MH1301 Discrete Mathematics

Tutorial 4

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Overview

This tutorial develops the standard counting techniques from Chapter 6.5, with emphasis on:

- arrangements under symmetry constraints (hexagonal seating around a table);
- stars-and-bars for equations and inequalities in non-negative integers;
- digit-sum counting with upper bounds handled by inclusion-exclusion;
- ordered selection, multisets, and assignments with repetition allowed;
- multinomial counting and equipartition of distinct books across labelled shelves;
- shelf-placement models with indistinguishable or distinguishable books;
- partitions of an integer into indistinguishable boxes with lower bounds.

Question map.

- Additional exercise: hexagonal seating with six boys and six girls.
- Ex. 6.5.11: restricted solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 21$.
- Ex. 6.5.14 and 6.5.19: inequality counting and digit sums.
- Ex. 6.5.1, 6.5.4, 6.5.12: repetition allowed, multinomial strings, equal shelf sizes.
- Ex. 6.5.32 and 6.5.59: shelf models and indistinguishable boxes.

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Question 1 (Additional Exercise: Hexagonal seating)

Problem

How many seating arrangements are there with six boys and six girls if the table is hexagonal with 2 seats on each side, out of which one must be occupied by a boy and the other one by a girl?

Solution

We follow the tutorial convention for circular seatings: arrangements that differ only by a rotation of the hexagon are identified, whereas mirror images are regarded as different unless a reflection is explicitly specified.

Method 1: Label the seats first, then quotient by rotational symmetry

Temporarily distinguish all 12 seat positions by fixing the six sides of the hexagon and the two seats on each side.

For each side, exactly one of its two seats must be occupied by a boy and the other by a girl. Hence, for each of the six sides, there are exactly

$$2$$

possible gender patterns. Therefore the total number of gender patterns over all six sides is

$$2^6.$$

Once the boy-seats and girl-seats have been determined, we may assign the six distinct boys to the six boy-seats in

$$6!$$

ways, and independently assign the six distinct girls to the six girl-seats in

$$6!$$

ways. Thus the number of seatings on a *fully labelled* hexagon is

$$2^6(6!)(6!).$$

Now pass from the labelled hexagon to the actual round-the-table arrangement. A rotation of the hexagon through $0°, 60°, 120°, 180°, 240°, 300°$ produces the same circular seating. Since all 12 people are distinct, no nontrivial rotation can fix a labelled seating. Consequently each circular seating is represented by exactly

$$6$$

labelled seatings.

Hence the required number is

$$\frac{2^6(6!)^2}{6} = \frac{64 \cdot 720^2}{6} = 5{,}529{,}600.$$

$$\boxed{5{,}529{,}600}$$

Method 2: Count labelled valid seatings, then divide by 6

$$N = \frac{\#\{\text{labelled valid seatings}\}}{6}.$$

For each of the 6 sides, there are 2 choices for which seat is occupied by a boy, so

$$\#\{\text{side-patterns}\} = 2^6.$$

Then assign the 6 distinct boys and 6 distinct girls:

$$\#\{\text{labelled valid seatings}\} = 2^6 \cdot 6! \cdot 6!.$$

Hence

$$N = \frac{2^6(6!)^2}{6} = 5{,}529{,}600.$$

$$\boxed{5{,}529{,}600}$$

Method 3: Burnside's lemma on the rotational action of the hexagon

Let X be the set of all seatings on a *labelled* hexagon satisfying the condition that on each side there is one boy and one girl. As in Method 1,

$$|X| = 2^6(6!)^2.$$

The rotation group of the hexagon is the cyclic group

$$C_6 = \{\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5\},$$

where ρ_j denotes rotation through $60j^\circ$.

