MH1201 Linear Algebra II

Tutorial 8 (Week 10) – Problems & Solutions

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Overview

This tutorial emphasizes:

- diagonalization as a computational tool for iterates T^n and for explicit closed forms (Fibonacci/Binet);
- eigenvalues/eigenspaces and diagonalizability for operators on $\mathcal{P}_2(\mathbb{R})$;
- functional calculus: eigenvalues of $p(A)$ and determinants of $\sum_{k=1}^{n} A^k$;
- similarity of diagonalizable operators with the same distinct eigenvalues;
- invertibility of $T - \lambda I$ when λ is not an eigenvalue, and solving affine equations $T(v) = 8v + b$.

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Question 1 (Diagonalization of F and Fibonacci)

Problem

Recall the linear transformation $F \in \mathcal{L}(\mathbb{R}^{2,1})$ defined in Problem 5 of Tutorial Problem Set 7:

$$\forall x, y \in \mathbb{R}: \qquad F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) :== \begin{bmatrix} y \\ x+y \end{bmatrix}.$$

(a) Derive an ordered basis β for $\mathbb{R}^{2,1}$ such that $[F]_\beta^\beta \in \mathbb{R}^{2,2}$ is a diagonal matrix.
Hint: Use your work for Part (b) of Problem 5 of Tutorial Problem Set 7.

(b) Derive, for any $n \in \mathbb{N}_0$ and any $x, y \in \mathbb{R}$, an explicit formula for

$$F^n\left(\begin{bmatrix} x \\ y \end{bmatrix}\right).$$

Hint: Use your work for Part (a) of this problem.

(c) Letting $(F_n)_{n\in\mathbb{N}_0}$ denote the Fibonacci sequence (with $F_0 = 0$ and $F_1 = 1$), show that

$$\forall n \in \mathbb{N}_0: \qquad F^n\left(\begin{bmatrix} F_0 \\ F_1 \end{bmatrix}\right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

Hint: Apply the Principle of Mathematical Induction.

(d) Hence, derive an explicit formula for the members of the Fibonacci sequence.

Solution

Method 1: Eigen-decomposition (diagonalize F and compute F^n)

With respect to the standard basis of $\mathbb{R}^{2,1}$, F has matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A\begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) **Diagonalize A.** Compute the characteristic polynomial:

$$\chi_A(\lambda) = \det(A - \lambda I_2) = \det\begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = \lambda^2 - \lambda - 1.$$

Hence the (real) eigenvalues are

$$\varphi := \frac{1+\sqrt{5}}{2}, \qquad \psi := \frac{1-\sqrt{5}}{2}.$$

For an eigenvalue λ, solve $(A - \lambda I_2)\binom{x}{y} = 0$:

$$-\lambda x + y = 0 \;\Rightarrow\; y = \lambda x.$$

Thus an eigenvector is $\binom{1}{\lambda}$. Therefore an eigenbasis is

$$\beta = \left(\begin{bmatrix} 1 \\ \varphi \end{bmatrix}, \begin{bmatrix} 1 \\ \psi \end{bmatrix} \right).$$

In this basis,

$$[F]_\beta^\beta = \operatorname{diag}(\varphi, \psi).$$

$$\boxed{\beta = \left(\begin{bmatrix} 1 \\ \varphi \end{bmatrix}, \begin{bmatrix} 1 \\ \psi \end{bmatrix} \right), \qquad [F]_\beta^\beta = \begin{bmatrix} \varphi & 0 \\ 0 & \psi \end{bmatrix}}.$$

(b) **Explicit formula for $F^n\binom{x}{y}$.** Write the vector in the eigenbasis:

$$\begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} 1 \\ \varphi \end{bmatrix} + b \begin{bmatrix} 1 \\ \psi \end{bmatrix}.$$

Solving

$$x = a + b, \qquad y = a\varphi + b\psi$$

gives

$$a = \frac{y - \psi x}{\varphi - \psi}, \qquad b = \frac{\varphi x - y}{\varphi - \psi}, \qquad \varphi - \psi = \sqrt{5}.$$

Since F scales eigenvectors,

$$F^n\left(\begin{bmatrix} 1 \\ \varphi \end{bmatrix} \right) = \varphi^n \begin{bmatrix} 1 \\ \varphi \end{bmatrix}, \qquad F^n\left(\begin{bmatrix} 1 \\ \psi \end{bmatrix} \right) = \psi^n \begin{bmatrix} 1 \\ \psi \end{bmatrix}.$$

