big(\lambda \cdot_V (v +_V \tilde{v}),\ \lambda \cdot_W (w +_W \tilde{w})\big).$$

Since scalar multiplication distributes over vector addition in V and in W,

$$\big(\lambda \cdot_V (v +_V \tilde{v}),\ \lambda \cdot_W (w +_W \tilde{w})\big) = (\lambda \cdot_V v +_V \lambda \cdot_V \tilde{v},\ \lambda \cdot_W w +_W \lambda \cdot_W \tilde{w}).$$

By definition of the operations on $V \times W$, this is

$$(\lambda \cdot_V v,\ \lambda \cdot_W w) + (\lambda \cdot_V \tilde{v},\ \lambda \cdot_W \tilde{w}) = \lambda \cdot x + \lambda \cdot y.$$

Therefore

$$\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y.$$

A8. Distributivity of scalar multiplication over scalar addition. We compute

$$(\lambda + \mu) \cdot x = ((\lambda + \mu) \cdot_V v,\ (\lambda + \mu) \cdot_W w).$$

Since scalar multiplication distributes over scalar addition in V and in W,

$$((\lambda + \mu) \cdot_V v,\ (\lambda + \mu) \cdot_W w) = (\lambda \cdot_V v +_V \mu \cdot_V v,\ \lambda \cdot_W w +_W \mu \cdot_W w).$$

By definition of the operations on $V \times W$, this is

$$(\lambda \cdot_V v,\ \lambda \cdot_W w) + (\mu \cdot_V v,\ \mu \cdot_W w) = \lambda \cdot x + \mu \cdot x.$$

Hence

$$(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x.$$

All eight vector space axioms hold. Therefore $V \times W$, equipped with the above operations, is a vector space over $\mathbb{F}$. This vector space is denoted by

$$V \oplus_{\text{ext}} W.$$

Thus

$$\boxed{V \oplus_{\text{ext}} W \text{ is a vector space over } \mathbb{F}.}$$

Method 2: Verification via coordinate projections

Define the coordinate projections

$$\pi_V : V \times W \to V, \qquad \pi_V(v, w) = v,$$

and

$$\pi_W : V \times W \to W, \qquad \pi_W(v, w) = w.$$

We use the elementary fact that for any $(v, w), (\tilde{v}, \tilde{w}) \in V \times W$,

$$(v, w) = (\tilde{v}, \tilde{w}) \iff v = \tilde{v} \text{ and } w = \tilde{w}.$$

Thus, to prove two ordered pairs are equal, it suffices to prove equality of their first components and equality of their second components.

Let

$$x = (v, w), \qquad y = (\tilde{v}, \tilde{w}), \qquad z = (\hat{v}, \hat{w}) \in V \times W,$$

and let $\lambda, \mu \in \mathbb{F}$. We verify the eight axioms by checking both coordinate projections.

A1. Associativity of addition. First components:

$$\pi_V((x + y) + z) = (v +_V \tilde{v}) +_V \hat{v} = v +_V (\tilde{v} +_V \hat{v}) = \pi_V(x + (y + z)).$$

Second components:

$$\pi_W((x + y) + z) = (w +_W \tilde{w}) +_W \hat{w} = w +_W (\tilde{w} +_W \hat{w}) = \pi_W(x + (y + z)).$$

Hence

$$(x + y) + z = x + (y + z).$$

A2. Commutativity of addition. First components:

$$\pi_V(x + y) = v +_V \tilde{v} = \tilde{v} +_V v = \pi_V(y + x).$$

Second components:

$$\pi_W(x + y) = w +_W \tilde{w} = \tilde{w} +_W w = \pi_W(y + x).$$

Thus

$$x + y = y + x.$$

A3. Existence of an additive identity. Take $(0_V, 0_W) \in V \times W$. Then

$$\pi_V(x + (0_V, 0_W)) = v +_V 0_V = v, \qquad \pi_W(x + (0_V, 0_W)) = w +_W 0_W = w.$$

Therefore

$$x + (0_V, 0_W) = x.$$

Similarly,

$$(0_V, 0_W) + x = x.$$

So $(0_V, 0_W)$ is an additive identity.