The required number of circular seatings is the number of orbits of X under this group action. By Burnside's lemma,

$$\#\text{orbits} = \frac{1}{6}\sum_{j=0}^{5}\big|\text{Fix}(\rho_j)\big|.$$

We compute the fixed-point counts:

- For the identity rotation ρ_0, every seating is fixed, so
$$\big|\text{Fix}(\rho_0)\big| = |X| = 2^6(6!)^2.$$
- For any nonidentity rotation ρ_j with $j \in \{1, 2, 3, 4, 5\}$, a seating fixed by ρ_j would have to repeat every person after a nontrivial rotation. Since all 12 people are distinct, that is impossible. Therefore
$$\big|\text{Fix}(\rho_j)\big| = 0 \qquad (j = 1, 2, 3, 4, 5).$$

Thus

$$\#\text{orbits} = \frac{1}{6}\Big(2^6(6!)^2 + 0 + 0 + 0 + 0 + 0\Big) = \frac{2^6(6!)^2}{6} = 5{,}529{,}600.$$

Therefore,

$$\boxed{5{,}529{,}600}$$

Question 2 (Ex. 6.5.11: Restricted non-negative solutions)

Problem

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = 1, 2, 3, 4, 5$, is a non-negative integer, such that

(a) $x_1 \geq 1$?

(b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?

(c) $0 \leq x_1 \leq 10$?

(d) $0 \leq x_1 \leq 3$ and $x_2 \geq 15$?

Solution

Method 1: Stars-and-bars with variable shifts and subtraction

(a) Since $x_1 \geq 1$, write

$$x_1 = 1 + y_1, \qquad y_1 \geq 0.$$

Then the equation becomes

$$y_1 + x_2 + x_3 + x_4 + x_5 = 20.$$

The number of non-negative integer solutions in five variables is, by stars-and-bars,

$$\binom{20 + 5 - 1}{5 - 1} = \binom{24}{4} = 10626.$$

Therefore

$$\boxed{10626}.$$

(b) Since every variable must satisfy $x_i \geq 2$, write

$$x_i = 2 + y_i \qquad (i = 1, 2, 3, 4, 5), \qquad y_i \geq 0.$$

Then

$$(2 + y_1) + \cdots + (2 + y_5) = 21$$

reduces to

$$y_1 + y_2 + y_3 + y_4 + y_5 = 11.$$

Hence the number of solutions is

$$\binom{11 + 5 - 1}{4} = \binom{15}{4} = 1365.$$

Therefore

$$\boxed{1365}.$$

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(c) First count all non-negative solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

which is

$$\binom{21+5-1}{4} = \binom{25}{4}.$$

Now subtract the solutions with $x_1 \geq 11$. For those, write

$$x_1 = 11 + y_1, \qquad y_1 \geq 0.$$

Then the equation becomes

$$y_1 + x_2 + x_3 + x_4 + x_5 = 10,$$

so the number of such solutions is

$$\binom{10+5-1}{4} = \binom{14}{4}.$$

Therefore the number with $0 \leq x_1 \leq 10$ is

$$\binom{25}{4} - \binom{14}{4} = 12650 - 1001 = 11649.$$

Hence

$$\boxed{11649}.$$

(d) First impose $x_2 \geq 15$ by writing

$$x_2 = 15 + y_2, \qquad y_2 \geq 0.$$

Then the equation becomes

$$x_1 + y_2 + x_3 + x_4 + x_5 = 6.$$

Ignoring for the moment the restriction $x_1 \leq 3$, the number of non-negative solutions is

$$\binom{6+5-1}{4} = \binom{10}{4} = 210.$$

Now subtract the solutions with $x_1 \geq 4$. Write

$$x_1 = 4 + z_1, \qquad z_1 \geq 0.$$

Then

$$z_1 + y_2 + x_3 + x_4 + x_5 = 2,$$

which has

$$\binom{2+5-1}{4} = \binom{6}{4} = 15$$

solutions. Therefore the number with $0 \leq x_1 \leq 3$ and $x_2 \geq 15$ is

$$\binom{10}{4} - \binom{6}{4} = 210 - 15 = 195.$$

Thus

$$\boxed{195}.$$

Method 2: Counting notation with a general lower-bound formula

$$\text{Let } N(\mathcal{C}) = \#\{x \in \mathbb{Z}^5_{\geq 0} : \ x_1 + \cdots + x_5 = 21, \ \mathcal{C}\}.$$

