MH1201 Linear Algebra II

Tutorial 5 (Week 6) – Problems & Solutions

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Overview

This tutorial emphasizes:

- vector space structure on spaces of linear maps $\mathcal{L}(V, W)$ and the matrix space $M_{n\times m}(\mathbb{F})$;
- linearity of the "matrix-of-a-map" operator and coordinate representations;
- computing standard matrices, ranges, kernels, ranks, nullities, and (non-)direct sum decompositions;
- degree properties of polynomials and linearity of substitution operators;
- composition: how injectivity/surjectivity of $T \circ S$ constrains S and T, and sharp counterexamples.

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Question 1 (Vector space structure of $\mathcal{L}(V, W)$)

Problem

Let V and W be vector spaces over a field $\mathbb{F}$. Show that $\mathcal{L}(V, W)$ is a vector space over $\mathbb{F}$ when equipped with

- pointwise addition: $(S + T)(v) \coloneqq S(v) + T(v)$,
- pointwise scalar multiplication: $(aT)(v) \coloneqq a\,T(v)$,

for all $S, T \in \mathcal{L}(V, W)$, $a \in \mathbb{F}$, and $v \in V$.

Remark: You are requested to verify all eight axioms for a vector space.

Solution

Method 1: Verify the axioms directly (pointwise)

We first check *closure* (so the operations are well-defined on $\mathcal{L}(V, W)$), then verify the axioms.

Step 0: Closure under addition and scalar multiplication. Let $S, T \in \mathcal{L}(V, W)$ and $a \in \mathbb{F}$. We must show $S + T \in \mathcal{L}(V, W)$ and $aT \in \mathcal{L}(V, W)$.

- For additivity: for all $u, v \in V$,

  $$(S+T)(u+v) = S(u+v)+T(u+v) = (S(u)+S(v))+(T(u)+T(v)) = (S+T)(u)+(S+T)(v),$$

  using linearity of S, T and associativity/commutativity of addition in W. Also, for all $c \in \mathbb{F}$ and $u \in V$,

  $$(S + T)(cu) = S(cu) + T(cu) = cS(u) + cT(u) = c(S(u) + T(u)) = c(S + T)(u).$$

  Thus $S + T$ is linear.

- For scalar multiplication: for all $u, v \in V$,

  $$(aT)(u + v) = a\,T(u + v) = a(T(u) + T(v)) = aT(u) + aT(v) = (aT)(u) + (aT)(v),$$

  and for all $c \in \mathbb{F}$,

  $$(aT)(cu) = a\,T(cu) = a(cT(u)) = (ac)T(u) = c(aT)(u),$$

  using associativity and commutativity of scalar multiplication in the field. Thus aT is linear.

  Hence the operations stay inside $\mathcal{L}(V, W)$.

Step 1: Verify the vector space axioms. Let $R, S, T \in \mathcal{L}(V, W)$ and $a, b \in \mathbb{F}$. Each axiom will be shown by evaluating at an arbitrary $v \in V$.

(A1) Commutativity of addition: For all $v \in V$,

$$(S + T)(v) = S(v) + T(v) = T(v) + S(v) = (T + S)(v),$$

so $S + T = T + S$.

(A2) Associativity of addition: For all $v \in V$,

$$\begin{aligned}((R+S)+T)(v) &= (R+S)(v)+T(v)\\ &= (R(v)+S(v))+T(v)\\ &= R(v)+(S(v)+T(v))\\ &= (R+(S+T))(v).\end{aligned}$$

So $(R+S)+T = R+(S+T)$.

(A3) Additive identity: Define the *zero map* $0 \in \mathcal{L}(V,W)$ by $0(v) \coloneqq 0_W$ for all $v \in V$. It is linear because $0(u+v) = 0_W = 0_W + 0_W = 0(u)+0(v)$ and $0(cv) = 0_W = c0_W = c0(v)$. Moreover, for all v,

$$(S+0)(v) = S(v)+0(v) = S(v)+0_W = S(v),$$

so $S+0 = S$.

(A4) Additive inverses: For each $S \in \mathcal{L}(V,W)$ define $(-S)(v) \coloneqq -S(v)$. This is linear since

$$(-S)(u+v) = -S(u+v) = -(S(u)+S(v)) = (-S(u))+(-S(v)),$$

and $(-S)(cv) = -S(cv) = -cS(v) = c(-S)(v)$. Also, for all v,

$$(S+(-S))(v) = S(v)+(-S(v)) = 0_W = 0(v),$$

so $S+(-S) = 0$.

(A5) Compatibility with scalar multiplication (associativity): For all v,

$$((ab)S)(v) = (ab)S(v) = a(bS(v)) = (a(bS))(v),$$

so $(ab)S = a(bS)$.

(A6) Identity scalar: For all v,

$$(1S)(v) = 1 \cdot S(v) = S(v),$$

so $1S = S$.

(A7) Distributivity over vector addition: For all v,

$$(a(S+T))(v) = a(S(v)+T(v)) = aS(v)+aT(v) = ((aS)+(aT))(v),$$

so $a(S+T) = aS+aT$.

(A8) Distributivity over scalar addition: For all v,

$$((a+b)S)(v) = (a+b)S(v) = aS(v)+bS(v) = ((aS)+(bS))(v),$$

so $(a+b)S = aS+bS$.

All required axioms hold. Therefore $\mathcal{L}(V,W)$ is a vector space over $\mathbb{F}$ with these operations.

$\boxed{\mathcal{L}(V,W) \text{ is a vector space over } \mathbb{F} \text{ under pointwise operations.}}$

Method 2: View $\mathcal{L}(V,W)$ as a subspace of the function space W^V

Let W^V denote the set of all functions $f : V \to W$. Define pointwise operations on W^V by

$$(f+g)(v) = f(v) + g(v), \qquad (af)(v) = af(v).$$

One checks the vector space axioms on W^V exactly as in Method 1, using only that W is a vector space.

