MH1201 Linear Algebra II

Tutorial 2 (Week 3) – Problems & Solutions

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Overview of This Tutorial Problem Set

This sheet develops standard tools for spans, linear independence, and structured subspaces:

- Direct sums and uniqueness of decomposition.
- Why span(S) is always a subspace (two fundamental viewpoints).
- Basic properties of span($\cdot$) (monotonicity, idempotence, and unions).
- Quick criteria for linear independence (especially for small sets).
- Polynomial subspaces defined by differential constraints.
- The subspace of symmetric matrices: basis construction and dimension counting.

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Question 1 (Direct sum criterion for n subspaces)

Problem

Let (i) $n \in \mathbb{N}$, (ii) V be a vector space, and (iii) $V_1, \ldots, V_n$ be vector subspaces of V. Show that the following two statements are equivalent (i.e. imply one another):

(a) $V_1 + \cdots + V_n = V_1 \oplus \cdots \oplus V_n$.

(b) If $v_1 \in V_1, \ldots, v_n \in V_n$ satisfy $0 = v_1 + \cdots + v_n$, then $v_1 = \cdots = v_n = 0$.

Solution

Method 1: Uniqueness of representation

Recall that $V_1 \oplus \cdots \oplus V_n$ is (by definition) the sum $V_1 + \cdots + V_n$ together with the additional requirement that every element of the sum has a *unique* representation as $v_1 + \cdots + v_n$ with $v_i \in V_i$.

(a) $\Rightarrow$ (b). Assume $V_1 + \cdots + V_n = V_1 \oplus \cdots \oplus V_n$. Let $v_i \in V_i$ satisfy

$$0 = v_1 + \cdots + v_n.$$

Also note that $0 = 0 + \cdots + 0$ with each $0 \in V_i$. Thus we have two decompositions of the same vector 0 as a sum of vectors from $V_1, \ldots, V_n$:

$$0 = v_1 + \cdots + v_n \quad \text{and} \quad 0 = 0 + \cdots + 0.$$

By uniqueness of decomposition in the direct sum, we must have $v_i = 0$ for every i. Hence (b) holds.

(b) $\Rightarrow$ (a). Assume (b). First, $V_1 + \cdots + V_n$ is already defined as the set of all sums $v_1 + \cdots + v_n$ with $v_i \in V_i$. So to show

$$V_1 + \cdots + V_n = V_1 \oplus \cdots \oplus V_n,$$

it suffices to show that every vector in $V_1 + \cdots + V_n$ has a *unique* such decomposition.

Let $x \in V_1 + \cdots + V_n$. Suppose x has two decompositions:

$$x = v_1 + \cdots + v_n \quad \text{and} \quad x = w_1 + \cdots + w_n,$$

with $v_i, w_i \in V_i$. Subtract the two equations:

$$0 = (v_1 - w_1) + \cdots + (v_n - w_n).$$

Since each V_i is a subspace, $v_i - w_i \in V_i$. Applying (b) to the relation above yields

$$v_1 - w_1 = \cdots = v_n - w_n = 0 \quad \Longrightarrow \quad v_i = w_i \text{ for all } i.$$

Thus the decomposition of x is unique, which is exactly the direct-sum property. Therefore (a) holds.

Hence the statements are equivalent, and the final conclusion is

$$\boxed{(a) \Longleftrightarrow (b).}$$

Method 2: Linear map with trivial kernel

Define the linear map

$$T : V_1 \times \cdots \times V_n \longrightarrow V, \qquad T(v_1, \ldots, v_n) \coloneqq v_1 + \cdots + v_n.$$

Linearity follows from pointwise addition and scalar multiplication in the product space:

$$\begin{aligned} T\big((v_1, \ldots, v_n) + (w_1, \ldots, w_n)\big) &= T(v_1 + w_1, \ldots, v_n + w_n) \\ &= (v_1 + w_1) + \cdots + (v_n + w_n) \\ &= (v_1 + \cdots + v_n) + (w_1 + \cdots + w_n) \\ &= T(v_1, \ldots, v_n) + T(w_1, \ldots, w_n), \end{aligned}$$

and similarly $T(\lambda v_1, \ldots, \lambda v_n) = \lambda T(v_1, \ldots, v_n)$.