$$x_i \geq l_i \ \Rightarrow \ x_i = l_i + y_i, \qquad y_i \geq 0.$$

$$N(x_1 \geq l_1, \ldots, x_5 \geq l_5) = \binom{25 - \sum_{i=1}^{5} l_i}{4}.$$

(a) Since $x_1 \geq 1$,

$$N(x_1 \geq 1) = \binom{24}{4} = 10626.$$

(b) Since $x_i \geq 2$ for all i,

$$N(x_1 \geq 2, \ldots, x_5 \geq 2) = \binom{15}{4} = 1365.$$

(c)

$$N(0 \leq x_1 \leq 10) = N(\text{all}) - N(x_1 \geq 11) = \binom{25}{4} - \binom{14}{4} = 11649.$$

(d)

$$N(0 \leq x_1 \leq 3, \ x_2 \geq 15) = N(x_2 \geq 15) - N(x_1 \geq 4, \ x_2 \geq 15).$$

$$N(x_2 \geq 15) = \binom{10}{4}, \qquad N(x_1 \geq 4, \ x_2 \geq 15) = \binom{6}{4}.$$

$$\binom{10}{4} - \binom{6}{4} = 195.$$

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Method 3: Generating functions

For non-negative integer solutions, the generating function for one unrestricted variable is

$$1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

Hence a system with five unrestricted non-negative variables summing to 21 is encoded by

$$\left(\frac{1}{1-x}\right)^5.$$

We use the standard coefficient identity

$$[x^m](1-x)^{-5} = \binom{m+4}{4}.$$

(a) The condition $x_1 \geq 1$ contributes generating function

$$x + x^2 + x^3 + \cdots = \frac{x}{1-x}.$$

Therefore the desired number is

$$[x^{21}]\,\frac{x}{(1-x)^5} = [x^{20}](1-x)^{-5} = \binom{24}{4} = 10626.$$

So

$$\boxed{10626}.$$

(b) The condition $x_i \geq 2$ for every i contributes generating function

$$\left(x^2 + x^3 + \cdots\right)^5 = \left(\frac{x^2}{1-x}\right)^5 = \frac{x^{10}}{(1-x)^5}.$$

Thus the number of solutions is

$$[x^{21}]\,\frac{x^{10}}{(1-x)^5} = [x^{11}](1-x)^{-5} = \binom{15}{4} = 1365.$$

Hence

$$\boxed{1365}.$$

(c) The condition $0 \leq x_1 \leq 10$ contributes

$$1 + x + x^2 + \cdots + x^{10} = \frac{1-x^{11}}{1-x}.$$

The other four variables contribute $(1-x)^{-4}$. Therefore the generating function is

$$\frac{1-x^{11}}{(1-x)^5}.$$

So the number of solutions is

$$[x^{21}]\,\frac{1-x^{11}}{(1-x)^5} = [x^{21}](1-x)^{-5} - [x^{10}](1-x)^{-5}.$$

Hence

$$\binom{25}{4} - \binom{14}{4} = 11649.$$

Therefore

$$\boxed{11649}.$$

(d) The condition $0 \le x_1 \le 3$ contributes

$$1 + x + x^2 + x^3 = \frac{1-x^4}{1-x},$$

and the condition $x_2 \ge 15$ contributes

$$x^{15} + x^{16} + \cdots = \frac{x^{15}}{1-x}.$$

The remaining three variables contribute $(1-x)^{-3}$. Thus the full generating function is

$$\frac{1-x^4}{1-x} \cdot \frac{x^{15}}{1-x} \cdot \frac{1}{(1-x)^3} = x^{15}(1-x^4)(1-x)^{-5}.$$

Hence the number of solutions is

$$[x^{21}]\,x^{15}(1-x^4)(1-x)^{-5} = [x^6](1-x^4)(1-x)^{-5}.$$

Therefore

$$[x^6](1-x)^{-5} - [x^2](1-x)^{-5} = \binom{10}{4} - \binom{6}{4} = 210 - 15 = 195.$$

Thus

$$\boxed{195}.$$

Unofficial QRS@NTU solutions

Question 3 (Ex. 6.5.14: A non-negative inequality)

Problem

How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, x_3 are non-negative integers?