Now observe:

- the zero function $0(v) = 0_W$ is linear, hence $0 \in \mathcal{L}(V,W)$;
- if $S, T \in \mathcal{L}(V,W)$ then $S+T$ is linear (closure under addition, shown in Method 1);
- if $T \in \mathcal{L}(V,W)$ and $a \in \mathbb{F}$ then aT is linear (closure under scalar multiplication, shown in Method 1).

Thus $\mathcal{L}(V,W)$ is a *subspace* of W^V. Hence it inherits the vector space axioms from W^V.

$$\boxed{\mathcal{L}(V,W) \leq W^V \text{ (a subspace), hence a vector space.}}$$

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Question 2 (Linearity of the "matrix of T" map)

Problem

Let V and W be finite-dimensional vector spaces over a field $\mathbb{F}$. Let α and β be ordered bases for V and W, respectively. Define a function

$$M : \mathcal{L}(V, W) \to \mathbb{F}^{\dim(W),\dim(V)}$$

by

$$\forall T \in \mathcal{L}(V, W) : \qquad M(T) \coloneqq [T]_\beta^\alpha,$$

the matrix of T with respect to α (domain basis) and β (codomain basis). Show that

$$M \in \mathcal{L}\big(\mathcal{L}(V, W),\ \mathbb{F}^{\dim(W),\dim(V)}\big).$$

Solution

Method 1: Use columns $[T(v_i)]_\beta$ and check additivity/homogeneity

Let $\alpha = (v_1, \ldots, v_m)$ with $m = \dim(V)$ and $\beta = (w_1, \ldots, w_n)$ with $n = \dim(W)$. By definition, the matrix $[T]_\beta^\alpha \in \mathbb{F}^{n\times m}$ has ith column equal to the coordinate vector

$$\big[T(v_i)\big]_\beta \in \mathbb{F}^n.$$

Let $S, T \in \mathcal{L}(V, W)$ and $a \in \mathbb{F}$.

Additivity. For each $i \in \{1, \ldots, m\}$,

$$[(S + T)(v_i)]_\beta = [S(v_i) + T(v_i)]_\beta = [S(v_i)]_\beta + [T(v_i)]_\beta,$$

since the coordinate map $[\,\cdot\,]_\beta : W \to \mathbb{F}^n$ is linear. Thus, column-by-column,

$$[S + T]_\beta^\alpha = [S]_\beta^\alpha + [T]_\beta^\alpha,$$

so

$$M(S + T) = M(S) + M(T).$$

Homogeneity. Similarly, for each i,

$$[(aT)(v_i)]_\beta = [aT(v_i)]_\beta = a[T(v_i)]_\beta.$$

Hence

$$[aT]_\beta^\alpha = a[T]_\beta^\alpha,$$

so

$$M(aT) = aM(T).$$

Therefore M is linear:

$$\boxed{M \in \mathcal{L}(\mathcal{L}(V, W), \mathbb{F}^{n\times m})}.$$

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Method 2: Using Coordinate Isomorphisms (Concise Version)

Let α and β be ordered bases of V and W with $\dim V = m$ and $\dim W = n$.

Define the coordinate isomorphisms

$$C_\alpha : V \to \mathbb{F}^m, \qquad C_\beta : W \to \mathbb{F}^n$$

by

$$C_\alpha(v) = [v]_\alpha, \qquad C_\beta(w) = [w]_\beta.$$

Both maps are linear isomorphisms.

For $T \in \mathcal{L}(V, W)$, define

$$\widetilde{T} \coloneqq C_\beta \circ T \circ C_\alpha^{-1} : \mathbb{F}^m \to \mathbb{F}^n.$$

Then $\widetilde{T}$ is linear, and by definition of matrix representation,

$$M(T) = [T]_\beta^\alpha = [\widetilde{T}].$$

Let $S, T \in \mathcal{L}(V, W)$ and $a \in \mathbb{F}$.

Additivity. Using distributivity of composition over addition,

$$\widetilde{S+T} = C_\beta(S+T)C_\alpha^{-1} = C_\beta S C_\alpha^{-1} + C_\beta T C_\alpha^{-1} = \widetilde{S} + \widetilde{T}.$$

Taking standard matrices gives

$$M(S+T) = [\widetilde{S+T}] = [\widetilde{S} + \widetilde{T}] = [\widetilde{S}] + [\widetilde{T}] = M(S) + M(T).$$

Homogeneity. Similarly,

$$\widetilde{aT} = C_\beta(aT)C_\alpha^{-1} = a(C_\beta T C_\alpha^{-1}) = a\widetilde{T}.$$

Hence

$$M(aT) = [\widetilde{aT}] = [a\widetilde{T}] = a[\widetilde{T}] = aM(T).$$

Therefore M is linear, i.e.

$$\boxed{M \in \mathcal{L}\big(\mathcal{L}(V, W), \mathbb{F}^{n\times m}\big).}$$

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Question 3 (A linear map on $\mathbb{R}^{2,2}$ defined using J and trace)

Problem

Define a function $T : \mathbb{R}^{2,2} \to \mathbb{R}^{2,2}$ by

$$\forall A \in \mathbb{R}^{2,2} : \qquad T(A) \coloneqq A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \operatorname{trace}(A) \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(a) Show that $T \in \mathcal{L}(\mathbb{R}^{2,2})$.

(b) Determine the standard matrix representation of T.

(c) Derive a basis for $\operatorname{range}(T)$ and hence determine $\operatorname{rank}(T)$.

(d) Derive a basis for $\operatorname{nullity}(T)$ and hence determine $\operatorname{nullity}(T)$.

(e) Determine whether or not $\mathbb{R}^{2,2} = \operatorname{nullity}(T) \oplus \operatorname{range}(T)$.

Solution

Method 1: Compute T on a general matrix and read off all answers

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then

$$AJ = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}, \qquad \operatorname{trace}(A) = a + d, \qquad \operatorname{trace}(A)\, J = \begin{bmatrix} a+d & a+d \\ a+d & a+d \end{bmatrix}.$$

Hence

$$\begin{aligned} T(A) &= AJ - \operatorname{trace}(A)\, J \\ &= \begin{bmatrix} (a+b)-(a+d) & (a+b)-(a+d) \\ (c+d)-(a+d) & (c+d)-(a+d) \end{bmatrix} \\ &= \begin{bmatrix} b-d & b-d \\ c-a & c-a \end{bmatrix}. \end{aligned}$$

(a) We show that T is linear.