Therefore

$$F^n\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = a\varphi^n \begin{bmatrix} 1 \\ \varphi \end{bmatrix} + b\psi^n \begin{bmatrix} 1 \\ \psi \end{bmatrix}.$$

Equivalently, componentwise,

$$F^n\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} a\varphi^n + b\psi^n \\ a\varphi^{n+1} + b\psi^{n+1} \end{bmatrix}, \quad a = \frac{y - \psi x}{\sqrt{5}},\ b = \frac{\varphi x - y}{\sqrt{5}}.$$

$$\boxed{F^n\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} \frac{y-\psi x}{\sqrt{5}}\,\varphi^n + \frac{\varphi x - y}{\sqrt{5}}\,\psi^n \\ \frac{y-\psi x}{\sqrt{5}}\,\varphi^{n+1} + \frac{\varphi x - y}{\sqrt{5}}\,\psi^{n+1} \end{bmatrix}}.$$

(c) **Show $F^n\binom{F_0}{F_1} = \binom{F_n}{F_{n+1}}$ by induction.** Here $\binom{F_0}{F_1} = \binom{0}{1}$.

Base case $n = 0$. $F^0 = \mathrm{id}$, so

$$F^0\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}.$$

Inductive step. Assume for some $n \in \mathbb{N}_0$ that

$$F^n\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

Apply F to both sides:

$$F^{n+1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = F\left(\begin{bmatrix}F_n\\F_{n+1}\end{bmatrix}\right) = \begin{bmatrix}F_{n+1}\\F_n+F_{n+1}\end{bmatrix} = \begin{bmatrix}F_{n+1}\\F_{n+2}\end{bmatrix},$$

using the Fibonacci recursion $F_{n+2} = F_{n+1} + F_n$. Thus the statement holds for $n+1$, and by induction it holds for all $n \in \mathbb{N}_0$.

$$\boxed{\forall n \in \mathbb{N}_0 :\ F^n\left(\begin{bmatrix}F_0\\F_1\end{bmatrix}\right) = \begin{bmatrix}F_n\\F_{n+1}\end{bmatrix}}.$$

(d) **Explicit Fibonacci formula (Binet).** Apply the boxed formula in (b) with $(x, y) = (0, 1)$. Then $a = \frac{1-\psi\cdot 0}{\sqrt{5}} = \frac{1}{\sqrt{5}}$ and $b = \frac{\varphi\cdot 0-1}{\sqrt{5}} = -\frac{1}{\sqrt{5}}$, hence

$$F^n\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}\frac{\varphi^n-\psi^n}{\sqrt{5}}\\ \frac{\varphi^{n+1}-\psi^{n+1}}{\sqrt{5}}\end{bmatrix}.$$

By part (c), the first component equals F_n. Therefore

$$\boxed{\forall n \in \mathbb{N}_0 :\quad F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \quad \text{where} \quad \varphi = \frac{1+\sqrt{5}}{2},\ \psi = \frac{1-\sqrt{5}}{2}.}$$

Method 2: Fibonacci matrix powers and a second route to (b)–(d)

Still let $A = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$ so that $F(v) = Av$.

Claim. For all $n \in \mathbb{N}$,

$$A^n = \begin{bmatrix}F_{n-1} & F_n\\F_n & F_{n+1}\end{bmatrix},$$

where (F_n) is the Fibonacci sequence with $F_0 = 0$, $F_1 = 1$.

Proof (induction). For $n = 1$, $A = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}F_0 & F_1\\F_1 & F_2\end{bmatrix}$ since $F_2 = 1$. Assume the identity holds for n. Then

$$A^{n+1} = A^n A = \begin{bmatrix}F_{n-1} & F_n\\F_n & F_{n+1}\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}F_n & F_{n-1}+F_n\\F_{n+1} & F_n+F_{n+1}\end{bmatrix} = \begin{bmatrix}F_n & F_{n+1}\\F_{n+1} & F_{n+2}\end{bmatrix},$$

which is the desired form for $n+1$. This completes the induction.