Solution

Method 1: Introduce an auxiliary variable

The condition

$$x_1 + x_2 + x_3 \leq 11$$

is equivalent to the existence of a non-negative integer x_4 such that

$$x_1 + x_2 + x_3 + x_4 = 11.$$

Indeed, given (x_1, x_2, x_3) with $x_1 + x_2 + x_3 \leq 11$, define

$$x_4 = 11 - (x_1 + x_2 + x_3) \geq 0.$$

Conversely, any non-negative solution of $x_1+x_2+x_3+x_4 = 11$ clearly satisfies $x_1+x_2+x_3 \leq 11$.

So the required number is exactly the number of non-negative solutions to a four-variable equation with total 11. By stars-and-bars,

$$\binom{11+4-1}{4-1} = \binom{14}{3} = 364.$$

Therefore

$$\boxed{364}.$$

Method 2: Sum over r

$$N(r) = \#\{(x_1, x_2, x_3) \in \mathbb{Z}^3_{\geq 0} : \ x_1 + x_2 + x_3 = r\} = \binom{r+2}{2}.$$

Thus

$$\#\{x_1 + x_2 + x_3 \leq 11\} = \sum_{r=0}^{11} N(r) = \sum_{r=0}^{11} \binom{r+2}{2} = \binom{14}{3} = 364.$$

Unofficial Question Solutions

Method 3: Sum over the possible values of the total

For each integer

$$s = x_1 + x_2 + x_3,$$

where $0 \leq s \leq 11$, the number of non-negative solutions of

$$x_1 + x_2 + x_3 = s$$

is

$$\binom{s+3-1}{3-1} = \binom{s+2}{2}.$$

Therefore the required count is

$$\sum_{s=0}^{11} \binom{s+2}{2}.$$

Now apply the hockey-stick identity:

$$\sum_{s=0}^{11} \binom{s+2}{2} = \binom{14}{3} = 364.$$

Hence

$$\boxed{364}.$$

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Question 4 (Ex. 6.5.19: Digit sum equal to 19)

Problem

How many positive integers strictly lesser than 1,000,000 have the sum of their digits equal to 19?

Solution

Every positive integer less than 1,000,000 can be written uniquely as a six-digit string

$$x_1x_2x_3x_4x_5x_6,$$

allowing leading zeros, with

$$0 \le x_i \le 9 \qquad (i = 1, 2, 3, 4, 5, 6).$$

The condition on the sum of digits is

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 19.$$

The number 000000 does not occur anyway, since its digit sum is 0, not 19. So we only need to count the six-tuples $(x_1, \ldots, x_6)$ satisfying the above equation and the digit bounds.

Method 1: Inclusion-exclusion with the upper bound $x_i \le 9$

If we temporarily ignore the constraints $x_i \le 9$, then the number of non-negative integer solutions of

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 19$$

is

$$\binom{19 + 6 - 1}{6 - 1} = \binom{24}{5}.$$

Now we must exclude the solutions in which at least one digit is at least 10.