A4. Existence of additive inverses. Let

$$-x := (-_V v,\ -_W w).$$

Then

$$\pi_V(x + (-x)) = v +_V (-_V v) = 0_V, \qquad \pi_W(x + (-x)) = w +_W (-_W w) = 0_W.$$

Hence

$$x + (-x) = (0_V, 0_W).$$

Similarly,

$$(-x) + x = (0_V, 0_W).$$

Thus every element has an additive inverse.

A5. Associativity of scalar multiplication. First components:

$$\pi_V((\lambda\mu) \cdot x) = (\lambda\mu) \cdot_V v = \lambda \cdot_V (\mu \cdot_V v) = \pi_V(\lambda \cdot (\mu \cdot x)).$$

Second components:

$$\pi_W((\lambda\mu) \cdot x) = (\lambda\mu) \cdot_W w = \lambda \cdot_W (\mu \cdot_W w) = \pi_W(\lambda \cdot (\mu \cdot x)).$$

Therefore

$$(\lambda\mu) \cdot x = \lambda \cdot (\mu \cdot x).$$

A6. Identity element of scalar multiplication. First components:

$$\pi_V(1 \cdot x) = 1 \cdot_V v = v.$$

Second components:

$$\pi_W(1 \cdot x) = 1 \cdot_W w = w.$$

Thus

$$1 \cdot x = x.$$

A7. Distributivity over vector addition. First components:

$$\pi_V(\lambda \cdot (x + y)) = \lambda \cdot_V (v +_V \tilde{v}) = \lambda \cdot_V v +_V \lambda \cdot_V \tilde{v} = \pi_V(\lambda \cdot x + \lambda \cdot y).$$

Second components:

$$\pi_W(\lambda \cdot (x + y)) = \lambda \cdot_W (w +_W \tilde{w}) = \lambda \cdot_W w +_W \lambda \cdot_W \tilde{w} = \pi_W(\lambda \cdot x + \lambda \cdot y).$$

Hence

$$\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y.$$

A8. Distributivity over scalar addition. First components:

$$\pi_V((\lambda + \mu) \cdot x) = (\lambda + \mu) \cdot_V v = \lambda \cdot_V v +_V \mu \cdot_V v = \pi_V(\lambda \cdot x + \mu \cdot x).$$

Second components:

$$\pi_W((\lambda + \mu) \cdot x) = (\lambda + \mu) \cdot_W w = \lambda \cdot_W w +_W \mu \cdot_W w = \pi_W(\lambda \cdot x + \mu \cdot x).$$

Therefore

$$(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x.$$

Thus all eight vector space axioms hold, so $V \times W$ with the given operations is a vector space over $\mathbb{F}$, denoted $V \oplus_{\text{ext}} W$. Hence

$$\boxed{V \oplus_{\text{ext}} W \text{ is a vector space over } \mathbb{F}.}$$

(b) Basis and dimension of $V \oplus_{\text{ext}} W$

Problem

Suppose now that V and W are finite-dimensional, with

$$\dim(V) = m, \qquad \dim(W) = n.$$

Let

$$B = \{b_1, \ldots, b_m\} \qquad \text{and} \qquad C = \{c_1, \ldots, c_n\}$$

be bases of V and W, respectively.

(i) Construct a basis for $V \oplus_{\text{ext}} W$ using $b_1, \ldots, b_m, c_1, \ldots, c_n$, and verify both spanning and linear independence.

(ii) Determine, with justification, $\dim(V \oplus_{\text{ext}} W)$.

Solution

Method 1: Explicit basis construction with direct spanning and independence proofs

(i) Define

$$\mathcal{D} := \{(b_1, 0_W), \ldots, (b_m, 0_W), (0_V, c_1), \ldots, (0_V, c_n)\}.$$

We claim that $\mathcal{D}$ is a basis of $V \oplus_{\text{ext}} W$.

