MH1201 Linear Algebra II - Cheat Sheet **QRS@NTU**

Vector Spaces (Ch1)

Subspaces, span, sums

Vector-space facts: $-(-u)=u$, $0_{\mathbb{F}}u=0_V$, $(-1)u=-u$, $c0_V=0_V$, $u-v=u+(-v)$.
Subspace test: $W\le V \iff W\neq\varnothing$ and $u,v\in W\Rightarrow u+v\in W$, $c\in\mathbb{F}$, $u\in W\Rightarrow cu\in W$.
One-line test: $W\le V\iff W\neq\varnothing$ and $cu+v\in W$ for all $u,v\in W$, $c\in\mathbb{F}$.
Fast disproof: $0\notin W\Rightarrow W\not\le V$. If $W\le V$, then $0\in W$, $u\in W\Rightarrow -u\in W$, $u,v\in W\Rightarrow u-v\in W$.
Kernel trick: $T:V\to U$ linear $\Rightarrow \operatorname{Ker}(T)=\{v:T(v)=0\}\le V$. Homogeneous linear conditions usually define subspaces.
Span: $\operatorname{span}(S)$ = set of all finite linear combinations of vectors in S; $\operatorname{span}(\varnothing)=\{0\}$. Smallest-subspace property: $S\subseteq W\le V\Rightarrow \operatorname{span}(S)\subseteq W$.
Span identities: $A\subseteq B\Rightarrow \operatorname{span}(A)\subseteq\operatorname{span}(B)$; $\operatorname{span}(\operatorname{span}(A))=\operatorname{span}(A)$; $\operatorname{span}(A\cup B)=\operatorname{span}(A)+\operatorname{span}(B)$.
Subspace ops: If $U,W\le V$, then $U\cap W\le V$ and $U+W=\operatorname{span}(U\cup W)\le V$.
Union criterion: $W_1\cup W_2\le V\iff (W_1\subseteq W_2)$ or $(W_2\subseteq W_1)$.
Affine/coset test: $\{\sum_i\lambda_i a_i+b\}\le V\iff b\in\operatorname{span}\{a_i\}$. If $W\le V$, then $w_0+W\le V\iff w_0\in W$ (then $w_0+W=W$).

Direct sums

Definition: $V_1\oplus\cdots\oplus V_n$ means every $v\in V_1+\cdots+V_n$ has a unique decomposition $v=v_1+\cdots+v_n$ with $v_i\in V_i$.
Zero-sum test: $V_1+\cdots+V_n$ is direct $\iff v_1+\cdots+v_n=0$, $v_i\in V_i\Rightarrow v_1=\cdots=v_n=0$.
2-subspace test: $V=U\oplus W\iff V=U+W$ and $U\cap W=\{0\}$.
Intersection form: $V_1+\cdots+V_n$ direct $\iff V_i\cap\left(\sum_{j\neq i}V_j\right)=\{0\}$ for all i.
Warning: Pairwise trivial intersections are not enough when $n\ge 3$.
LI link: If S is LI and $S=A\sqcup B$, then $\operatorname{span}(S)=\operatorname{span}(A)\oplus\operatorname{span}(B)$.
Sum-map view: $\phi:V_1\times\cdots\times V_n\to V$, $\phi(v_1,\dots,v_n)=\sum_i v_i$. Then $\operatorname{im}(\phi)=\sum_i V_i$ and the sum is direct $\iff\operatorname{Ker}(\phi)=\{0\}$.

Finite-Dimensional Spaces (Ch2)

LI, basis, coordinates

LI: S is LI $\iff \lambda_1v_1+\cdots+\lambda_kv_k=0$ (distinct $v_i\in S$) implies $\lambda_1=\cdots=\lambda_k=0$.
Dependence criterion: S dependent $\iff\exists v\in S$ with $v\in\operatorname{span}(S\setminus\{v\})$. In particular, $0\in S\Rightarrow S$ dependent.
Quick facts: $\varnothing$ is LI; $\{v\}$ is LI $\iff v\neq 0$; LI $\Rightarrow$ every subset LI; dependent $\Rightarrow$ every superset dependent.
Column test: For $A=[v_1\ \cdots\ v_n]$ with $v_i\in\mathbb{F}^m$: $\{v_i\}$ LI $\iff A\lambda=0$ has only the trivial solution $\iff\operatorname{rref}(A)$ has a pivot in every column.
Basis: B basis of $V\iff B$ is LI and $\operatorname{span}(B)=V$.
Unique expansion: If $\beta=(b_1,\dots,b_n)$ is an ordered basis and $v=\sum_i c_ib_i$, then $[v]_\beta=(c_1,\dots,c_n)^{\mathsf T}$. Order matters.
Determining a basis: In an n-dimensional space, any n vectors form a basis $\iff$ they are LI $\iff$ they span. If $W\le V$ with $\dim W=k$, then any k LI vectors in W form a basis of W, and any k spanning vectors of W form a basis of W.