Observe:

$$\operatorname{Im}(T) = V_1 + \cdots + V_n.$$

Also,

$$\ker(T) = \{(v_1, \ldots, v_n) \in V_1 \times \cdots \times V_n \mid v_1 + \cdots + v_n = 0\}.$$

Thus statement (b) is precisely:

$$\ker(T) = \{(0, \ldots, 0)\}.$$

Now, $V_1 + \cdots + V_n$ is a direct sum (i.e. equals $V_1 \oplus \cdots \oplus V_n$) exactly when each $x \in V_1 + \cdots + V_n$ has a *unique* preimage under T, i.e. when T is injective. But a linear map is injective if and only if its kernel is trivial. Therefore (a) holds if and only if (b) holds.

So again,

$$\boxed{(a) \iff (b).}$$

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Question 2 (Span is a subspace)

Problem

Let (i) V be a vector space, and (ii) S be a set of V-vectors (not necessarily finite). Show that $\operatorname{span}(S)$ is a vector subspace of V.

Solution

Method 1: Closure using finite linear combinations

By definition,

$$\operatorname{span}(S) = \left\{ \sum_{k=1}^{m} \alpha_k s_k \;\middle|\; m \in \mathbb{N},\ \alpha_k \in \mathbb{R},\ s_k \in S \right\}.$$

(Here each sum is a *finite* linear combination.)

Zero vector. Take $m = 1$, $s_1 \in S$ arbitrary (if $S = \varnothing$, then $\operatorname{span}(S) = \{0\}$ by convention; the argument below still leads to the same conclusion), and $\alpha_1 = 0$. Then $0 \cdot s_1 = 0 \in \operatorname{span}(S)$.

Closure under addition. Let $x, y \in \operatorname{span}(S)$. Then there exist $m, \ell \in \mathbb{N}$, scalars $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_\ell \in \mathbb{R}$, and vectors $s_1, \ldots, s_m, t_1, \ldots, t_\ell \in S$ such that

$$x = \sum_{i=1}^{m} \alpha_i s_i, \qquad y = \sum_{j=1}^{\ell} \beta_j t_j.$$

Then

$$x + y = \sum_{i=1}^{m} \alpha_i s_i + \sum_{j=1}^{\ell} \beta_j t_j$$

is again a finite linear combination of vectors from S, hence $x + y \in \operatorname{span}(S)$.

Closure under scalar multiplication. Let $\lambda \in \mathbb{R}$ and $x \in \operatorname{span}(S)$ with $x = \sum_{i=1}^{m} \alpha_i s_i$. Then

$$\lambda x = \sum_{i=1}^{m} (\lambda \alpha_i) s_i$$

is again a finite linear combination of vectors from S, hence $\lambda x \in \operatorname{span}(S)$.

Therefore $\operatorname{span}(S)$ is a subspace, and the final answer is

$$\boxed{\operatorname{span}(S) \text{ is a vector subspace of } V.}$$

Method 2: Span as an intersection of subspaces

Let $\mathcal{F}$ be the collection of all subspaces of V that contain S:

$$\mathcal{F} \coloneqq \{\, W \leq V \mid S \subseteq W \,\}.$$

Define

$$U \coloneqq \bigcap_{W \in \mathcal{F}} W.$$

Claim 1: U is a subspace of V. Take $x, y \in U$. Then for every $W \in \mathcal{F}$, we have $x \in W$ and $y \in W$. Because each W is a subspace, $x + y \in W$ and $\lambda x \in W$ for every scalar λ. Thus $x + y \in U$ and $\lambda x \in U$, and $0 \in U$ as well. So U is a subspace.