Consequently, for any $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$F^n\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = A^n\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}F_{n-1}x + F_n y\\F_n x + F_{n+1} y\end{bmatrix}.$$

This provides another explicit formula for $F^n(\binom{x}{y})$ (and immediately yields part (c) by taking $(x, y) = (0, 1)$). Combining this with the diagonalization from Method 1 recovers Binet's formula in part (d).

Question 2 (Diagonalization and iterates of an operator on $\mathcal{P}_2(\mathbb{R})$)

Problem

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}))$ as follows:

$$\forall P \in \mathcal{P}_2(\mathbb{R}): \qquad T(P) :== P(1) + P'(0)X + \big(P'(0) + P''(0)\big)X^2.$$

(a) Compute the eigenvalues of T.

(b) For each eigenvalue λ of T, determine its corresponding eigenspace $E_{T,\lambda}$.

(c) Explain why T is diagonalizable.

(d) Derive an ordered basis β for $\mathcal{P}_2(\mathbb{R})$ such that $[T]_\beta^\beta \in \mathbb{R}^{3,3}$ is a diagonal matrix.

(e) Hence, derive, for any $n \in \mathbb{N}_0$ and any $a, b, c \in \mathbb{R}$, an explicit formula for

$$T^n\big(a + bX + cX^2\big).$$

Solution

Method 1: Compute the matrix of T in the standard basis and diagonalize

Let $P(X) = a + bX + cX^2 \in \mathcal{P}_2(\mathbb{R})$. Then

$$P(1) = a + b + c, \qquad P'(X) = b + 2cX \Rightarrow P'(0) = b, \qquad P''(X) = 2c \Rightarrow P''(0) = 2c.$$

Hence

$$T(P) = (a + b + c) + bX + (b + 2c)X^2.$$

Relative to the standard ordered basis $(1, X, X^2)$, the coordinate map $(a, b, c)^\mathsf{T} \mapsto (a', b', c')^\mathsf{T}$ is

$$\begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

So

$$[T]_{(1,X,X^2)}^{(1,X,X^2)} = M := \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

(a) **Eigenvalues.** Compute $\chi_M(\lambda) = \det(M - \lambda I_3)$:

$$\chi_M(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 1 & 2-\lambda \end{bmatrix} = (1-\lambda)\det \begin{bmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{bmatrix} = (1-\lambda)^2(2-\lambda).$$

Thus the eigenvalues are

$$\boxed{\lambda = 1 \text{ (algebraic multiplicity 2)}, \quad \lambda = 2}.$$

(b) **Eigenspaces.** Solve $(M - \lambda I_3)\boldsymbol{v} = 0$.

Case $\lambda = 2$.

$$M - 2I_3 = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow b = 0,\ a = c.$$

So a basis vector is $(1, 0, 1)^{\mathsf{T}}$, which corresponds to polynomial $1 + X^2$. Hence

$$\boxed{E_{T,2} = \text{span}\{\, 1 + X^2 \,\}}.$$

Case $\lambda = 1$.

$$M - I_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow b = -c,\ a \text{ free}.$$

So eigenvectors are $(a, -c, c)^{\mathsf{T}} = a(1, 0, 0)^{\mathsf{T}} + c(0, -1, 1)^{\mathsf{T}}$, corresponding to

$$1, \qquad X^2 - X.$$

Therefore

$$\boxed{E_{T,1} = \text{span}\{\, 1,\ X^2 - X \,\}}.$$

(c) **Diagonalizability.** We have

$$\dim(E_{T,1}) + \dim(E_{T,2}) = 2 + 1 = 3 = \dim(\mathcal{P}_2(\mathbb{R})).$$

Hence $\mathcal{P}_2(\mathbb{R})$ has a basis of eigenvectors, so T is diagonalizable.

$$\boxed{T \text{ is diagonalizable since eigenspaces span } \mathcal{P}_2(\mathbb{R}).}$$

(d) **An eigenbasis β making $[T]_\beta^\beta$ diagonal.** Take

$$\beta = (\, 1,\ X^2 - X,\ 1 + X^2 \,).$$

Then T acts by 1 on the first two basis vectors and by 2 on the third, so

$$[T]_\beta^\beta = \text{diag}(1, 1, 2).$$

$$\boxed{\beta = (1,\ X^2 - X,\ 1 + X^2), \qquad [T]_\beta^\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}.$$