Dimension logic

Bounds: If $\dim V=n$, then every LI set has size $\le n$ and every spanning set has size $\ge n$.
Extension/extraction: Any LI set extends to a basis; any spanning set contains a basis.
Add-one-more: If S is LI, then $S\cup\{v\}$ is LI $\iff v\notin\operatorname{span}(S)$.
Subspace dim: $W\le V$, $\dim V<\infty\Rightarrow\dim W\le\dim V$, with equality $\iff W=V$.
Dimension formula: $\dim(U+W)=\dim U+\dim W-\dim(U\cap W)$. Hence $\dim(U\cap W)\ge\dim U+\dim W-\dim V$.
Direct sum dim: $V=U\oplus W\iff V=U+W$ and $\dim V=\dim U+\dim W$.
Standard dims: $\dim\mathbb{F}^n=n$, $\dim\mathbb{F}^{m\times n}=mn$, $\dim P_n(\mathbb{F})=n+1$, $\dim P(\mathbb{F})=\infty$.
Common matrix dims: $\dim\{A\in\mathbb{R}^{n\times n}:A^{\mathsf T}=A\}=\frac{n(n+1)}{2}$, $\dim\{A\in\mathbb{R}^{n\times n}:A^{\mathsf T}=-A\}=\frac{n(n-1)}{2}$, $\dim\{A\in\mathbb{F}^{n\times n}:\operatorname{tr}(A)=0\}=n^2-1$.
Codim-1 kernel: If $\ell:W\to\mathbb{F}$ is nonzero linear and $\dim W<\infty$, then $\dim\operatorname{Ker}(\ell)=\dim W-1$ and, for any p with $\ell(p)\neq 0$, $W=\operatorname{Ker}(\ell)\oplus\operatorname{span}\{p\}$.

Linear Transformations (Ch3)

Kernel, range, rank-nullity

Linearity: $T:V\to W$ linear $\iff T(u+v)=T(u)+T(v)$ and $T(cu)=cT(u)\iff T(\alpha u+\beta v)=\alpha T(u)+\beta T(v)$.
Basic consequences: $T(0)=0$, $T(-v)=-T(v)$, and $T(\sum_i c_iv_i)=\sum_i c_iT(v_i)$.
Determined by basis: If $\beta=(b_i)$ is a basis of V, specifying $T(b_i)$ determines a unique linear map T.
Kernel/range: $\operatorname{Ker}(T)=\{v:T(v)=0\}\le V$, $\operatorname{Range}(T)=\{T(v):v\in V\}\le W$.
Rank-nullity: If $\dim V<\infty$, then $\dim V=\operatorname{rank}(T)+\operatorname{nullity}(T)$, where $\operatorname{rank}(T)=\dim\operatorname{Range}(T)$ and $\operatorname{nullity}(T)=\dim\operatorname{Ker}(T)$.
Injective tests: T injective $\iff\operatorname{Ker}(T)=\{0\}\iff\operatorname{nullity}(T)=0\iff\operatorname{rank}(T)=\dim V$.
Surjective tests: T onto $W\iff\operatorname{Range}(T)=W\iff\operatorname{rank}(T)=\dim W$.
Equal-dim rule: If $\dim V=\dim W<\infty$, then injective $\iff$ surjective $\iff$ bijective $\iff$ invertible.
Square-matrix chain: For $A\in\mathbb{F}^{n\times n}$: $\det A\neq 0\iff A$ invertible $\iff\operatorname{rank}A=n\iff\operatorname{nullity}A=0\iff\operatorname{Ker}A=\{0\}\iff\operatorname{Range}A=\mathbb{F}^{n,1}\iff 0$ is not an eigenvalue of A.
Composition: $T\circ S$ injective $\Rightarrow S$ injective; $T\circ S$ surjective $\Rightarrow T$ surjective; if S,T invertible, then $(T\circ S)^{-1}=S^{-1}\circ T^{-1}$.
Preimages: If $U\le W$, then $T^{-1}(U)=\{v:T(v)\in U\}\le V$.
Isomorphism theorem: $V/\operatorname{Ker}(T)\cong\operatorname{Range}(T)$.