We first prove that $\mathcal{D}$ spans $V \oplus_{\text{ext}} W$. Let $(v, w) \in V \oplus_{\text{ext}} W$. Since B is a basis of V, there exist scalars $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ such that

$$v = \sum_{i=1}^{m} \alpha_i b_i.$$

Similarly, since C is a basis of W, there exist scalars $\beta_1, \ldots, \beta_n \in \mathbb{F}$ such that

$$w = \sum_{j=1}^{n} \beta_j c_j.$$

Using the decomposition

$$(v, w) = (v, 0_W) + (0_V, w),$$

we obtain

$$(v, 0_W) = \left(\sum_{i=1}^{m} \alpha_i b_i, 0_W \right) = \sum_{i=1}^{m} \alpha_i (b_i, 0_W),$$

and

$$(0_V, w) = \left(0_V, \sum_{j=1}^{n} \beta_j c_j \right) = \sum_{j=1}^{n} \beta_j (0_V, c_j).$$

Hence

$$(v, w) = \sum_{i=1}^{m} \alpha_i (b_i, 0_W) + \sum_{j=1}^{n} \beta_j (0_V, c_j).$$

So every element of $V \oplus_{\text{ext}} W$ is a linear combination of vectors in $\mathcal{D}$, and therefore $\mathcal{D}$ spans $V \oplus_{\text{ext}} W$.

Next we prove that $\mathcal{D}$ is linearly independent. Suppose

$$\sum_{i=1}^{m} \alpha_i (b_i, 0_W) + \sum_{j=1}^{n} \beta_j (0_V, c_j) = (0_V, 0_W).$$

Then

$$\left(\sum_{i=1}^{m} \alpha_i b_i,\ \sum_{j=1}^{n} \beta_j c_j \right) = (0_V, 0_W),$$

so

$$\sum_{i=1}^{m} \alpha_i b_i = 0_V \qquad \text{and} \qquad \sum_{j=1}^{n} \beta_j c_j = 0_W.$$

Since B is linearly independent, we obtain

$$\alpha_1 = \cdots = \alpha_m = 0.$$

Since C is linearly independent, we obtain

$$\beta_1 = \cdots = \beta_n = 0.$$

Thus all coefficients are zero, and $\mathcal{D}$ is linearly independent.

Therefore $\mathcal{D}$ is a basis of $V \oplus_{\text{ext}} W$. Hence

$$\boxed{\mathcal{D} = \{(b_1, 0_W), \ldots, (b_m, 0_W), (0_V, c_1), \ldots, (0_V, c_n)\} \text{ is a basis of } V \oplus_{\text{ext}} W.}$$

(ii) Since the basis $\mathcal{D}$ consists of m vectors of the form $(b_i, 0_W)$ and n vectors of the form $(0_V, c_j)$, it contains exactly $m + n$ vectors. Therefore

$$\dim(V \oplus_{\text{ext}} W) = m + n.$$

Hence

$$\boxed{\dim(V \oplus_{\text{ext}} W) = m + n.}$$

Method 2: Isomorphism from $\mathbb{F}^{m+n}$

(i) Define a map

$$\Phi : \mathbb{F}^{m+n} \to V \oplus_{\text{ext}} W$$

by

$$\Phi(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) = \left(\sum_{i=1}^{m} \alpha_i b_i,\ \sum_{j=1}^{n} \beta_j c_j \right).$$

We prove that Φ is a vector space isomorphism.

First, Φ is linear. Let

$$u = (\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n), \qquad u' = (\alpha_1', \ldots, \alpha_m', \beta_1', \ldots, \beta_n'),$$

and let $\lambda \in \mathbb{F}$. Then

$$\begin{aligned} \Phi(u + u') &= \Phi(\alpha_1 + \alpha_1', \ldots, \alpha_m + \alpha_m', \\ &\qquad \beta_1 + \beta_1', \ldots, \beta_n + \beta_n') \\ \Phi(u + u') &= \left(\sum_{i=1}^{m} (\alpha_i + \alpha_i') b_i, \sum_{j=1}^{n} (\beta_j + \beta_j') c_j \right) \\ &= \left(\sum_{i=1}^{m} \alpha_i b_i, \sum_{j=1}^{n} \beta_j c_j \right) + \left(\sum_{i=1}^{m} \alpha_i' b_i, \sum_{j=1}^{n} \beta_j' c_j \right) \\ &= \Phi(u) + \Phi(u'). \end{aligned}$$

Also,

$$\Phi(\lambda u) = \Phi(\lambda\alpha_1, \ldots, \lambda\alpha_m, \lambda\beta_1, \ldots, \lambda\beta_n) = \left(\sum_{i=1}^{m} \lambda\alpha_i b_i,\ \sum_{j=1}^{n} \lambda\beta_j c_j \right) = \lambda\Phi(u).$$

Hence Φ is linear.