Claim 2: $U = \operatorname{span}(S)$. First, $\operatorname{span}(S)$ is a subspace that contains S, hence $\operatorname{span}(S) \in \mathcal{F}$. Therefore, by definition of intersection,

$$U = \bigcap_{W \in \mathcal{F}} W \subseteq \operatorname{span}(S).$$

Conversely, let $x \in \operatorname{span}(S)$. Then x is a finite linear combination of elements of S. If $W \in \mathcal{F}$, then $S \subseteq W$, so W contains each element of S and hence contains all finite linear combinations of elements of S, i.e. it contains $\operatorname{span}(S)$. Thus $x \in W$ for every $W \in \mathcal{F}$, so $x \in U$. Hence $\operatorname{span}(S) \subseteq U$.

So $U = \operatorname{span}(S)$, and since U is a subspace, $\operatorname{span}(S)$ is a subspace. Therefore,

$$\boxed{\operatorname{span}(S) \text{ is a vector subspace of } V.}$$

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Question 3 (Basic span identities)

Problem

Let (i) V be a vector space, and (ii) A and B be sets of V-vectors (not necessarily finite). Establish the following four statements:

(a) $A \subseteq \text{span}(A)$.

(b) If $A \subseteq B$, then $\text{span}(A) \subseteq \text{span}(B)$.

(c) $\text{span}(A \cup B) = \text{span}(A) + \text{span}(B)$.

(d) $\text{span}(\text{span}(A)) = \text{span}(A)$.

Solution

Method 1: Directly from finite linear combinations

(a) Take any $a \in A$. Then $a = 1 \cdot a$, which is a finite linear combination of elements of A. Hence $a \in \text{span}(A)$, so $A \subseteq \text{span}(A)$.

(b) Assume $A \subseteq B$. Take $x \in \text{span}(A)$. Then for some $m \in \mathbb{N}$,

$$x = \sum_{k=1}^{m} \alpha_k a_k \quad \text{with} \quad \alpha_k \in \mathbb{R},\ a_k \in A.$$

Since $A \subseteq B$, each $a_k \in B$, so the same expression shows $x \in \text{span}(B)$. Thus $\text{span}(A) \subseteq \text{span}(B)$.

(c) We prove both inclusions.

($\subseteq$) Take $x \in \text{span}(A \cup B)$. Then

$$x = \sum_{k=1}^{m} \alpha_k s_k \quad \text{for some } s_k \in A \cup B.$$

Split the index set $\{1, \ldots, m\}$ into those k with $s_k \in A$ and those with $s_k \in B$. Write

$$x = \sum_{s_k \in A} \alpha_k s_k \ + \sum_{s_k \in B} \alpha_k s_k.$$

Define

$$x_A \coloneqq \sum_{s_k \in A} \alpha_k s_k \in \text{span}(A), \qquad x_B \coloneqq \sum_{s_k \in B} \alpha_k s_k \in \text{span}(B),$$

(with the convention that an empty sum equals 0). Then $x = x_A + x_B \in \text{span}(A) + \text{span}(B)$. Hence $\text{span}(A \cup B) \subseteq \text{span}(A) + \text{span}(B)$.

Unofficial Question Solutions

($\supseteq$) Take $x \in \text{span}(A) + \text{span}(B)$. Then $x = u + v$ for some $u \in \text{span}(A)$ and $v \in \text{span}(B)$. Write

$$u = \sum_{i=1}^{m} \alpha_i a_i, \quad a_i \in A, \qquad v = \sum_{j=1}^{\ell} \beta_j b_j, \quad b_j \in B.$$

Then

$$x = u + v = \sum_{i=1}^{m} \alpha_i a_i + \sum_{j=1}^{\ell} \beta_j b_j,$$

which is a finite linear combination of vectors from $A \cup B$. Thus $x \in \text{span}(A \cup B)$, so $\text{span}(A) + \text{span}(B) \subseteq \text{span}(A \cup B)$.

Therefore $\text{span}(A \cup B) = \text{span}(A) + \text{span}(B)$.

(d) Since $\text{span}(A)$ is a subspace, the span of $\text{span}(A)$ should give nothing new. Formally, apply (a) and (b) twice.

First, by (a) with A replaced by $\text{span}(A)$,

$$\text{span}(A) \subseteq \text{span}(\text{span}(A)).$$

Second, because $A \subseteq \text{span}(A)$ by (a), apply (b) to get

$$\text{span}(A) \subseteq \text{span}(\text{span}(A)) \quad \text{(already obtained)},$$

and also, applying (b) with the inclusion $\text{span}(A) \subseteq \text{span}(A)$ is trivial.