High-yield templates

Find T from basis values: Write $v=\sum_i c_ib_i$, then $T(v)=\sum_i c_iT(b_i)$.
Kernel/range workflow: Solve $T(v)=0$ for a basis of $\operatorname{Ker}(T)$; parametrize $T(v)$ or use pivot columns / images of basis vectors for a basis of $\operatorname{Range}(T)$.
Idempotent maps: $P^2=P\Rightarrow V=\operatorname{Ker}(P)\oplus\operatorname{Range}(P)$ and $v=(v-Pv)+Pv$. Also $Q=I-P$ satisfies $Q^2=Q$, $\operatorname{Range}(Q)=\operatorname{Ker}(P)$, $\operatorname{Ker}(Q)=\operatorname{Range}(P)$.
Hyperplane trick: If $\ell\neq 0$ linear and $H=\operatorname{Ker}(\ell)$, then $\dim H=\dim V-1$. If $\operatorname{Range}(T)\subseteq H$ and $\dim\operatorname{Range}(T)\ge\dim H$, then $\operatorname{Range}(T)=H$.

Matrices / Coordinates / Similarity (Ch4)

Coordinate maps and matrix reps

Coordinate map: If $\beta=(b_1,\dots,b_n)$ is an ordered basis of V, then $[\cdot]_\beta:V\to\mathbb{F}^{n,1}$ is a linear isomorphism.
Matrix rep: For $T:V\to W$, bases $\beta=(b_1,\dots,b_n)$ of V, $\gamma=(c_1,\dots,c_m)$ of W: $[T]^\gamma_\beta=\left[[T(b_1)]_\gamma\ \cdots\ [T(b_n)]_\gamma\right]\in\mathbb{F}^{m\times n}$.
Fundamental identity: $[T(v)]_\gamma=[T]^\gamma_\beta[v]_\beta$.
Composition: $[T\circ S]^\gamma_\alpha=[T]^\gamma_\beta[S]^\beta_\alpha$ (the middle basis must match).
Inverse: If T invertible, then $[T^{-1}]^\beta_\gamma=([T]^\gamma_\beta)^{-1}$.
Standard matrix: If β,γ are standard bases, $[T]^\gamma_\beta$ is the standard matrix of T. In polynomial spaces: columns are coordinate vectors of images of $1,X,X^2,\dots$.
RREF criteria: For $A=[T]^\gamma_\beta\in\mathbb{F}^{m\times n}$: T injective $\iff\operatorname{rref}(A)$ has a pivot in every column; T onto $W\iff\operatorname{rref}(A)$ has a pivot in every row.

Change of basis / similarity

Coord conversion: $[v]_\gamma=[I_V]^\gamma_\beta[v]_\beta$ and $[I_V]^\beta_\gamma=([I_V]^\gamma_\beta)^{-1}$.
Columns of $[I]$: The j-th column of $[I_V]^\gamma_\beta$ is $[b_j]_\gamma$. It converts β-coordinates to γ-coordinates.
Using standard basis: If $P_\beta=[I_V]^\sigma_\beta=\left[[b_1]_\sigma\ \cdots\ [b_n]_\sigma\right]$, then $[I_V]^\gamma_\beta=P_\gamma^{-1}P_\beta$.
Change-of-rep formula: $[T]^\gamma_\gamma=[I_V]^\gamma_\beta[T]^\beta_\beta[I_V]^\beta_\gamma$. If $Q=[I_V]^\beta_\gamma$, then $[T]^\gamma_\gamma=Q^{-1}[T]^\beta_\beta Q$.
Similarity: $A\sim B\iff\exists Q\in GL_n(\mathbb{F})$ with $B=Q^{-1}AQ$. Similarity = same operator in different bases.
Similarity invariants: det, tr, characteristic polynomial, eigenvalues (with multiplicity), rank, nullity. One differing invariant proves $A\not\sim B$.
Useful identities: $\det(Q^{-1}AQ)=\det(A)$, $\operatorname{tr}(Q^{-1}AQ)=\operatorname{tr}(A)$, $\operatorname{tr}(XY)=\operatorname{tr}(YX)$.
2×2 char poly: If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then $p_A(\lambda)=\lambda^2-\operatorname{tr}(A)\lambda+\det(A)$.