The map $A \mapsto AJ$ is linear, since for all $A, B \in \mathbb{R}^{2,2}$ and $\lambda \in \mathbb{R}$,

$$(A+B)J = AJ + BJ, \qquad (\lambda A)J = \lambda(AJ).$$

Also, the trace map $A \mapsto \operatorname{trace}(A)$ is linear, and the map $x \mapsto xJ$ from $\mathbb{R}$ to $\mathbb{R}^{2,2}$ is linear. Therefore $A \mapsto \operatorname{trace}(A)\, J$ is linear. Since T is the difference of two linear maps,

$$T \in \mathcal{L}(\mathbb{R}^{2,2}).$$

(b) We use the standard ordered basis

$$(E_{11}, E_{12}, E_{21}, E_{22}),$$

where

$$E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

For

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

its coordinate vector relative to this basis is

$$[A]_{(E_{11},E_{12},E_{21},E_{22})} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

From the formula for $T(A)$,

$$[T(A)]_{(E_{11},E_{12},E_{21},E_{22})} = \begin{bmatrix} b-d \\ b-d \\ c-a \\ c-a \end{bmatrix}.$$

Thus

$$[T]_{(E_{11},E_{12},E_{21},E_{22})} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

Therefore

$$\boxed{[T]_{(E_{11},E_{12},E_{21},E_{22})} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.}$$

(c) From

$$T(A) = \begin{bmatrix} u & u \\ v & v \end{bmatrix} \quad \text{with} \quad u = b - d,\ v = c - a,$$

we obtain

$$\text{range}(T) = \left\{ \begin{bmatrix} u & u \\ v & v \end{bmatrix} : u, v \in \mathbb{R} \right\}.$$

Hence

$$\text{range}(T) = \text{span} \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

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These two matrices are linearly independent, so they form a basis of range(T). Therefore

$$\boxed{\text{range}(T) = \text{span}\left\{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right\}, \qquad \text{rank}(T) = 2.}$$

(d) We have $T(A) = 0$ if and only if

$$b - d = 0, \qquad c - a = 0,$$

that is,

$$d = b, \qquad c = a.$$

Hence

$$\text{nullity}(T) = \left\{\begin{bmatrix}a & b\\ a & b\end{bmatrix} : a, b \in \mathbb{R}\right\} = \text{span}\left\{\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right\}.$$

These two matrices are linearly independent, so they form a basis of nullity(T). Therefore

$$\boxed{\text{nullity}(T) = \text{span}\left\{\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right\}, \qquad \text{nullity}(T) = 2.}$$

(e) To have

$$\mathbb{R}^{2,2} = \text{nullity}(T) \oplus \text{range}(T),$$

we would need

$$\text{nullity}(T) \cap \text{range}(T) = \{0\}.$$

However,

$$J = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

lies in range(T) (take $u = v = 1$) and also lies in nullity(T) (its two rows are equal). Since $J \neq 0$, we have

$$\text{nullity}(T) \cap \text{range}(T) \neq \{0\}.$$

Therefore the sum is not direct. Hence

$$\boxed{\mathbb{R}^{2,2} \neq \text{nullity}(T) \oplus \text{range}(T).}$$

Method 2: Equal-columns and equal-rows subspaces

Let

$$U \coloneqq \left\{ \begin{bmatrix} u & u \\ v & v \end{bmatrix} : u, v \in \mathbb{R} \right\}, \qquad S \coloneqq \left\{ \begin{bmatrix} a & b \\ a & b \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

Thus U is the subspace of matrices with equal columns, and S is the subspace of matrices with equal rows.

Let

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Since

$$J = \mathbf{1}\mathbf{1}^T,$$

we have

$$AJ = (A\mathbf{1})\mathbf{1}^T.$$

Thus AJ always has equal columns, so $AJ \in U$. Also $\operatorname{trace}(A)J \in \operatorname{span}\{J\} \subseteq U$. Therefore

$$T(A) = AJ - \operatorname{trace}(A)J \in U \qquad \text{for all } A \in \mathbb{R}^{2,2},$$

and hence

$$\operatorname{range}(T) \subseteq U.$$

Now let

$$M = \begin{bmatrix} u & u \\ v & v \end{bmatrix} \in U.$$

Take

$$A = \begin{bmatrix} 0 & u \\ v & 0 \end{bmatrix}.$$

Then $\operatorname{trace}(A) = 0$, and

$$AJ = \begin{bmatrix} u & u \\ v & v \end{bmatrix} = M.$$

Hence

$$T(A) = M.$$

So every element of U lies in $\operatorname{range}(T)$, and therefore

$$\operatorname{range}(T) = U.$$

Since

$$U = \operatorname{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\},$$

we get

$$\boxed{\operatorname{range}(T) = \operatorname{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}, \qquad \operatorname{rank}(T) = 2.}$$

Next, $T(A) = 0$ if and only if

$$AJ = \operatorname{trace}(A)J.$$

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Write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$AJ = \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}, \qquad \operatorname{trace}(A)J = \begin{bmatrix} a+d & a+d \\ a+d & a+d \end{bmatrix}.$$

Equality of these matrices is equivalent to

$$a+b = a+d, \qquad c+d = a+d.$$

Hence

$$b = d, \qquad c = a.$$

Therefore

$$\operatorname{nullity}(T) = \left\{ \begin{bmatrix} a & b \\ a & b \end{bmatrix} : a, b \in \mathbb{R} \right\} = S.$$

Thus

$$\boxed{\operatorname{nullity}(T) = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}, \qquad \operatorname{nullity}(T) = 2.}$$

For linearity, the map $A \mapsto AJ$ is linear and the map $A \mapsto \operatorname{trace}(A)J$ is linear, so $T \in \mathcal{L}(\mathbb{R}^{2,2})$.