Let A_i be the set of solutions with $x_i \ge 10$. To count $|A_i|$, write

$$x_i = 10 + y_i, \qquad y_i \ge 0.$$

Then the equation becomes

$$y_i + \sum_{j \ne i} x_j = 9,$$

so

$$|A_i| = \binom{9 + 6 - 1}{5} = \binom{14}{5}.$$

There are 6 choices of i, so the first correction term is

$$6\binom{14}{5}.$$

Next we examine double intersections. If both $x_i \geq 10$ and $x_j \geq 10$ with $i \neq j$, then

$$x_i + x_j \geq 20,$$

which is impossible because the total sum is only 19. Hence every double intersection is empty, and therefore all higher intersections are also empty.

Thus inclusion-exclusion stops after the first subtraction, giving

$$\binom{24}{5} - 6\binom{14}{5}.$$

Now

$$\binom{24}{5} = 42504, \qquad \binom{14}{5} = 2002,$$

so

$$42504 - 6(2002) = 42504 - 12012 = 30492.$$

Therefore

$$\boxed{30492}.$$

Method 2: Generating functions

For one digit $x_i \in \{0, 1, 2, \ldots, 9\}$, the generating function recording its possible contributions to the digit sum is

$$1 + x + x^2 + \cdots + x^9 = \frac{1 - x^{10}}{1 - x}.$$

Since there are six digits, the generating function for the sum of the digits is

$$\left(1 + x + x^2 + \cdots + x^9\right)^6 = \left(\frac{1 - x^{10}}{1 - x}\right)^6.$$

Hence the required number is the coefficient of x^{19}:

$$[x^{19}] \left(\frac{1 - x^{10}}{1 - x}\right)^6.$$

Expand the numerator:

$$(1 - x^{10})^6 = 1 - 6x^{10} + 15x^{20} - \cdots.$$

Because we only need the coefficient of x^{19}, all terms involving $x^{20}, x^{30}, \ldots$ are irrelevant. Therefore

$$[x^{19}] \left(\frac{1 - x^{10}}{1 - x}\right)^6 = [x^{19}](1 - x)^{-6} - 6[x^9](1 - x)^{-6}.$$

Using

$$[x^m](1 - x)^{-6} = \binom{m + 5}{5},$$

we obtain

$$\binom{24}{5} - 6\binom{14}{5} = 30492.$$

Hence

$$\boxed{30492}.$$

Question 5 (Ex. 6.5.1, 6.5.4, additional exercise)

Problem

(a) In how many different ways can five elements be selected in *order* from a set with three elements when repetition is allowed?

(b) How many ways are there to select five unordered elements from a set with three elements when repetition is allowed?

(c) How many ways are there to assign three distinct jobs to five (of course distinct) employees if each employee can be given more than one jobs?

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Solution

Method 1: Direct encoding formulas

(a)

$$\#\{(x_1, \ldots, x_5) \in \{a, b, c\}^5\} = 3^5 = 243.$$

$$\boxed{243}$$

(b)

$$\#\{(n_1, n_2, n_3) \in \mathbb{Z}^3_{\geq 0} : \ n_1 + n_2 + n_3 = 5\} = \binom{5+3-1}{3-1} = \binom{7}{2} = 21.$$

$$\boxed{21}$$

(c)

$$\#\{f : \{J_1, J_2, J_3\} \to E\} = 5^3 = 125,$$

where E is the set of 5 employees.

$$\boxed{125}$$

Method 2: Direct counting by product rule and stars-and-bars

(a) Let the underlying set be $\{a, b, c\}$. We are forming an ordered 5-tuple

$$(x_1, x_2, x_3, x_4, x_5),$$

where each x_i may be any of the three elements, and repetition is allowed. For each of the five positions there are 3 choices, independently. Hence by the product rule,

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5 = 243.$$

Therefore

$$\boxed{243}.$$

(b) An unordered selection of five elements from three available types, with repetition allowed, is equivalent to choosing non-negative integers (n_1, n_2, n_3) such that

$$n_1 + n_2 + n_3 = 5,$$

where n_i records how many times the ith element is chosen. By stars-and-bars, the number of solutions is

$$\binom{5+3-1}{3-1} = \binom{7}{2} = 21.$$

Hence

$$\boxed{21}.$$

(c) Let the three distinct jobs be J_1, J_2, J_3. For each job, there are 5 possible employees to whom it may be assigned. Since an employee may receive more than one job, these choices are independent. Therefore the number of assignments is

$$5 \cdot 5 \cdot 5 = 5^3 = 125.$$

Thus

$$\boxed{125}.$$

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Method 3: Functions and generating functions