Eigenvalues / Diagonalization (Ch5)

Eigenvalues and eigenspaces

Definitions: λ eigenvalue of $T\iff\exists v\neq 0$ with $T(v)=\lambda v\iff E_{T,\lambda}=\operatorname{Ker}(T-\lambda I)\neq\{0\}$.
Matrix test: If $A=[T]^\beta_\beta$ on an n-dimensional space, then λ eigenvalue $\iff\det(A-\lambda I_n)=0$.
Characteristic poly: $p_T(\lambda)=\det(A-\lambda I_n)$ is basis-independent. Its roots in the base field are the eigenvalues.
Quick consequences: 0 eigenvalue $\iff T$ not injective. If $V=W$ finite-dim, then 0 eigenvalue $\iff T$ not invertible $\iff\det(A)=0$.
Triangular matrices: If A is triangular, its eigenvalues are its diagonal entries (with algebraic multiplicities).
Distinct eigenvalues: Eigenvectors for distinct eigenvalues are LI; hence $E_{\lambda_1}+\cdots+E_{\lambda_k}=E_{\lambda_1}\oplus\cdots\oplus E_{\lambda_k}$.
Field warning: Over $\mathbb{R}$, a matrix/operator may have no real eigenvalues even if it has complex eigenvalues.

Diagonalizability criteria

Definition: T diagonalizable $\iff V$ has a basis of eigenvectors $\iff[T]^\beta_\beta$ is diagonal for some basis β.
Constructing PDP^{-1}: If columns of P are eigenvectors $v_1,\dots,v_n$ and corresponding eigenvalues are $\lambda_1,\dots,\lambda_n$, then $A=PDP^{-1}$ with $D=\operatorname{diag}(\lambda_1,\dots,\lambda_n)$.
Core criterion: T diagonalizable $\iff\dim V=\sum_\lambda\dim E_{T,\lambda}$.
Multiplicity criterion: If p_T splits over $\mathbb{F}$, then T diagonalizable $\iff$ geomult$(\lambda)=$ algmult(λ) for every eigenvalue λ. Always $1\le$ geomult$(\lambda)\le$ algmult(λ).
Easy sufficient test: If $\dim V=n$ and T has n distinct eigenvalues in $\mathbb{F}$, then T is diagonalizable.
Repeated-root warning: Repeated eigenvalue does not decide diagonalizability; compute eigenspaces. Distinct eigenvalues are sufficient, not necessary.
Powers: If $A=PDP^{-1}$, then $A^k=PD^kP^{-1}$ with $D^k=\operatorname{diag}(\lambda_1^k,\dots,\lambda_n^k)$.
Polynomial calculus: If $Av=\lambda v$, then $p(A)v=p(\lambda)v$. If $A=PDP^{-1}$, then $p(A)=P\operatorname{diag}(p(\lambda_1),\dots,p(\lambda_n))P^{-1}$.
Distinct-root annihilator: If $p(T)=0$ and p splits into distinct linear factors over $\mathbb{F}$, then T is diagonalizable. In particular, $T^2=I$ implies diagonalizable when $\operatorname{char}(\mathbb{F})\neq 2$.

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Nilpotent facts: T nilpotent means $T^k = 0$ for some k. Then the only eigenvalue is 0. A nilpotent operator is diagonalizable $\iff T = 0$; hence any nonzero nilpotent operator is not diagonalizable.

Inner Product Spaces (Ch6)