(e) **Explicit formula for $T^n(a + bX + cX^2)$.** Decompose $P = a + bX + cX^2$ in the eigenbasis:

$$a + bX + cX^2 = \alpha \cdot 1 + \beta \cdot (X^2 - X) + \gamma \cdot (1 + X^2).$$

Expanding the right-hand side gives

$$(\alpha + \gamma) + (-\beta)X + (\beta + \gamma)X^2.$$

Matching coefficients yields

$$\beta = -b, \qquad \gamma = b + c, \qquad \alpha = a - \gamma = a - b - c.$$

Since eigenvalues are $1, 1, 2$ on $(1,\ X^2 - X,\ 1 + X^2)$, we get

$$T^n(P) = \alpha \cdot 1^n \cdot 1 + \beta \cdot 1^n \cdot (X^2 - X) + \gamma \cdot 2^n \cdot (1 + X^2).$$

Thus

$$T^n(a + bX + cX^2) = (a - b - c) + (-b)(X^2 - X) + (b + c)2^n(1 + X^2).$$

In the standard form,

$$T^n(a + bX + cX^2) = \big(a - b - c + (b + c)2^n\big) + bX + \big(-b + (b + c)2^n\big)X^2.$$

$$\boxed{T^n(a + bX + cX^2) = \big(a - b - c + (b + c)2^n\big) + bX + \big(-b + (b + c)2^n\big)X^2 \quad (n \in \mathbb{N}_0).}$$

Method 2: Functional viewpoint (eigen-decomposition without explicit 3×3 determinants)

From the formula

$$T(a + bX + cX^2) = (a + b + c) + bX + (b + 2c)X^2,$$

observe directly:

$$T(1) = 1, \qquad T(X^2 - X) = X^2 - X, \qquad T(1 + X^2) = 2(1 + X^2).$$

Thus 1 and $X^2 - X$ are eigenvectors with eigenvalue 1, and $1 + X^2$ is an eigenvector with eigenvalue 2. These three polynomials are linearly independent (their coefficient vectors are independent), hence form a basis; therefore T is diagonalizable. The iterate formula in (e) follows immediately by expanding $a + bX + cX^2$ in this eigenbasis (as done in Method 1).

Question 3 (Determinant of $\sum_{k=1}^{n} A^k$)

Problem

Let $A \in \mathbb{C}^{3,3}$ satisfy the property that $A - I_3$, $A - eI_3$, and $A - iI_3$ are not invertible. Derive, for any $n \in \mathbb{N}$, an explicit formula for

$$\det\left(\sum_{k=1}^{n} A^k\right).$$

Solution

Method 1: Diagonalize A and evaluate the geometric sum on eigenvalues

Since $A - \lambda I_3$ is not invertible iff λ is an eigenvalue, the hypotheses imply that

$$1,\ e,\ i$$

are eigenvalues of A. They are distinct complex numbers, and A is 3×3, so the characteristic polynomial has degree 3 and hence the eigenvalues of A are exactly $\{1, e, i\}$ (each with algebraic multiplicity 1). Therefore A is diagonalizable over $\mathbb{C}$.

Thus there exists an invertible $P \in \mathbb{C}^{3,3}$ such that

$$A = PDP^{-1}, \qquad D = \operatorname{diag}(1, e, i)$$

(after a suitable ordering of the eigenvectors).

Define

$$S_n \coloneqq \sum_{k=1}^{n} A^k.$$

Then

$$S_n = \sum_{k=1}^{n} (PD^kP^{-1}) = P\left(\sum_{k=1}^{n} D^k\right)P^{-1}.$$

Taking determinants and using $\det(PBP^{-1}) = \det(B)$ gives

$$\det(S_n) = \det\left(\sum_{k=1}^{n} D^k\right).$$

But $\sum_{k=1}^{n} D^k$ is diagonal:

$$\sum_{k=1}^{n} D^k = \operatorname{diag}\left(\sum_{k=1}^{n} 1^k,\ \sum_{k=1}^{n} e^k,\ \sum_{k=1}^{n} i^k\right) = \operatorname{diag}\left(n,\ \sum_{k=1}^{n} e^k,\ \sum_{k=1}^{n} i^k\right).$$