For the standard matrix representation, since

$$T(E_{11}) = \begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}, \quad T(E_{12}) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad T(E_{21}) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad T(E_{22}) = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix},$$

their coordinate columns relative to $(E_{11}, E_{12}, E_{21}, E_{22})$ are

$$\begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

so

$$\boxed{[T]_{(E_{11},E_{12},E_{21},E_{22})} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.}$$

Finally, since

$$J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \in U \cap S$$

and $J \neq 0$, we have

$$\operatorname{nullity}(T) \cap \operatorname{range}(T) \neq \{0\},$$

so

$$\boxed{\mathbb{R}^{2,2} \neq \operatorname{nullity}(T) \oplus \operatorname{range}(T).}$$

Method 3: Standard matrix, then rank–nullity

We again write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

As in Method 1,

$$T(A) = \begin{bmatrix} b-d & b-d \\ c-a & c-a \end{bmatrix}.$$

Hence relative to the standard ordered basis $(E_{11}, E_{12}, E_{21}, E_{22})$,

$$[A] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \qquad [T(A)] = \begin{bmatrix} b-d \\ b-d \\ c-a \\ c-a \end{bmatrix},$$

so

$$[T] = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.$$

(a) Since matrix multiplication by $[T]$ defines a linear map on coordinate vectors, T is linear. Hence

$$T \in \mathcal{L}(\mathbb{R}^{2,2}).$$

(b) The standard matrix representation is

$$\boxed{[T]_{(E_{11},E_{12},E_{21},E_{22})} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}.}$$

(c) Row-reducing $[T]$,

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so

$$\operatorname{rank}([T]) = 2.$$

Hence

$$\operatorname{rank}(T) = 2.$$

A basis for the column space of $[T]$ is given by the first and second columns of the original matrix:

$$\begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Translating back to matrices gives the basis

$$\begin{bmatrix} 0 & 0 \\ -1 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Equivalently, one may use the basis

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Thus

$$\boxed{\text{range}(T) = \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}, \qquad \text{rank}(T) = 2.}$$

(d) By the Rank–Nullity Theorem,

$$\dim(\mathbb{R}^{2,2}) = \text{rank}(T) + \text{nullity}(T).$$

Since $\dim(\mathbb{R}^{2,2}) = 4$ and $\text{rank}(T) = 2$, we obtain

$$\text{nullity}(T) = 2.$$

To determine the null space explicitly, solve

$$[T] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0.$$

This gives

$$b - d = 0, \qquad c - a = 0.$$

So

$$d = b, \qquad c = a,$$

and hence

$$\text{nullity}(T) = \left\{ \begin{bmatrix} a & b \\ a & b \end{bmatrix} : a, b \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

Therefore

$$\boxed{\text{nullity}(T) = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}, \qquad \text{nullity}(T) = 2.}$$

(e) Since

$$J = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

belongs to both $\text{range}(T)$ and $\text{nullity}(T)$, the intersection is nontrivial. Hence

$$\boxed{\mathbb{R}^{2,2} \neq \text{nullity}(T) \oplus \text{range}(T).}$$

Method 4: Factorization $T(A) = x(A)\,[1\ 1]$

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

From direct multiplication,

$$T(A) = \begin{bmatrix} b-d & b-d \\ c-a & c-a \end{bmatrix}.$$

Therefore

$$T(A) = \begin{bmatrix} b-d \\ c-a \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Set

$$x(A) \coloneqq \begin{bmatrix} b-d \\ c-a \end{bmatrix} \in \mathbb{R}^2.$$

Then

$$T(A) = x(A) \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

(a) The map $A \mapsto x(A)$ is linear in the entries of A, and the map

$$x \mapsto x \begin{bmatrix} 1 & 1 \end{bmatrix}$$

from $\mathbb{R}^2$ to $\mathbb{R}^{2,2}$ is linear. Hence T is linear, so

$$T \in \mathcal{L}(\mathbb{R}^{2,2}).$$

(b) Relative to the standard basis $(E_{11}, E_{12}, E_{21}, E_{22})$,

$$[A] = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \qquad [T(A)] = \begin{bmatrix} b-d \\ b-d \\ c-a \\ c-a \end{bmatrix}.$$

Thus

$$\boxed{[T]_{(E_{11},E_{12},E_{21},E_{22})} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}}.$$

(c) Since every image has the form

$$x \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} u & u \\ v & v \end{bmatrix},$$

we have

$$\operatorname{range}(T) = \left\{ \begin{bmatrix} u & u \\ v & v \end{bmatrix} : u, v \in \mathbb{R} \right\}.$$

Hence

$$\text{range}(T) = \text{span}\left\{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right\},$$

so

$$\boxed{\text{range}(T) = \text{span}\left\{\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\right\}, \qquad \text{rank}(T) = 2.}$$

(d) We have

$$T(A) = 0 \iff x(A) = 0 \iff \begin{bmatrix}b-d\\ c-a\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}.$$

Thus

$$b = d, \qquad c = a.$$

So

$$\text{nullity}(T) = \left\{\begin{bmatrix}a & b\\ a & b\end{bmatrix} : a, b \in \mathbb{R}\right\} = \text{span}\left\{\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right\}.$$

Therefore

$$\boxed{\text{nullity}(T) = \text{span}\left\{\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right\}, \qquad \text{nullity}(T) = 2.}$$

(e) Again

$$J = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

belongs to both $\text{range}(T)$ and $\text{nullity}(T)$, so

$$\text{nullity}(T) \cap \text{range}(T) \neq \{0\}.$$

Hence

$$\boxed{\mathbb{R}^{2,2} \neq \text{nullity}(T) \oplus \text{range}(T).}$$

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Question 4 (Polynomial degree identities and a substitution map)

Problem

(a) Let $P, Q \in \mathcal{P}(\mathbb{F}) \setminus \{0\}$ be nonzero polynomials. Prove:

(i) $\deg(PQ) = \deg(P) + \deg(Q)$.