Inner products

Definition: $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ is an inner product iff:
(1) Linearity / Sesquilinearity: Real: bilinear;
Complex: conjugate-linear in first $\langle au_1 + bu_2, v\rangle = \overline{a}\langle u_1, v\rangle + \overline{b}\langle u_2, v\rangle$, linear in second $\langle u, av_1 + bv_2\rangle = a\langle u, v_1\rangle + b\langle u, v_2\rangle$
(2) Symmetry: Real: $\langle u, v\rangle = \langle v, u\rangle$; Complex: $\langle u, v\rangle = \overline{\langle v, u\rangle}$
(3) Positive definite: $\langle u, u\rangle \geq 0$, and $= 0 \iff u = 0$
Fast test (matrix form): $\langle x, y\rangle = x^\top G y$ (real), $x^\dagger H y$ (complex)
Real: inner product $\iff G = G^\top$ and $G \succ 0$;
Complex: $H = H^\dagger$ and $H \succ 0$
Sylvester's Criterion: $H \succ 0 \iff \det(H_k) > 0\ \forall k$ (Hermitian H)
Spectral Theorem: $H = H^\dagger \Rightarrow H = U\Lambda U^\dagger$, so $x^\dagger H x = \sum \lambda_i |y_i|^2$ in eigenbasis
Eigenvalue test: $H \succ 0 \iff \lambda_i(H) > 0\ \forall i$; $H \succeq 0 \iff \lambda_i(H) \geq 0$
Gram Matrix: $G_{ij} = \langle v_i, v_j\rangle \Rightarrow x^\dagger G x = \|\sum x_i v_i\|^2 \geq 0 \Rightarrow G \succeq 0$
$A^\dagger A$ **PSD:** $x^\dagger (A^\dagger A) x = \|Ax\|^2 \geq 0 \Rightarrow A^\dagger A \succeq 0$
Cholesky Decomposition:
$H \succ 0 \iff H = LL^\dagger$ with L lower triangular
(constructive PD test) $H = \begin{bmatrix} a & b \\ \overline{b} & c \end{bmatrix} \Rightarrow L = \begin{bmatrix} \sqrt{a} & 0 \\ \dfrac{b}{\sqrt{a}} & \sqrt{c - \dfrac{|b|^2}{a}} \end{bmatrix}$
then $x^\dagger H x = \|Lx\|^2 > 0$ for $x \neq 0$.
Equiv. $H = R^\dagger R$ with R invertible upper triangular;
then $x^\dagger H x = \|Rx\|^2 > 0$ for $x \neq 0$.
Cauchy–Schwarz: $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$
Rayleigh Quotient: $R(x) = \frac{x^\dagger H x}{x^\dagger x}$ satisfies $\lambda_{\min} \leq R(x) \leq \lambda_{\max}$
Polarisation link: Norm determines inner product via polarisation; see Norms section. Useful when proving two inner products are equal from equal norms.

Norms and identities

Norm & distance: $\|u\| = \sqrt{\langle u, u\rangle}$, $\quad d(u, v) = \|u - v\|$
Norm expansion (±): $\|u \pm v\|^2 = \|u\|^2 + \|v\|^2 \pm 2\,\mathrm{Re}\langle u, v\rangle$
Real: $\mathrm{Re}\langle u, v\rangle = \langle u, v\rangle$
Parallelogram Identity: $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$
Polarisation Identity: Recover $\langle x, y\rangle$ from norm.
$\Re\langle x, y\rangle = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$; $\Im\langle x, y\rangle = \frac{1}{4}(\|x + iy\|^2 - \|x - iy\|^2)$
Cauchy–Schwarz: $|\langle u, v\rangle| \leq \|u\|\,\|v\|$, equality $\iff u, v$ LD
Triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$
Pythagoras: $u \perp v \Rightarrow \|u + v\|^2 = \|u\|^2 + \|v\|^2$

Orthogonality and expansions

Orthogonality: $u \perp v \iff \langle u, v\rangle = 0$; orthogonal set = pairwise orthogonal; orthonormal (ON) = orthogonal + $\|u\| = 1$; orthogonal nonzero set $\Rightarrow$ LI
ONB expansion: If $\beta = (e_i)$ is ONB, then $v = \sum_i \langle e_i, v\rangle e_i$; Fourier coefficients: $c_i = \langle e_i, v\rangle$
Orthogonal basis expansion: If $\{u_i\}$ is orthogonal and nonzero, then $v = \sum_i \dfrac{\langle u_i, v\rangle}{\langle u_i, u_i\rangle} u_i$
Fourier coefficients: ONB: $c_i = \langle e_i, v\rangle$; orthogonal basis: $c_i = \dfrac{\langle u_i, v\rangle}{\langle u_i, u_i\rangle}$.
Use after normalizing GS outputs; no need to solve a coordinate system.
Parity shortcut: On $[-a, a]$, odd integrals vanish; even–odd functions are orthogonal.
For $\langle f, g\rangle = \int_{-1}^{1} fg$, GS on $(1, x, x^2, x^3)$ splits into even block $(1, x^2)$ and odd block (x, x^3).
Legendre shortcut: For $\langle f, g\rangle = \int_{-1}^{1} fg$, GS on $(1, x, x^2, \dots)$ gives Legendre polynomials.
$L_0 = 1,\ L_1 = x,\ L_2 = \frac{1}{2}(3x^2 - 1),\ L_3 = \frac{1}{2}(5x^3 - 3x)$;
$\int_{-1}^{1} L_m L_n = 0$ if $m \neq n$, and $\|L_n\|^2 = \dfrac{2}{2n + 1}$.
Legendre ONB: $\phi_n = \sqrt{\dfrac{2n+1}{2}} L_n$ is ON under $\int_{-1}^{1} fg$.
Useful identities: $x^2 = \dfrac{1}{3} + \dfrac{2}{3}L_2$, $x^3 = \dfrac{3}{5}L_1 + \dfrac{2}{5}L_3$.