Next we show that Φ is injective. Suppose

$$\Phi(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n) = (0_V, 0_W).$$

Then

$$\left(\sum_{i=1}^{m} \alpha_i b_i,\ \sum_{j=1}^{n} \beta_j c_j \right) = (0_V, 0_W),$$

so

$$\sum_{i=1}^{m} \alpha_i b_i = 0_V \qquad \text{and} \qquad \sum_{j=1}^{n} \beta_j c_j = 0_W.$$

Since B and C are bases, they are linearly independent. Therefore

$$\alpha_1 = \cdots = \alpha_m = 0 \qquad \text{and} \qquad \beta_1 = \cdots = \beta_n = 0.$$

Thus Φ is injective.

We now show that Φ is surjective. Let $(v, w) \in V \oplus_{\text{ext}} W$. Since B is a basis of V, there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$ such that

$$v = \sum_{i=1}^{m} \alpha_i b_i.$$

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Since C is a basis of W, there exist $\beta_1, \ldots, \beta_n \in \mathbb{F}$ such that

$$w = \sum_{j=1}^{n} \beta_j c_j.$$

Therefore

$$(v, w) = \left(\sum_{i=1}^{m} \alpha_i b_i,\ \sum_{j=1}^{n} \beta_j c_j \right) = \Phi(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n).$$

So Φ is surjective.

Hence Φ is an isomorphism. Let $e_1, \ldots, e_{m+n}$ be the standard basis of $\mathbb{F}^{m+n}$. Then

$$\Phi(e_1) = (b_1, 0_W),\ \ldots,\ \Phi(e_m) = (b_m, 0_W),$$

and

$$\Phi(e_{m+1}) = (0_V, c_1),\ \ldots,\ \Phi(e_{m+n}) = (0_V, c_n).$$

Since the image of a basis under an isomorphism is again a basis, it follows that

$$\mathcal{D} = \{(b_1, 0_W), \ldots, (b_m, 0_W), (0_V, c_1), \ldots, (0_V, c_n)\}$$

is a basis of $V \oplus_{\text{ext}} W$. Hence

$$\boxed{\mathcal{D} = \{(b_1, 0_W), \ldots, (b_m, 0_W), (0_V, c_1), \ldots, (0_V, c_n)\} \text{ is a basis of } V \oplus_{\text{ext}} W.}$$

(ii) Since $\Phi : \mathbb{F}^{m+n} \to V \oplus_{\text{ext}} W$ is an isomorphism, the two vector spaces have the same dimension. Therefore

$$\dim(V \oplus_{\text{ext}} W) = \dim(\mathbb{F}^{m+n}) = m + n.$$

Hence

$$\boxed{\dim(V \oplus_{\text{ext}} W) = m + n.}$$

Method 3: Internal direct sum of the coordinate subspaces

(i) Define

$$U := \{(v, 0_W) : v \in V\}, \qquad Z := \{(0_V, w) : w \in W\}.$$

We first show that U and Z are subspaces of $V \oplus_{\text{ext}} W$.

Let $(v, 0_W), (\tilde{v}, 0_W) \in U$ and let $\lambda \in \mathbb{F}$. Then

$$(v, 0_W) + (\tilde{v}, 0_W) = (v +_V \tilde{v},\ 0_W +_W 0_W) = (v +_V \tilde{v}, 0_W) \in U,$$

and

$$\lambda \cdot (v, 0_W) = (\lambda \cdot_V v,\ \lambda \cdot_W 0_W) = (\lambda \cdot_V v, 0_W) \in U.$$

Thus $U \leq V \oplus_{\text{ext}} W$.

Similarly, if $(0_V, w), (0_V, \tilde{w}) \in Z$ and $\lambda \in \mathbb{F}$, then

$$(0_V, w) + (0_V, \tilde{w}) = (0_V +_V 0_V,\ w +_W \tilde{w}) = (0_V, w +_W \tilde{w}) \in Z,$$

and

$$\lambda \cdot (0_V, w) = (\lambda \cdot_V 0_V,\ \lambda \cdot_W w) = (0_V, \lambda \cdot_W w) \in Z.$$

Hence $Z \leq V \oplus_{\text{ext}} W$.