(ii) $\deg(P(Q)) = \deg(P) \cdot \deg(Q)$, where $P(Q)$ is obtained by replacing each occurrence of X in P by Q.

(b) Let $n \in \mathbb{N}$ and $Q \in \mathcal{P}(\mathbb{F})$. Define

$$T : \mathcal{P}_n(\mathbb{F}) \to \mathcal{P}_{n\cdot\deg(Q)}(\mathbb{F}), \qquad T(P) \coloneqq P(Q).$$

Show that $T \in \mathcal{L}\big(\mathcal{P}_n(\mathbb{F}), \mathcal{P}_{n\cdot\deg(Q)}(\mathbb{F})\big)$.

Remark

The statement in part (a)(ii) requires Q to be nonconstant. Indeed, if Q is constant, then $P(Q)$ may be the zero polynomial; for example,

$$P(X) = X - 1, \qquad Q(X) = 1,$$

gives $P(Q) = 0$. Thus the correct general statement is

$$\deg(P(Q)) \le \deg(P)\deg(Q),$$

with equality when $\deg(Q) \ge 1$.

Solution

Method 1: Leading term / leading coefficient argument

(a) Let

$$\deg(P) = m, \qquad \deg(Q) = k,$$

and write

$$P(X) = a_m X^m + a_{m-1}X^{m-1} + \cdots + a_0, \qquad a_m \neq 0,$$
$$Q(X) = b_k X^k + b_{k-1}X^{k-1} + \cdots + b_0, \qquad b_k \neq 0.$$

(i) The highest-degree term of PQ is

$$(a_m X^m)(b_k X^k) = a_m b_k X^{m+k}.$$

Since $\mathbb{F}$ is a field, $a_m b_k \neq 0$. Hence the coefficient of X^{m+k} in PQ is nonzero, and no term of higher degree occurs. Therefore

$$\boxed{\deg(PQ) = m + k = \deg(P) + \deg(Q).}$$

(ii) Assume now that Q is nonconstant, so $k = \deg(Q) \geq 1$.

We have

$$P(Q) = a_m Q^m + a_{m-1} Q^{m-1} + \cdots + a_0.$$

By part (i),

$$\deg(Q^m) = m \deg(Q) = mk.$$

Hence the term $a_m Q^m$ has degree mk, with leading coefficient $a_m (b_k)^m \neq 0$.

For each $i < m$,

$$\deg(Q^i) = ik < mk.$$

Thus every other summand has degree strictly less than mk, so no cancellation can remove the leading term of $a_m Q^m$. Therefore

$$\boxed{\deg(P(Q)) = mk = \deg(P) \deg(Q).}$$

(b) We show that

$$T : \mathcal{P}_n(\mathbb{F}) \to \mathcal{P}_{n \deg(Q)}(\mathbb{F}), \qquad T(P) = P(Q),$$

is linear.

Let $P_1, P_2 \in \mathcal{P}_n(\mathbb{F})$ and $c \in \mathbb{F}$. Then

$$T(P_1 + P_2) = (P_1 + P_2)(Q) = P_1(Q) + P_2(Q) = T(P_1) + T(P_2),$$

and

$$T(cP_1) = (cP_1)(Q) = c\, P_1(Q) = c\, T(P_1).$$

So T is linear.

It remains to check that $T(P) \in \mathcal{P}_{n \deg(Q)}(\mathbb{F})$ for every $P \in \mathcal{P}_n(\mathbb{F})$. If $P = 0$, this is clear. If $P \neq 0$, then

$$\deg(P) \leq n.$$

If Q is nonconstant, then by part (a)(ii),

$$\deg(P(Q)) = \deg(P) \deg(Q) \leq n \deg(Q).$$

If Q is constant, then $P(Q)$ is constant, hence has degree at most $0 = n \deg(Q)$. Therefore $T(P) \in \mathcal{P}_{n \deg(Q)}(\mathbb{F})$.

Thus

$$\boxed{T \in \mathcal{L}\big(\mathcal{P}_n(\mathbb{F}), \mathcal{P}_{n \deg(Q)}(\mathbb{F})\big).}$$

Method 2: Substitution as an $\mathbb{F}$-algebra homomorphism

(a) Define

$$\Phi_Q : \mathbb{F}[X] \to \mathbb{F}[X], \qquad \Phi_Q(P) = P(Q).$$

Then Φ_Q is an $\mathbb{F}$-algebra homomorphism: for all $P_1, P_2 \in \mathbb{F}[X]$ and $c \in \mathbb{F}$,

$$\Phi_Q(P_1 + P_2) = P_1(Q) + P_2(Q) = \Phi_Q(P_1) + \Phi_Q(P_2),$$

$$\Phi_Q(cP_1) = cP_1(Q) = c\,\Phi_Q(P_1),$$

and

$$\Phi_Q(P_1P_2) = (P_1P_2)(Q) = P_1(Q)P_2(Q) = \Phi_Q(P_1)\Phi_Q(P_2).$$

(i) Since $\mathbb{F}[X]$ is an integral domain, the degree of a product of two nonzero polynomials adds:

$$\boxed{\deg(PQ) = \deg(P) + \deg(Q).}$$

(ii) Assume Q is nonconstant. Let $\deg(P) = m$. Write

$$P(X) = a_mX^m + \cdots + a_0, \qquad a_m \neq 0.$$

Applying Φ_Q,

$$P(Q) = a_mQ^m + \cdots + a_0.$$

Now $\deg(Q^m) = m\deg(Q)$ by repeated application of part (i), and for every $i < m$,

$$\deg(Q^i) = i\deg(Q) < m\deg(Q).$$

Hence the term a_mQ^m uniquely contributes the highest degree, so

$$\boxed{\deg(P(Q)) = \deg(P)\deg(Q).}$$

(b) Since Φ_Q is linear on $\mathbb{F}[X]$, its restriction to the subspace $\mathcal{P}_n(\mathbb{F})$ is also linear.