To get the reverse inclusion, use the fact that $\text{span}(A)$ is a subspace containing A. Any element of $\text{span}(\text{span}(A))$ is a finite linear combination of elements of $\text{span}(A)$, and since $\text{span}(A)$ is closed under linear combinations, that combination lies in $\text{span}(A)$. Thus $\text{span}(\text{span}(A)) \subseteq \text{span}(A)$.

Combining both inclusions gives $\text{span}(\text{span}(A)) = \text{span}(A)$.

Therefore the final results are

(a) $A \subseteq \text{span}(A)$, (b) $A \subseteq B \Rightarrow \text{span}(A) \subseteq \text{span}(B)$,
(c) $\text{span}(A \cup B) = \text{span}(A) + \text{span}(B)$, (d) $\text{span}(\text{span}(A)) = \text{span}(A)$.

Method 2: Use the "intersection of all containing subspaces" characterisation

Recall the characterisation:

$$\text{span}(S) = \bigcap \{\, W \leq V \mid S \subseteq W \,\}.$$

Let $\mathcal{W}_A \coloneqq \{W \leq V \mid A \subseteq W\}$ and similarly $\mathcal{W}_B$.

(a) Every subspace W with $A \subseteq W$ contains A. Hence A is contained in the intersection of all such W, i.e. $A \subseteq \text{span}(A)$.

(b) If $A \subseteq B$, then any subspace that contains B also contains A. So $\mathcal{W}_B \subseteq \mathcal{W}_A$. Taking intersections reverses inclusions of index families:

$$\operatorname{span}(A) = \bigcap_{W \in \mathcal{W}_A} W \subseteq \bigcap_{W \in \mathcal{W}_B} W = \operatorname{span}(B).$$

(c) Let $U \coloneqq \operatorname{span}(A) + \operatorname{span}(B)$. Since $\operatorname{span}(A)$ and $\operatorname{span}(B)$ are subspaces, U is a subspace of V. Also $A \subseteq \operatorname{span}(A) \subseteq U$ and $B \subseteq \operatorname{span}(B) \subseteq U$, so $A \cup B \subseteq U$. Therefore, by minimality of span,

$$\operatorname{span}(A \cup B) \subseteq U = \operatorname{span}(A) + \operatorname{span}(B).$$

Conversely, $\operatorname{span}(A \cup B)$ is a subspace containing $A \cup B$, hence it contains A and contains B. Thus $\operatorname{span}(A) \subseteq \operatorname{span}(A \cup B)$ and $\operatorname{span}(B) \subseteq \operatorname{span}(A \cup B)$ by monotonicity (part (b)). Adding these inclusions gives

$$\operatorname{span}(A) + \operatorname{span}(B) \subseteq \operatorname{span}(A \cup B).$$

So equality holds.

(d) Since $\operatorname{span}(A)$ is already a subspace containing A, it is one of the subspaces being intersected to form $\operatorname{span}(A)$. In particular, $\operatorname{span}(A) \subseteq \operatorname{span}(\operatorname{span}(A))$ from (a).

On the other hand, $\operatorname{span}(\operatorname{span}(A))$ is the smallest subspace containing $\operatorname{span}(A)$, but $\operatorname{span}(A)$ itself is a subspace containing $\operatorname{span}(A)$. Therefore the smallest such subspace must be $\operatorname{span}(A)$ itself, i.e. $\operatorname{span}(\operatorname{span}(A)) \subseteq \operatorname{span}(A)$.

Thus $\operatorname{span}(\operatorname{span}(A)) = \operatorname{span}(A)$.

So the same boxed conclusions follow:

(a) $A \subseteq \operatorname{span}(A)$,	(b) $A \subseteq B \Rightarrow \operatorname{span}(A) \subseteq \operatorname{span}(B)$,
(c) $\operatorname{span}(A \cup B) = \operatorname{span}(A) + \operatorname{span}(B)$,	(d) $\operatorname{span}(\operatorname{span}(A)) = \operatorname{span}(A)$.