Projection, Gram–Schmidt, complements

Proj onto vector: $\mathrm{proj}_u(v) = \frac{\langle u, v\rangle}{\langle u, u\rangle} u$, $u \neq 0$; $v - \mathrm{proj}_u(v) \perp u$
Proj onto subspace: If ONB (b_i) of W, then $P_W(v) = \sum_i \langle b_i, v\rangle b_i$ and $v - P_W(v) \in W^\perp$
Best approx: $P_W(v)$ is the unique minimizer of $\|v - w\|$ over $w \in W$; condition: $v - w \in W^\perp$
Gram–Schmidt: $v_1 = b_1$, $v_k = b_k - \sum_{j<k} \mathrm{proj}_{v_j}(b_k)$; $g_k = v_k / \|v_k\|$ gives ONB.
Keep v_k unnormalized until the end to simplify coefficients.
GS on orthogonal list: If (b_i) already orthogonal, all projection terms vanish; non-normalized GS returns $v_i = b_i$.
If normalized GS is required, output is $g_i = b_i / \|b_i\|$.
Nested spans: GS output satisfies $\mathrm{span}(v_1, \dots, v_k) = \mathrm{span}(b_1, \dots, b_k)$ for all k
GS preserves flags: If $F_k = \mathrm{span}(b_1, \dots, b_k)$, then GS gives $\mathrm{span}(g_1, \dots, g_k) = F_k$.
So GS changes the basis but preserves the same nested subspaces.
Upper triangular flag: For basis $\beta = (b_i)$, $[T]_\beta^\beta$ upper triangular $\iff T(F_k) \subseteq F_k$ for all k, where $F_k = \mathrm{span}(b_1, \dots, b_k)$.
Flag-slice ONB: For a flag $F_1 \subset \cdots \subset F_n$, choose $g_k \in F_k \cap F_{k-1}^\perp$, $\|g_k\| = 1$.
Then (g_k) is ON and $\mathrm{span}(g_1, \dots, g_k) = F_k$.
QR: GS on columns of A gives $A = QR$, where Q has ON columns and R is upper triangular; $r_{ij} = \langle q_i, a_j\rangle$ for $i \leq j$
QR + triangularity: If $B = [T]_\beta^\beta$ is upper triangular and the GS change matrix is upper triangular, then $[T]_\gamma^\gamma = RBR^{-1}$ is upper triangular.
Orthogonal complement: $W^\perp = \{v : \langle w, v\rangle = 0\ \forall w \in W\}$; $A^\perp = (\mathrm{span}\, A)^\perp$
Finite-dim facts: $V = W \oplus W^\perp$; $\dim W + \dim W^\perp = \dim V$; $W \cap W^\perp = \{0\}$; $(W^\perp)^\perp = W$
Double complement warning: Always $W \subseteq W^{\perp\perp}$. Finite dim: $W^{\perp\perp} = W$.
Hilbert space: $W^{\perp\perp} = \overline{W}$; equality can fail if W is not closed.
Kernel-row theorem: For real matrix A, $\mathrm{Null}(A)^\perp = \mathrm{Row}(A)$.
Thus $W = \{x : Ax = 0\} \Rightarrow W^\perp = \mathrm{Row}(A)$.
Hyperplane normal: If $W = \{x : n \cdot x = 0\} = n^\perp$, then $W^\perp = \mathrm{span}\{n\}$.
One equation $\Rightarrow$ orthogonal complement is the span of the coefficient/normal vector.
Hyperplane projection: If $W = \{x : n \cdot x = 0\}$ and $p = a + b$, $a \in W$, $b \in W^\perp$, then
$b = \dfrac{p \cdot n}{n \cdot n} n, \qquad a = p - b.$
Least squares: $y - Ab \in \mathrm{Range}(A)^\perp \iff A^\top(y - Ab) = 0 \iff A^\top Ab = A^\top y$