It remains to check the codomain. If $P \in \mathcal{P}_n(\mathbb{F})$, then $\deg(P) \leq n$. Therefore:

- if Q is nonconstant, then
$$\deg(P(Q)) \leq n\deg(Q);$$
- if Q is constant, then $P(Q)$ is constant, so again
$$\deg(P(Q)) \leq 0 = n\deg(Q).$$

Hence

$$\Phi_Q(\mathcal{P}_n(\mathbb{F})) \subseteq \mathcal{P}_{n\deg(Q)}(\mathbb{F}).$$

Therefore

$$T = \Phi_Q|_{\mathcal{P}_n(\mathbb{F})} \in \mathcal{L}\big(\mathcal{P}_n(\mathbb{F}), \mathcal{P}_{n\deg(Q)}(\mathbb{F})\big).$$

Thus

$$\boxed{T \in \mathcal{L}\big(\mathcal{P}_n(\mathbb{F}), \mathcal{P}_{n\deg(Q)}(\mathbb{F})\big).}$$

Method 3: Basis and degree-filtration viewpoint

(a) Let $\deg(P) = m$ and $\deg(Q) = k$.

(i) Write

$$P(X) = a_m X^m + \cdots + a_0, \qquad Q(X) = b_k X^k + \cdots + b_0$$

with $a_m, b_k \neq 0$. Then

$$PQ = a_m b_k X^{m+k} + (\text{lower-degree terms}).$$

Since $a_m b_k \neq 0$, we get

$$\boxed{\deg(PQ) = m + k.}$$

(ii) Assume Q is nonconstant. Express P in the basis $(1, X, \ldots, X^m)$:

$$P(X) = \sum_{i=0}^{m} a_i X^i, \qquad a_m \neq 0.$$

Then

$$P(Q) = \sum_{i=0}^{m} a_i Q^i.$$

Since $\deg(Q^i) = ik$, these summands have pairwise distinct degrees

$$0, k, 2k, \ldots, mk.$$

In particular, the term $a_m Q^m$ has strictly larger degree than all the other terms, so it determines the degree of the sum.

$$\boxed{\deg(P(Q)) = mk = \deg(P)\deg(Q).}$$

(b) The space $\mathcal{P}_n(\mathbb{F})$ has basis

$$1, X, \ldots, X^n.$$

Define

$$T(P) = P(Q).$$

Then on basis elements,

$$T(X^i) = Q^i \qquad (0 \leq i \leq n).$$

Hence T is linear by the theorem that a map defined on a basis extends uniquely to a linear map.

Moreover, for each i,

$$\deg(Q^i) = i \deg(Q) \leq n \deg(Q)$$

when Q is nonconstant, while if Q is constant then each Q^i is constant. Therefore

$$T(X^i) \in \mathcal{P}_{n \deg(Q)}(\mathbb{F}) \qquad (0 \leq i \leq n).$$

So T indeed maps $\mathcal{P}_n(\mathbb{F})$ into $\mathcal{P}_{n \deg(Q)}(\mathbb{F})$. Therefore

$$\boxed{T \in \mathcal{L}\big(\mathcal{P}_n(\mathbb{F}), \mathcal{P}_{n \deg(Q)}(\mathbb{F})\big).}$$

Question 5 (A linear operator on $\mathcal{P}_2(\mathbb{R})$ involving composition and differentiation)

Problem

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ by

$$\forall P \in \mathcal{P}_2(\mathbb{R}): \qquad T(P) \coloneqq P(X+1) \; - \; \big(P(X^2)\big)''.$$

(a) Determine the standard matrix representation of T (with respect to the standard ordered basis $(1, X, X^2)$ of $\mathcal{P}_2(\mathbb{R})$).

(b) Determine whether or not T is bijective; if it is bijective, then find T^{-1}.

Solution

Method 1: Compute T on the standard basis directly

We use the standard ordered basis

$$(1, X, X^2)$$

of $\mathcal{P}_2(\mathbb{R})$.

(a) We compute T on each basis vector.

For 1,

$$T(1) = 1 - \big(1\big)'' = 1.$$

For X,

$$T(X) = (X+1) - \big(X^2\big)'' = (X+1) - 2 = X - 1.$$

For X^2,

$$T(X^2) = (X+1)^2 - \big(X^4\big)'' = (X^2 + 2X + 1) - 12X^2 = 1 + 2X - 11X^2.$$

Hence the coordinate vectors relative to $(1, X, X^2)$ are

$$[T(1)]_{(1,X,X^2)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad [T(X)]_{(1,X,X^2)} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \qquad [T(X^2)]_{(1,X,X^2)} = \begin{bmatrix} 1 \\ 2 \\ -11 \end{bmatrix}.$$

Therefore the standard matrix representation of T is

$$[T]_{(1,X,X^2)} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{bmatrix}.$$

Thus

$$\boxed{[T]_{(1,X,X^2)} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{bmatrix}.}$$

(b) Since $[T]_{(1,X,X^2)}$ is upper triangular, its determinant is the product of its diagonal entries:

$$\det\big([T]_{(1,X,X^2)}\big) = 1 \cdot 1 \cdot (-11) = -11 \neq 0.$$

Hence $[T]_{(1,X,X^2)}$ is invertible, so T is bijective.

To determine T^{-1}, let

$$Q(X) = \alpha + \beta X + \gamma X^2 \in \mathcal{P}_2(\mathbb{R}),$$

and seek

$$P(X) = a + bX + cX^2 \in \mathcal{P}_2(\mathbb{R})$$

such that

$$T(P) = Q.$$

Using the values of $T(1), T(X), T(X^2)$, we have

$$T(a + bX + cX^2) = a\,T(1) + b\,T(X) + c\,T(X^2).$$

Thus

$$\begin{aligned} T(P) &= a(1) + b(X-1) + c(1 + 2X - 11X^2) \\ &= (a - b + c) + (b + 2c)X - 11cX^2. \end{aligned}$$