Question 4 (0 in a set prevents linear independence)

Problem

Let (i) V be a vector space, and (ii) S be a set of V-vectors (not necessarily finite). Show that if $0 \in S$, then S cannot be linearly independent.

Solution

Method 1: Directly from the definition

Assume $0 \in S$. Consider the finite subset $\{0\} \subseteq S$. Take the (nontrivial) scalar choice $\alpha = 1$. Then

$$\alpha \cdot 0 = 1 \cdot 0 = 0.$$

This is a linear combination of elements of S that equals 0 but uses a coefficient $\alpha \neq 0$. Therefore S is linearly dependent.

Hence,

$$\boxed{0 \in S \implies S \text{ is linearly dependent (not linearly independent).}}$$

Method 2: "Unique representation" viewpoint

If S were linearly independent and contained 0, then the vector $0 \in V$ would have at least two distinct representations as a linear combination of vectors in S:

$$0 = 0 \quad \text{(the empty/zero combination)} \qquad \text{and} \qquad 0 = 1 \cdot 0.$$

The second representation uses a nonzero coefficient and a vector from S, so it is genuinely different from the trivial representation. But linear independence exactly forbids a nontrivial linear combination equalling 0. Thus S cannot be linearly independent.

So again,

$$\boxed{0 \in S \implies S \text{ is not linearly independent.}}$$

Question 5 (Two-vector linear independence)

Problem

Let (i) V be a vector space, and (ii) u and v be distinct V-vectors. Show that the following two statements are equivalent:

(a) $\{u, v\}$ is linearly independent.

(b) u is not a scalar multiple of v, and v is not a scalar multiple of u.

Solution

Method 1: Solve the linear dependence equation

($\Rightarrow$). Assume $\{u, v\}$ is linearly independent. Suppose, for contradiction, that u is a scalar multiple of v, i.e. $u = \lambda v$ for some $\lambda \in \mathbb{R}$. Then

$$u - \lambda v = 0$$

is a linear combination of u and v with coefficients 1 and $-\lambda$, not both zero (since the coefficient of u is 1). This contradicts linear independence. So u is not a scalar multiple of v. The same argument (swapping u and v) shows v is not a scalar multiple of u.

($\Leftarrow$). Assume u is not a scalar multiple of v and v is not a scalar multiple of u. To prove $\{u, v\}$ is linearly independent, let $\alpha, \beta \in \mathbb{R}$ satisfy

$$\alpha u + \beta v = 0.$$

If $\alpha = 0$, then $\beta v = 0$, so $\beta = 0$ (because $v \neq 0$; indeed if $v = 0$, then $u \neq v$ would force $u \neq 0$, and $\{u, 0\}$ is certainly dependent; the condition "not a scalar multiple" would also fail since $0 = 0 \cdot u$). So in the meaningful case, $\alpha \neq 0$. Then we can rearrange:

$$u = -\frac{\beta}{\alpha} v,$$

which says u is a scalar multiple of v, contradicting the assumption. Therefore $\alpha \neq 0$ is impossible, so $\alpha = 0$, and then $\beta = 0$. Thus the only solution is $\alpha = \beta = 0$, proving linear independence.

Hence,

$$\boxed{(a) \Longleftrightarrow (b).}$$

Method 2: Span/dimension viewpoint for two vectors

($\Rightarrow$). If $\{u, v\}$ is linearly independent, then $v \notin \text{span}\{u\}$ and $u \notin \text{span}\{v\}$. But $\text{span}\{u\} = \{\lambda u : \lambda \in \mathbb{R}\}$ is exactly the set of scalar multiples of u, and similarly for v. So neither vector is a scalar multiple of the other.

($\Leftarrow$). Assume neither is a scalar multiple of the other. Then $v \notin \text{span}\{u\}$. If $\{u, v\}$ were dependent, then one of them would lie in the span of the other (for two vectors, dependence

means one is a scalar multiple of the other), which contradicts the assumption. Therefore $\{u, v\}$ must be independent.