(a) Choosing five ordered elements from a three-element set with repetition allowed is exactly the same as specifying a function

$$f : \{1, 2, 3, 4, 5\} \to \{a, b, c\},$$

where $f(i)$ is the element chosen in the ith position. A function from a five-element set to a three-element set is determined by choosing one of 3 images for each of the 5 inputs, so there are $3^5 = 243$ such functions. Hence

$$\boxed{243}.$$

(b) For unordered selection with repetition, the ordinary generating function for one element type is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x},$$

because we may choose that type $0, 1, 2, \ldots$ times. With three element types, the generating function is

$$\frac{1}{(1-x)^3}.$$

Therefore the number of ways to choose five elements is

$$[x^5](1-x)^{-3} = \binom{5+3-1}{3-1} = \binom{7}{2} = 21.$$

So

$$\boxed{21}.$$

(c) Assigning three distinct jobs to five employees is the same as specifying a function

$$g : \{J_1, J_2, J_3\} \to E,$$

where E is the five-element set of employees. Since there are 5 choices for each of the 3 jobs, the number of such functions is

$$5^3 = 125.$$

Therefore

$$\boxed{125}.$$

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Question 6 (Ex. 6.5.12, additional exercise)

Problem

(a) How many strings of 10 ternary digits (0, 1, or 2) are there that contain exactly two 0's, three 1's, and five 2's?

(b) Let n be a multiple of k. How many ways can n books be placed on k distinguishable shelves if no two books are the same, the positions of the books on the shelves do not matter, and each shelf has exactly n/k books?

Solution

Method 1: Multinomial form / equal-block form

(a)

$$\# = \binom{10}{2,3,5} = \frac{10!}{2!\,3!\,5!} = 2520.$$

$$\boxed{2520}$$

(b) Let

$$m = \frac{n}{k}.$$

Then a valid placement is a partition of the n distinct books into k labelled groups of size m, so

$$\# = \binom{n}{\underbrace{m,\ldots,m}_{k \text{ times}}} = \frac{n!}{(m!)^k} = \frac{n!}{((n/k)!)^k}.$$

$$\boxed{\frac{n!}{((n/k)!)^k}}$$

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Method 2: Choose positions / choose groups successively

(a) First choose the 2 positions occupied by 0's. This can be done in

$$\binom{10}{2}$$

ways. Among the remaining 8 positions, choose the 3 positions occupied by 1's. This can be done in

$$\binom{8}{3}$$

ways. The remaining 5 positions are then forced to be 2's. Thus the total number of strings is

$$\binom{10}{2}\binom{8}{3} = 45 \cdot 56 = 2520.$$

Hence

$$\boxed{2520}.$$

(b) Write

$$m = \frac{n}{k},$$

which is an integer by hypothesis. Since each shelf must contain exactly m books, we choose the books shelf by shelf.

Choose the m books for shelf 1 in

$$\binom{n}{m}$$

ways. Then choose the m books for shelf 2 from the remaining $n - m$ books in

$$\binom{n-m}{m}$$

ways. Continue in this way. The final count is

$$\binom{n}{m}\binom{n-m}{m}\binom{n-2m}{m}\cdots\binom{m}{m}.$$

Now simplify this product:

$$\begin{aligned}\binom{n}{m}\binom{n-m}{m}\cdots\binom{m}{m} &= \frac{n!}{m!(n-m)!}\cdot\frac{(n-m)!}{m!(n-2m)!}\cdots\frac{m!}{m!0!} \\ &= \frac{n!}{(m!)^k}.\end{aligned}$$