Matching coefficients with

$$Q(X) = \alpha + \beta X + \gamma X^2$$

gives

$$a - b + c = \alpha, \qquad b + 2c = \beta, \qquad -11c = \gamma.$$

We solve from the last equation upward:

$$c = -\frac{\gamma}{11},$$

$$b = \beta - 2c = \beta + \frac{2\gamma}{11},$$

$$a = \alpha + b - c = \alpha + \beta + \frac{3\gamma}{11}.$$

Therefore

$$T^{-1}(\alpha + \beta X + \gamma X^2) = \left(\alpha + \beta + \frac{3\gamma}{11}\right) + \left(\beta + \frac{2\gamma}{11}\right) X - \frac{\gamma}{11} X^2.$$

Hence

$$\boxed{T \text{ is bijective, and } T^{-1}(\alpha + \beta X + \gamma X^2) = \left(\alpha + \beta + \frac{3\gamma}{11}\right) + \left(\beta + \frac{2\gamma}{11}\right) X - \frac{\gamma}{11} X^2.}$$

Method 2: Coefficient chasing on a general quadratic

Let

$$P(X) = a + bX + cX^2 \in \mathcal{P}_2(\mathbb{R}).$$

(a) We first compute $T(P)$.

Since

$$\begin{aligned} P(X+1) &= a + b(X+1) + c(X+1)^2 \\ &= a + bX + b + c(X^2 + 2X + 1) \\ &= (a+b+c) + (b+2c)X + cX^2, \end{aligned}$$

and

$$P(X^2) = a + bX^2 + cX^4,$$

we get

$$\big(P(X^2)\big)' = 2bX + 4cX^3, \qquad \big(P(X^2)\big)'' = 2b + 12cX^2.$$

Therefore

$$\begin{aligned} T(P) &= P(X+1) - \big(P(X^2)\big)'' \\ &= \big((a+b+c) + (b+2c)X + cX^2\big) - \big(2b + 12cX^2\big) \\ &= (a-b+c) + (b+2c)X - 11cX^2. \end{aligned}$$

Thus relative to the standard basis $(1, X, X^2)$,

$$[T(P)]_{(1,X,X^2)} = \begin{bmatrix} a-b+c \\ b+2c \\ -11c \end{bmatrix}.$$

Hence

$$[T]_{(1,X,X^2)} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{bmatrix},$$

since

$$\begin{bmatrix} a-b+c \\ b+2c \\ -11c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Therefore

$$\boxed{[T]_{(1,X,X^2)} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{bmatrix}.}$$

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(b) Since the standard matrix is upper triangular with nonzero diagonal entries $1, 1, -11$, we have

$$\det\big([T]_{(1,X,X^2)}\big) = -11 \neq 0.$$

Hence T is bijective.

To compute T^{-1}, let

$$Q(X) = \alpha + \beta X + \gamma X^2.$$

We solve

$$T(a + bX + cX^2) = Q(X).$$

Using

$$T(a + bX + cX^2) = (a - b + c) + (b + 2c)X - 11cX^2,$$

we obtain

$$a - b + c = \alpha, \qquad b + 2c = \beta, \qquad -11c = \gamma.$$

Thus

$$c = -\frac{\gamma}{11}, \qquad b = \beta + \frac{2\gamma}{11}, \qquad a = \alpha + \beta + \frac{3\gamma}{11}.$$

So

$$T^{-1}(\alpha + \beta X + \gamma X^2) = \left(\alpha + \beta + \frac{3\gamma}{11}\right) + \left(\beta + \frac{2\gamma}{11}\right) X - \frac{\gamma}{11} X^2.$$

Hence

$$\boxed{T \text{ is bijective, and } T^{-1}(\alpha + \beta X + \gamma X^2) = \left(\alpha + \beta + \frac{3\gamma}{11}\right) + \left(\beta + \frac{2\gamma}{11}\right) X - \frac{\gamma}{11} X^2.}$$

Unofficial Solution

Method 3: Triangular operator viewpoint + back-substitution

We use the ordered basis

$$(1, X, X^2)$$

and the theorem that if a linear operator has an upper triangular matrix with respect to some basis, then its diagonal entries are the coefficients of the highest-degree basis vectors in the images of those basis vectors.

(a) We first determine the leading-term behaviour of T on the basis vectors.

For 1,

$$T(1) = 1,$$

so the coefficient of the basis vector 1 is 1.

For X,

$$T(X) = (X+1) - \left(X^2\right)'' = X - 1,$$

so the coefficient of the basis vector X is 1, and only lower-degree terms are added.

For X^2,

$$T(X^2) = (X+1)^2 - \left(X^4\right)'' = (X^2 + 2X + 1) - 12X^2 = 1 + 2X - 11X^2,$$

so the coefficient of the basis vector X^2 is -11, and only lower-degree terms are added.

Hence the matrix of T relative to $(1, X, X^2)$ is upper triangular with diagonal entries

$$1,\ 1,\ -11.$$

To determine the full matrix, we record the exact images:

$$T(1) = 1, \qquad T(X) = X - 1, \qquad T(X^2) = 1 + 2X - 11X^2.$$

Therefore

$$[T]_{(1,X,X^2)} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{bmatrix}.$$

Thus

$$\boxed{[T]_{(1,X,X^2)} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -11 \end{bmatrix}.}$$

(b) Since T has an upper triangular matrix with diagonal entries $1, 1, -11$, all of which are nonzero, the matrix is invertible. Hence T is bijective.

To determine T^{-1}, let

$$Q(X) = \alpha + \beta X + \gamma X^2$$

and seek

$$P(X) = a + bX + cX^2$$

such that

$$T(P) = Q.$$

Using the basis images,

$$T(P) = a\,T(1) + b\,T(X) + c\,T(X^2),$$

so

$$T(P) = a(1) + b(X - 1) + c(1 + 2X - 11X^2).$$

Hence

$$T(P) = (a - b + c) + (b + 2c)X - 11cX^2.$$

Because the matrix is triangular, we solve by back-substitution, starting from the highest-degree coefficient.