Hence

$$\det(S_n) = n\left(\sum_{k=1}^{n} e^k\right)\left(\sum_{k=1}^{n} i^k\right).$$

Evaluate the scalar geometric sums (valid since $e \neq 1$ and $i \neq 1$):

$$\sum_{k=1}^{n} e^k = \frac{e^{n+1} - e}{e - 1}, \qquad \sum_{k=1}^{n} i^k = \frac{i^{n+1} - i}{i - 1}.$$

Therefore

$$\boxed{\det\left(\sum_{k=1}^{n} A^k\right) = n \cdot \frac{e^{n+1} - e}{e - 1} \cdot \frac{i^{n+1} - i}{i - 1} \quad (n \in \mathbb{N}).}$$

Method 2: Polynomial functional calculus (eigenvalues of $p(A)$)

Let

$$p_n(t) \coloneqq \sum_{k=1}^{n} t^k.$$

Then $\sum_{k=1}^{n} A^k = p_n(A)$. For any eigenpair $Av = \lambda v$ with $v \neq 0$,

$$p_n(A)v = \left(\sum_{k=1}^{n} A^k\right) v = \sum_{k=1}^{n} A^k v = \sum_{k=1}^{n} \lambda^k v = p_n(\lambda)v.$$

Hence $p_n(\lambda)$ is an eigenvalue of $p_n(A)$ for each eigenvalue λ of A. Since A has three distinct eigenvalues $1, e, i$ (thus A is diagonalizable), $p_n(A)$ is also diagonalizable in the same eigenbasis, so the eigenvalues of $p_n(A)$ are exactly $p_n(1), p_n(e), p_n(i)$.

Therefore,

$$\det\left(\sum_{k=1}^{n} A^k\right) = \det(p_n(A)) = p_n(1)\, p_n(e)\, p_n(i) = n \left(\sum_{k=1}^{n} e^k\right) \left(\sum_{k=1}^{n} i^k\right),$$

and evaluating the scalar sums yields the same boxed formula as in Method 1.

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Question 4 (Similarity from shared distinct eigenvalues in dimension 3)

Problem

Let V be a 3-dimensional vector space over $\mathbb{R}$. Let $S, T \in \mathcal{L}(V)$ each have 2, 6, and 7 as eigenvalues. Show that there exists an invertible $R \in \mathcal{L}(V)$ such that

$$S = R^{-1} \circ T \circ R.$$

Solution

Method 1: Build R from aligned eigenbases

Since $2, 6, 7$ are distinct real eigenvalues, both S and T are diagonalizable over $\mathbb{R}$.

Step 1: Choose eigenbases with matched eigenvalue order. Choose nonzero eigenvectors $s_2, s_6, s_7 \in V$ such that

$$S(s_\lambda) = \lambda s_\lambda \quad (\lambda \in \{2, 6, 7\}),$$

and similarly choose nonzero eigenvectors $t_2, t_6, t_7 \in V$ such that

$$T(t_\lambda) = \lambda t_\lambda \quad (\lambda \in \{2, 6, 7\}).$$

Because eigenvalues are distinct, each eigenspace is 1-dimensional, and the three eigenvectors for each operator form a basis:

$$\beta_S = (s_2, s_6, s_7) \text{ is a basis of } V, \qquad \beta_T = (t_2, t_6, t_7) \text{ is a basis of } V.$$

Step 2: Define R by sending the T-eigenbasis to the S-eigenbasis. Define $R : V \to V$ by its action on the basis β_T:

$$R(t_2) = s_2, \qquad R(t_6) = s_6, \qquad R(t_7) = s_7,$$

and extend linearly. Then $R \in \mathcal{L}(V)$ and is invertible because it maps a basis to a basis.

Step 3: Verify the conjugacy identity on a basis. It suffices to verify $S \circ R = R \circ T$ on the basis vectors t_λ:

$$(S \circ R)(t_\lambda) = S(s_\lambda) = \lambda s_\lambda, \qquad (R \circ T)(t_\lambda) = R(\lambda t_\lambda) = \lambda R(t_\lambda) = \lambda s_\lambda.$$

Hence $(S \circ R)(t_\lambda) = (R \circ T)(t_\lambda)$ for $\lambda \in \{2, 6, 7\}$, so $S \circ R = R \circ T$ on V. Right-compose by R^{-1} to obtain

$$S = R \circ T \circ R^{-1}.$$

Equivalently,

$$S = R^{-1} \circ T \circ R$$

after renaming $R \leftarrow R^{-1}$ (both are invertible). Therefore such an invertible operator exists.