Therefore the number of ways is

$$\boxed{\frac{n!}{((n/k)!)^k}}.$$

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Method 3: Multinomial viewpoint / permutation-and-cut viewpoint

(a) A valid string is a permutation of the multiset

$$\{0, 0, 1, 1, 1, 2, 2, 2, 2, 2\}.$$

If the ten symbols were all distinct, there would be 10! permutations. But the two 0's are indistinguishable, the three 1's are indistinguishable, and the five 2's are indistinguishable, so we divide by

$$2!\,3!\,5!.$$

Hence the number of strings is

$$\frac{10!}{2!3!5!} = 2520.$$

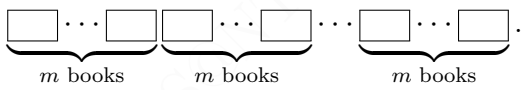
Therefore

$$\boxed{2520}.$$

(b) Again write $m = n/k$. Arrange all n distinct books in a line. This can be done in

$$n!$$

ways. Now cut this ordered list into k consecutive labelled blocks of size m:



The first block is assigned to shelf 1, the second to shelf 2, and so on. This procedure overcounts each shelf placement because the books on a given shelf are *unordered*: permuting the m books within any one block does not change the shelf placement. Therefore we divide by $m!$ for each shelf, giving a factor of $(m!)^k$.

Hence the number of placements is

$$\frac{n!}{(m!)^k} = \frac{n!}{((n/k)!)^k}.$$

Thus

$$\boxed{\frac{n!}{((n/k)!)^k}}.$$

Question 7 (Ex. 6.5.32 modified: Books on distinguishable shelves)

Problem

How many ways can n books be placed on k distinguishable shelves

(a) if the books are indistinguishable copies of the same title?

(b) if no two books are the same, and the positions of the books on the shelves matter?

Solution

Empty shelves are allowed unless otherwise stated.

Method 1: Counting-vector / separator encoding

(a)

$$\#\{(x_1,\dots,x_k)\in\mathbb{Z}_{\geq 0}^k:\ x_1+\cdots+x_k=n\}=\binom{n+k-1}{k-1}=\binom{n+k-1}{n}.$$

$$\boxed{\binom{n+k-1}{n}}$$

(b) A placement is equivalent to:

- a permutation of the n distinct books;
- together with $k-1$ identical separators.

Hence

$$\#=n!\binom{n+k-1}{k-1}=n!\binom{n+k-1}{n}.$$

$$\boxed{n!\binom{n+k-1}{n}}$$

Unofficial QRS@NTU solutions

Method 2: Stars-and-bars, then refine by ordering the books

(a) Let x_i denote the number of books placed on shelf i. Because the books are indistinguishable and only the number on each shelf matters, every placement corresponds uniquely to a non-negative integer solution of

$$x_1 + x_2 + \cdots + x_k = n.$$

Conversely, every such solution determines a placement. Therefore the number of placements equals the number of non-negative integer solutions of this equation, namely

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

Hence

$$\boxed{\binom{n+k-1}{n}}.$$

(b) First choose how many books go onto each shelf. As in part (a), the number of possible shelf-size vectors

$$(x_1, \ldots, x_k) \quad \text{with} \quad x_1 + \cdots + x_k = n, \quad x_i \geq 0,$$

is

$$\binom{n+k-1}{n}.$$

Now fix such a vector $(x_1, \ldots, x_k)$. Because the books are distinct and the order on each shelf matters, we may proceed as follows: take any permutation of the n books, then place the first x_1 books on shelf 1 in that order, the next x_2 books on shelf 2 in that order, and so on. For the fixed vector $(x_1, \ldots, x_k)$, this gives exactly

$$n!$$

placements.