Matching the X^2-coefficient:

$$-11c = \gamma \quad \Longrightarrow \quad c = -\frac{\gamma}{11}.$$

Matching the X-coefficient:

$$b + 2c = \beta \quad \Longrightarrow \quad b = \beta - 2c = \beta + \frac{2\gamma}{11}.$$

Matching the constant term:

$$a - b + c = \alpha \quad \Longrightarrow \quad a = \alpha + b - c = \alpha + \beta + \frac{3\gamma}{11}.$$

Therefore

$$T^{-1}(\alpha + \beta X + \gamma X^2) = \left(\alpha + \beta + \frac{3\gamma}{11}\right) + \left(\beta + \frac{2\gamma}{11}\right) X - \frac{\gamma}{11} X^2.$$

Hence

$$\boxed{T \text{ is bijective, and } T^{-1}(\alpha + \beta X + \gamma X^2) = \left(\alpha + \beta + \frac{3\gamma}{11}\right) + \left(\beta + \frac{2\gamma}{11}\right) X - \frac{\gamma}{11} X^2.}$$

Question 6 (Injectivity/surjectivity under composition)

Problem

Let U, V, W be vector spaces over the same field. Let $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$. Establish:

(1) If $T \circ S$ is injective, then S is injective.

(2) If $T \circ S$ is surjective onto W, then T is surjective onto W.

Solution

Method 1: Directly from definitions (kernels and images)

(1) Assume $T \circ S$ is injective. To prove S is injective, let $u_1, u_2 \in U$ and assume $S(u_1) = S(u_2)$. Apply T to both sides:

$$(T \circ S)(u_1) = T(S(u_1)) = T(S(u_2)) = (T \circ S)(u_2).$$

Since $T \circ S$ is injective, $u_1 = u_2$. Hence S is injective.

Equivalently, $\ker(S) \subseteq \ker(T \circ S)$, and if $\ker(T \circ S) = \{0\}$ then $\ker(S) = \{0\}$.

$$\boxed{T \circ S \text{ injective } \Rightarrow S \text{ injective.}}$$

(2) Assume $T \circ S$ is surjective onto W. Let $w \in W$ be arbitrary. Since $T \circ S$ is surjective, there exists $u \in U$ such that

$$(T \circ S)(u) = T(S(u)) = w.$$

Let $v = S(u) \in V$. Then $T(v) = w$. Since w was arbitrary, T is surjective onto W.

Equivalently, $\text{range}(T \circ S) \subseteq \text{range}(T)$, and if $\text{range}(T \circ S) = W$ then $\text{range}(T) = W$.

$$\boxed{T \circ S \text{ surjective onto } W \Rightarrow T \text{ surjective onto } W.}$$

Method 2: Contrapositives (construct a failure of the composition)

(1) **Contrapositive:** If S is not injective, then $T \circ S$ is not injective.

If S is not injective, there exist $u_1 \neq u_2$ such that $S(u_1) = S(u_2)$. Then

$$(T \circ S)(u_1) = T(S(u_1)) = T(S(u_2)) = (T \circ S)(u_2),$$

so $T \circ S$ is not injective. This proves the original statement.

(2) **Contrapositive:** If T is not surjective onto W, then $T \circ S$ is not surjective onto W.

If T is not surjective, then $\text{range}(T) \subsetneq W$. But $\text{range}(T \circ S) \subseteq \text{range}(T)$ always (since outputs of $T \circ S$ are outputs of T), so $\text{range}(T \circ S) \subsetneq W$ as well. Hence $T \circ S$ is not surjective.

Question 7 (Are the converses true?)

Problem

Let U, V, W be vector spaces over the same field. Let $S \in \mathcal{L}(U, V)$ and $T \in \mathcal{L}(V, W)$.

(a) If $T \circ S$ is injective, is T necessarily injective?

(b) If $T \circ S$ is surjective onto W, is S necessarily surjective onto V?

Solution

Method 1: Explicit counterexamples (sharp "no" for both)

(a) **No.** Take $U = \mathbb{R}$, $V = \mathbb{R}^2$, $W = \mathbb{R}$. Define

$$S : \mathbb{R} \to \mathbb{R}^2, \quad S(t) = (t, 0), \qquad T : \mathbb{R}^2 \to \mathbb{R}, \quad T(x, y) = x.$$

Then S is injective. The composition is

$$(T \circ S)(t) = T(t, 0) = t,$$

which is injective. However, T is *not* injective because $T(0, 1) = 0 = T(0, 0)$ while $(0, 1) \neq (0, 0)$. So $T \circ S$ injective does *not* force T injective.

$$\boxed{\text{(a) No.}}$$

(b) **No.** Take $U = \mathbb{R}$, $V = \mathbb{R}^2$, $W = \mathbb{R}$. Define

$$S : \mathbb{R} \to \mathbb{R}^2, \quad S(t) = (t, 0), \qquad T : \mathbb{R}^2 \to \mathbb{R}, \quad T(x, y) = x + y.$$

Then T is surjective (e.g. $T(w, 0) = w$ for any $w \in \mathbb{R}$). The composition is

$$(T \circ S)(t) = T(t, 0) = t,$$

which is surjective onto $\mathbb{R}$. But S is *not* surjective onto $\mathbb{R}^2$ because no t satisfies $S(t) = (0, 1)$. So surjectivity of $T \circ S$ does *not* force S surjective.

$$\boxed{\text{(b) No.}}$$

Method 2: Conceptual explanation (why the converses fail)

(a) If T has a nontrivial kernel, it can still happen that $S(U)$ avoids that kernel except at 0. Then T may fail to be injective on all of V, while T restricted to $S(U)$ is injective, making $T \circ S$ injective.

Formally, $T \circ S$ injective means: if $S(u) \in \ker(T)$ then $u = 0$. This does *not* imply $\ker(T) = \{0\}$; it only implies

$$S(U) \cap \ker(T) = \{0\}.$$

(b) If S is not surjective, then $S(U)$ is a proper subspace (or subset) of V. It is still possible that T maps $S(U)$ *onto* W, i.e. $T(S(U)) = W$, even though $S(U) \neq V$. Thus $T \circ S$ can be surjective without S being surjective.

Formally, $T \circ S$ surjective means $T(S(U)) = W$, which does not imply $S(U) = V$.

Neither converse holds in general.

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