$$\boxed{\exists\, R \in \mathcal{L}(V) \text{ invertible such that } S = R^{-1} \circ T \circ R.}$$

Method 2: Reduce both operators to the same diagonal matrix

Because S is diagonalizable with eigenvalues $2, 6, 7$, there exists an ordered basis β_S of V such that

$$[S]_{\beta_S}^{\beta_S} = \text{diag}(2, 6, 7).$$

Similarly, there exists an ordered basis β_T such that

$$[T]_{\beta_T}^{\beta_T} = \text{diag}(2, 6, 7).$$

Let $C : V \to V$ be the change-of-basis isomorphism that maps β_T-coordinates to β_S-coordinates. Then the matrices satisfy the similarity relation

$$[S]_{\beta_S}^{\beta_S} = [C]_{\beta_S}^{\beta_T} [T]_{\beta_T}^{\beta_T} [C]_{\beta_T}^{\beta_S}.$$

Translating back to operators gives $S = C \circ T \circ C^{-1}$, i.e. S is conjugate to T by an invertible map.

$$\boxed{S \sim T \text{ since both are similar to } \text{diag}(2, 6, 7).}$$

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Question 5 (Solving $T(v) = 8v + b$ using $8 \notin \sigma(T)$)

Problem

Let $T \in \mathcal{L}(\mathbb{R}^3)$ be non-diagonalizable with 6 and 7 as its only eigenvalues. Show that there exist $x, y, z \in \mathbb{R}$ such that

$$T(x, y, z) = (6 + 8x,\ 7 + 8y,\ 13 + 8z).$$

Solution

Method 1: Invertibility of $T - 8I$ (since 8 is not an eigenvalue)

Let $b \coloneqq (6, 7, 13) \in \mathbb{R}^3$. The desired conclusion

$$T(x, y, z) = (6 + 8x,\ 7 + 8y,\ 13 + 8z)$$

is equivalent to: there exists $v = (x, y, z) \in \mathbb{R}^3$ such that

$$T(v) = 8v + b \quad \Longleftrightarrow \quad (T - 8I_3)v = b.$$

Thus it suffices to prove that $T - 8I_3$ is invertible (so its range is all of $\mathbb{R}^3$ and every b is attained).

If $T - 8I_3$ were not invertible, then $\det(T - 8I_3) = 0$, i.e. 8 would be an eigenvalue of T. But we are told the *only* eigenvalues of T are 6 and 7, so 8 is not an eigenvalue. Therefore $T - 8I_3$ is invertible. Hence there exists a (unique) $v \in \mathbb{R}^3$ such that $(T - 8I_3)v = b$.

Writing $v = (x, y, z)$ yields the required statement.

$$\boxed{\exists (x, y, z) \in \mathbb{R}^3 :\ T(x, y, z) = (6 + 8x,\ 7 + 8y,\ 13 + 8z).}$$

Method 2: Determinant via the characteristic polynomial (nonzero at 8)

Let $\chi_T(\lambda)$ be the characteristic polynomial of T (degree 3). Since the only eigenvalues are 6 and 7, the characteristic polynomial must be

$$\chi_T(\lambda) = (\lambda - 6)^m (\lambda - 7)^{3-m} \quad \text{for some } m \in \{1, 2\}.$$

(Indeed, T is non-diagonalizable, so one of the eigenvalues has algebraic multiplicity 2 and geometric multiplicity 1, but the following argument does not require specifying which one.)

Now use

$$\det(T - 8I_3) = (-1)^3 \chi_T(8) = -\chi_T(8).$$

Compute:

$$\chi_T(8) = (8 - 6)^m (8 - 7)^{3-m} = 2^m \cdot 1^{3-m} \neq 0.$$

Hence $\det(T - 8I_3) \neq 0$, so $T - 8I_3$ is invertible. Therefore, for $b = (6, 7, 13)$, the linear system

$$(T - 8I_3)v = b$$

has a solution $v = (x, y, z)$, which rearranges to the required identity $T(v) = 8v + b$.

$$\boxed{\det(T - 8I_3) \neq 0 \ \Rightarrow\ \exists v \in \mathbb{R}^3 :\ (T - 8I_3)v = (6, 7, 13)}.$$