Now let $(v, w) \in V \oplus_{\text{ext}} W$. Then

$$(v, w) = (v, 0_W) + (0_V, w),$$

where $(v, 0_W) \in U$ and $(0_V, w) \in Z$. Therefore

$$V \oplus_{\text{ext}} W = U + Z.$$

Next we show that

$$U \cap Z = \{(0_V, 0_W)\}.$$

Let $(x, y) \in U \cap Z$. Since $(x, y) \in U$, we must have $y = 0_W$. Since $(x, y) \in Z$, we must have $x = 0_V$. Hence

$$(x, y) = (0_V, 0_W).$$

So

$$U \cap Z = \{(0_V, 0_W)\}.$$

Therefore

$$V \oplus_{\text{ext}} W = U \oplus Z.$$

We now determine bases for U and Z. We claim that

$$\{(b_1, 0_W), \ldots, (b_m, 0_W)\}$$

is a basis of U. Indeed, if $(v, 0_W) \in U$, then since B is a basis of V, we may write

$$v = \sum_{i=1}^{m} \alpha_i b_i,$$

and hence

$$(v, 0_W) = \sum_{i=1}^{m} \alpha_i (b_i, 0_W).$$

So these vectors span U. If

$$\sum_{i=1}^{m} \alpha_i (b_i, 0_W) = (0_V, 0_W),$$

then

$$\sum_{i=1}^{m} \alpha_i b_i = 0_V.$$

Since B is linearly independent, all $\alpha_i = 0$. Hence

$$\{(b_1, 0_W), \ldots, (b_m, 0_W)\}$$

is a basis of U.

Similarly,

$$\{(0_V, c_1), \ldots, (0_V, c_n)\}$$

is a basis of Z.

Since $V \oplus_{\text{ext}} W = U \oplus Z$, the union of a basis of U and a basis of Z is a basis of $V \oplus_{\text{ext}} W$. Therefore

$$\mathcal{D} = \{(b_1, 0_W), \ldots, (b_m, 0_W), (0_V, c_1), \ldots, (0_V, c_n)\}$$

is a basis of $V \oplus_{\text{ext}} W$. Hence

$$\boxed{\mathcal{D} = \{(b_1, 0_W), \ldots, (b_m, 0_W), (0_V, c_1), \ldots, (0_V, c_n)\} \text{ is a basis of } V \oplus_{\text{ext}} W.}$$

(ii) Since

$$V \oplus_{\text{ext}} W = U \oplus Z,$$

and since U has basis $\{(b_1, 0_W), \ldots, (b_m, 0_W)\}$ while Z has basis $\{(0_V, c_1), \ldots, (0_V, c_n)\}$, the direct sum $U \oplus Z$ has a basis consisting of $m + n$ vectors. Therefore

$$\dim(V \oplus_{\text{ext}} W) = m + n.$$

Hence

$$\boxed{\dim(V \oplus_{\text{ext}} W) = m + n.}$$

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Mark Scheme (25 marks)

- **(a) (16 marks)**
  - (2) Operations are correctly used on $V \times W$.
  - (2) Additive associativity is verified.
  - (2) Additive commutativity is verified.
  - (2) An additive identity is exhibited and verified.
  - (2) Additive inverses are exhibited and verified.
  - (2) Scalar associativity $(\lambda\mu)x = \lambda(\mu x)$ is verified.
  - (2) Scalar identity $1x = x$ is verified.
  - (2) Both distributive laws are verified:

$$\lambda(x+y) = \lambda x + \lambda y, \qquad (\lambda+\mu)x = \lambda x + \mu x.$$

- **(b)(i) (7 marks)**
  - (3) Correctly constructs

$$\mathcal{D} = \{(b_1, 0_W), \ldots, (b_m, 0_W), (0_V, c_1), \ldots, (0_V, c_n)\}$$

    and proves that it spans $V \oplus_{\text{ext}} W$.
  - (4) Proves linear independence of $\mathcal{D}$.
- **(b)(ii) (2 marks)**
  - (2) Concludes that

$$\dim(V \oplus_{\text{ext}} W) = m + n$$

    because the constructed basis has $m + n$ vectors.

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