Thus again,

$$\boxed{(a) \Longleftrightarrow (b).}$$

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Question 6 (Polynomial subspace defined by $P''' = P''$)

Problem

Define a vector subspace V of $\mathcal{P}_3(\mathbb{R})$ as follows:

$$V \coloneqq \{P \in \mathcal{P}_3(\mathbb{R}) \mid P''' = P''\}.$$

Find a basis for V and hence determine $\dim(V)$.

Solution

Method 1: Coefficient comparison in the monomial basis

Let

$$P(t) = at^3 + bt^2 + ct + d \in \mathcal{P}_3(\mathbb{R}).$$

Compute derivatives:

$$P'(t) = 3at^2 + 2bt + c, \qquad P''(t) = 6at + 2b, \qquad P'''(t) = 6a.$$

The condition $P''' = P''$ means equality as polynomials:

$$6a \equiv 6at + 2b \quad \text{for all } t \in \mathbb{R}.$$

Comparing coefficients of t gives

$$6a = 0 \quad \Longrightarrow \quad a = 0.$$

Then the constant term condition becomes

$$6a = 2b \quad \Longrightarrow \quad 0 = 2b \quad \Longrightarrow \quad b = 0.$$

So the only polynomials satisfying $P''' = P''$ are

$$P(t) = ct + d,$$

i.e. all polynomials of degree at most 1. Therefore

$$V = \operatorname{span}\{1, t\}.$$

A basis is $\{1, t\}$, and the dimension is 2:

$$\boxed{\text{A basis of } V \text{ is } \{1, t\}, \quad \dim(V) = 2.}$$

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Method 2: Reduce to a differential equation for P''

Let $Q(t) \coloneqq P''(t)$. Then Q is a polynomial of degree at most 1 (since $P \in \mathcal{P}_3(\mathbb{R})$), and the condition $P''' = P''$ becomes

$$Q'(t) = Q(t) \quad \text{for all } t \in \mathbb{R}.$$

Write the most general degree-≤ 1 polynomial as

$$Q(t) = \alpha t + \beta.$$

Then $Q'(t) = \alpha$. So $Q' = Q$ becomes

$$\alpha = \alpha t + \beta \quad \text{for all } t \in \mathbb{R}.$$

Comparing coefficients yields $\alpha = 0$ and $\beta = \alpha = 0$, so $Q(t) \equiv 0$. Thus

$$P''(t) = 0 \quad \text{for all } t.$$

Integrate twice:

$$P'(t) = C_1, \qquad P(t) = C_1 t + C_0,$$

for constants $C_0, C_1 \in \mathbb{R}$. Hence P is linear, so $V = \operatorname{span}\{1, t\}$ and

$$\boxed{\text{A basis of } V \text{ is } \{1, t\}, \;\; \dim(V) = 2.}$$

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Question 7 (Symmetric matrices in $\mathbb{R}^{n,n}$)

Problem

Let $n \in \mathbb{N}$, and let V denote the set of symmetric $n \times n$ matrices with real entries, i.e.,

$$V = \{M \in \mathbb{R}^{n,n} \mid M = M^\top\}.$$

(a) Show that V is a vector subspace of $\mathbb{R}^{n,n}$.

(b) Find a basis for V and hence determine $\dim(V)$.

Solution

Method 1: Entrywise verification and explicit basis construction

(a) **Zero matrix.** The zero matrix satisfies $\mathbf{0}^\top = \mathbf{0}$, so $\mathbf{0} \in V$.

Closure under addition. If $A, B \in V$, then $A^\top = A$ and $B^\top = B$, hence

$$(A+B)^\top = A^\top + B^\top = A + B,$$

so $A + B \in V$.

Closure under scalar multiplication. If $A \in V$ and $\lambda \in \mathbb{R}$, then

$$(\lambda A)^\top = \lambda A^\top = \lambda A,$$

so $\lambda A \in V$.

Therefore V is a vector subspace of $\mathbb{R}^{n,n}$.