Standard workflows

Subspace / direct sum: To show $W \leq V$: use subspace test, kernel trick, or $W = \mathrm{span}(S)$.
To show $V = U \oplus W$: prove $V = U + W$ and $U \cap W = \{0\}$, or use dimensions.
LI / basis / dim: Put vectors as columns and row-reduce. In an n-dimensional space: n vectors form a basis iff LI iff spanning. Use extension/extraction aggressively.
Kernel / range: Solve $T(v) = 0$ for $\mathrm{Ker}(T)$. For $\mathrm{Range}(T)$, use pivot columns of a matrix rep or images of basis vectors, then apply rank-nullity.
Matrix rep: Compute T on the domain basis vectors, express each image in the codomain basis, place those coordinate vectors as columns.
Change of basis: Build $[I_V]_\beta^\gamma$ from columns $[b_j]_\gamma$; if both bases are in standard coordinates, use $P_\gamma^{-1} P_\beta$.
Diagonalize: Compute $p_A(\lambda)$, find eigenvalues, compute each eigenspace, compare $\sum_\lambda \dim E_\lambda$ with n; then assemble P from eigenvectors and D from matching eigenvalues.
Projection / least squares: Find an ONB of W first. Then $P_W(v) = \sum \langle b_i, v\rangle b_i$.
For $Ax \approx y$, use $y - Ab \in \mathrm{Range}(A)^\perp \iff A^\top(y - Ab) = 0 \iff A^\top A\, b = A^\top y$.
Gram–Schmidt: Keep unnormalized v_k until the end; projection coefficients are $\langle v_j, b_k\rangle / \langle v_j, v_j\rangle$; normalize only after orthogonalization.
Fast IP-space recognition: Already orthogonal $\Rightarrow$ GS unchanged; matrix IP $\Rightarrow$ check symmetric/Hermitian PD;
hyperplane $n \cdot x = 0 \Rightarrow W^\perp = \mathrm{span}\{n\}$; Frobenius diagonal complement $\Rightarrow$ zero diagonal.
Fast theorem choices: Upper triangular $\Rightarrow$ invariant flag; polynomial IP on $[-1, 1] \Rightarrow$ parity/Legendre;
finite dim complement $\Rightarrow W^{\perp\perp} = W$; infinite dim $\Rightarrow$ check closure.
Inner product proof workflow: Direct definition: check linear/sesquilinear, symmetry/conjugate symmetry, positive definite.
Matrix form: check $G = G^\top$ or $H = H^\dagger$, then use Sylvester/eigenvalues/Cholesky/squares for PD.
Orthogonal complement workflow: If $W = \mathrm{span}(S)$, solve $\langle x, s\rangle = 0$ for all $s \in S$.
If $W = \{x : Ax = 0\}$, use $W^\perp = \mathrm{Row}(A)$.
If $W = n^\perp$, use $W^\perp = \mathrm{span}\{n\}$.

High-yield traps

Subspaces: Must contain 0. Kernels and homogeneous solution spaces are subspaces; affine translates usually are not.
Coordinates: Distinguish vectors from coordinate columns. Order of basis matters. $[T]_\beta^\gamma$ means input in β-coordinates, output in γ-coordinates.
Maps vs matrices: Operator statements are basis-free; matrices depend on bases. Similar matrices represent the same operator in different bases.
Rank-nullity: Injective $\Leftrightarrow$ surjective requires equal finite dimensions.
Diagonalization: Repeated eigenvalue does not imply diagonalizable. Check eigenspace dimensions; compare geometric vs algebraic multiplicity.
Orthogonality: Orthogonal $\neq$ orthonormal. Fourier coefficient formula $v = \sum \langle u, v\rangle u$ needs an ONB; for merely orthogonal bases divide by $\langle u_i, u_i\rangle$. Projection onto u requires $u \neq 0$.
Field issues: Over $\mathbb{R}$, some matrices/operators have no real eigenvalues. Over $\mathbb{C}$, keep conjugation in the correct slot of the inner product.