Therefore, multiplying by the number of possible shelf-size vectors, the total number of placements is

$$n!\binom{n+k-1}{n}.$$

Hence

$$\boxed{n!\binom{n+k-1}{n}}.$$

Method 3: Separator model

(a) Represent a placement by writing n identical stars for the books and $k-1$ bars as separators between shelves. For example, with $k=4$, a bar pattern of the form

$$\star\star \,||\, \star\star\star \,|\, \star$$

means that shelf 1 gets 2 books, shelf 2 gets 0, shelf 3 gets 3, and shelf 4 gets 1. Thus the number of placements equals the number of ways to choose the positions of the $k-1$ bars among the total $n+k-1$ symbols. This is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}.$$

So

$$\boxed{\binom{n+k-1}{n}}.$$

(b) Arrange the n distinct books in a line; this gives

$$n!$$

possible ordered lists. Now insert $k-1$ identical separators into the resulting list, allowing separators to be adjacent and allowing them to appear at the ends. The books before the first separator go to shelf 1, the books between the first and second separators go to shelf 2, and so on, while the order in which books appear in the list is exactly the order in which they appear on their shelves.

Thus a placement is equivalent to a sequence consisting of the n distinct books together with $k-1$ identical separators. To form such a sequence, once the order of the n books is chosen, we choose the positions of the $k-1$ separators among the $n+k-1$ total slots. Hence the number of possibilities is

$$n!\binom{n+k-1}{k-1} = n!\binom{n+k-1}{n}.$$

Therefore

$$\boxed{n!\binom{n+k-1}{n}}.$$

Question 8 (Ex. 6.5.59: Identical DVDs in indistinguishable boxes)

Problem

How many ways are there to pack nine identical DVDs into three indistinguishable boxes so that each box contains at least two DVDs?

Solution

Because the boxes are indistinguishable, only the multiset of box sizes matters. So we are counting partitions of 9 into exactly three parts, each at least 2.

Method 1: Direct partition analysis

Let the box sizes be

$$a \geq b \geq c \geq 2, \qquad a + b + c = 9.$$

Since each part is at least 2, the smallest part c can be either 2 or 3.

- If $c = 3$, then $a + b = 6$ with $a \geq b \geq 3$. The only possibility is

$$(a, b, c) = (3, 3, 3).$$

- If $c = 2$, then $a + b = 7$ with $a \geq b \geq 2$. The possibilities are

$$(5, 2, 2), \qquad (4, 3, 2).$$

The combination $(6, 1, 2)$ is not allowed because each box must contain at least 2 DVDs.

Therefore the only valid packings are

$$5 + 2 + 2, \qquad 4 + 3 + 2, \qquad 3 + 3 + 3.$$

So the number of packings is

$$\boxed{3}.$$

Unofficial Question Solutions

Method 2: Shift by 2, then partition 3

$$\#\{a \geq b \geq c \geq 2 : \ a+b+c=9\} = \#\{u \geq v \geq w \geq 0 : \ u+v+w=3\},$$

via

$$u = a-2, \qquad v = b-2, \qquad w = c-2.$$

The partitions of 3 into at most 3 parts are

$$3, \qquad 2+1, \qquad 1+1+1.$$

So the shifted triples are

$$(3,0,0), \qquad (2,1,0), \qquad (1,1,1),$$

hence the original triples are

$$(5,2,2), \qquad (4,3,2), \qquad (3,3,3).$$

Therefore

$$\boxed{3}.$$

Method 3: Subtract the lower bound first, then count partitions of 3

Place 2 DVDs into each of the three boxes first. This uses

$$2+2+2=6$$

DVDs, leaving

$$9-6=3$$

DVDs still to distribute among the same three indistinguishable boxes.

Because the boxes are indistinguishable, we are now counting partitions of 3 into at most three parts. The partitions of 3 are

$$3, \qquad 2+1, \qquad 1+1+1.$$

Adding back the initial baseline of 2 DVDs per box gives, respectively,

$$(5,2,2), \qquad (4,3,2), \qquad (3,3,3).$$

These are all distinct and all valid. Hence there are exactly

$$\boxed{3}$$

possible packings.