(b) For $1 \le i \le n$, let E_{ii} denote the matrix with a 1 in entry (i,i) and zeros elsewhere. For $1 \le i < j \le n$, define

$$F_{ij} \coloneqq E_{ij} + E_{ji},$$

where E_{ij} has a 1 in entry (i,j) and zeros elsewhere. Each E_{ii} is symmetric, and each F_{ij} is symmetric.

Spanning. Let $A = (a_{ij}) \in V$. Since A is symmetric, $a_{ij} = a_{ji}$. Then

$$A = \sum_{i=1}^{n} a_{ii} E_{ii} + \sum_{1 \le i < j \le n} a_{ij} F_{ij}.$$

So the set

$$\mathcal{B} \coloneqq \{E_{11}, \ldots, E_{nn}\} \cup \{F_{ij} : 1 \le i < j \le n\}$$

spans V.

Linear independence. Suppose

$$\sum_{i=1}^{n} \alpha_{ii} E_{ii} + \sum_{1 \le i < j \le n} \beta_{ij} F_{ij} = 0.$$

Look at diagonal entry (k,k). Only E_{kk} contributes there, so $\alpha_{kk} = 0$ for each k. Now look at an off-diagonal entry (i,j) with $i < j$. Only F_{ij} contributes there (with coefficient 1), so $\beta_{ij} = 0$ for each $i < j$. Thus all coefficients are zero, so $\mathcal{B}$ is linearly independent.

Therefore $\mathcal{B}$ is a basis. Its size is

$$n \ + \ \binom{n}{2} \ = \ n + \frac{n(n-1)}{2} \ = \ \frac{n(n+1)}{2}.$$

Hence

$$\boxed{\dim(V) = \frac{n(n+1)}{2} \ \text{ and a basis is } \{E_{ii}\}_{i=1}^{n} \cup \{E_{ij} + E_{ji}\}_{1 \le i < j \le n}.}$$

Method 2: Linear-map kernel and decomposition with skew-symmetric matrices

(a) Define a linear map $T : \mathbb{R}^{n,n} \to \mathbb{R}^{n,n}$ by

$$T(M) \coloneqq M - M^{\top}.$$

Linearity holds since transpose is linear:

$$T(A+B) = (A+B) - (A+B)^{\top} = (A - A^{\top}) + (B - B^{\top}) = T(A) + T(B),$$

and $T(\lambda A) = \lambda T(A)$.

Now observe:

$$\ker(T) = \{M \in \mathbb{R}^{n,n} \mid M - M^{\top} = 0\} = \{M \in \mathbb{R}^{n,n} \mid M = M^{\top}\} = V.$$

A kernel of a linear map is a subspace, hence V is a subspace.

(b) Let W be the subspace of skew-symmetric matrices:

$$W \coloneqq \{M \in \mathbb{R}^{n,n} \mid M^{\top} = -M\}.$$

Every matrix $M \in \mathbb{R}^{n,n}$ decomposes as

$$M = \underbrace{\frac{M + M^{\top}}{2}}_{\text{symmetric}} + \underbrace{\frac{M - M^{\top}}{2}}_{\text{skew-symmetric}},$$

and these two parts lie in V and W, respectively.

Moreover, $V \cap W = \{0\}$, because if $M \in V \cap W$, then $M = M^{\top}$ and $M = -M^{\top}$, so $M = -M$, hence $2M = 0$, so $M = 0$. Therefore,

$$\mathbb{R}^{n,n} = V \oplus W.$$

Taking dimensions,

$$n^2 = \dim(\mathbb{R}^{n,n}) = \dim(V) + \dim(W).$$

Now compute $\dim(W)$: a skew-symmetric matrix has zeros on the diagonal and satisfies $m_{ji} = -m_{ij}$, so the free parameters are exactly the entries above the diagonal, i.e. $\binom{n}{2} = \frac{n(n-1)}{2}$ degrees of freedom. Thus

$$\dim(W) = \frac{n(n-1)}{2},$$

and therefore

$$\dim(V) = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

This agrees with the explicit basis count in Method 1. Hence,

$$\boxed{\dim(V) = \frac{n(n+1)}{2}.}$$

(An explicit basis is exactly the one constructed in Method 1.)

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