MH1301 Discrete Mathematics

Tutorial 8

Academic Year 2025/2026, Semester 2

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Overview

This tutorial develops the basic theory of graphs, with emphasis on:

- strong and weak connectivity in directed graphs;
- degree sequences and graph isomorphism;
- longest simple paths in connected graphs;
- Euler circuits and Euler trails;
- complements of graphs and self-complementary graphs;
- Hamilton circuits;
- edge bounds in terms of the number of connected components.

Question map.

- Questions 1–3: connectivity, degree sequences, and longest paths.
- Questions 4–7: Euler circuits and one-stroke drawings.
- Question 8: Hamilton circuits in two displayed graphs.
- Question 9: upper and lower edge bounds for a graph with k connected components.

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Question 1 (Strong and weak connectivity of directed graphs)

Problem

Draw and determine whether each of these graphs is strongly connected and, if not, whether it is weakly connected.

(a) $G = (V, E)$ with $V = \{a, b, c, d, e\}$, and directed edges

$$E = \{(b, a), (b, c), (b, e), (e, a), (d, b), (c, d)\}.$$

(b) $G = (V, E)$ with $V = \{a, b, c, d, e\}$, and directed edges

$$E = \{(a, d), (a, e), (b, a), (b, c), (d, b), (d, c), (e, b), (e, d)\}.$$

(c) $G = (V, E)$ with $V = \{a, b, c, d, e, f, g\}$, and directed edges

$$E = \{(a, c), (a, g), (g, d), (d, c), (e, b), (b, f), (f, e)\}.$$

Solution

Method 1: Direct reachability analysis

(a) The digraph is *not* strongly connected. Indeed, the vertex a has out-degree 0, so there is no directed path from a to any other vertex. Therefore not every ordered pair of vertices is mutually reachable.

To decide weak connectivity, ignore edge directions. The resulting undirected graph has edges

$$\{a, b\}, \{b, c\}, \{b, e\}, \{a, e\}, \{b, d\}, \{c, d\},$$

and every vertex is connected to b or to a vertex adjacent to b. Thus the underlying undirected graph is connected.

Hence

Graph (a) is not strongly connected, but it is weakly connected.

(b) This digraph is also *not* strongly connected, because c has no outgoing edges. Hence there is no directed path from c to any other vertex, so strong connectivity fails.

Ignoring directions, the edges become

$$\{a, d\}, \{a, e\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, e\}, \{d, e\}.$$

This undirected graph is connected: for example, c is adjacent to both b and d, and the vertices a, b, d, e are all linked together.

Therefore

Graph (b) is not strongly connected, but it is weakly connected.

(c) This digraph is *not* strongly connected. In fact it is not even weakly connected.

Ignoring directions, the edges become

$$\{a, c\}, \{a, g\}, \{g, d\}, \{d, c\}, \{e, b\}, \{b, f\}, \{f, e\}.$$

So the underlying undirected graph has two connected components:

$$\{a, c, d, g\} \qquad \text{and} \qquad \{b, e, f\}.$$

Since the underlying undirected graph is disconnected, the digraph is not weakly connected; consequently it cannot be strongly connected either.

Hence

$$\boxed{\text{Graph (c) is neither strongly connected nor weakly connected.}}$$

Method 2: Strongly connected components and condensation

(a) In graph (a), the vertices b, c, d form a directed cycle

$$b \to c \to d \to b,$$

so $\{b, c, d\}$ is one strongly connected component. The vertex e has a directed edge to a, but there is no path back from a to e, so e is its own strongly connected component. The vertex a is a sink, so $\{a\}$ is another strongly connected component. Thus the strongly connected components are

$$\{b, c, d\}, \qquad \{e\}, \qquad \{a\}.$$

Since there is more than one strongly connected component, the graph is not strongly connected. But the condensation graph is connected as an undirected graph, so the original digraph is weakly connected.

Therefore

$$\boxed{\text{(a) is not strongly connected, but is weakly connected.}}$$

(b) In graph (b), the vertices a, b, d, e lie in one strongly connected component:

$$a \to d \to b \to a, \qquad a \to e \to b \to a, \qquad e \to d \to b \to a \to e,$$

so every one of a, b, d, e can reach every other. The vertex c is a sink, so $\{c\}$ is a separate strongly connected component. Hence the strongly connected components are

$$\{a, b, d, e\}, \qquad \{c\}.$$

Thus the graph is not strongly connected. The condensation graph consists of one edge from the first component to $\{c\}$, so its underlying undirected graph is connected. Hence the digraph is weakly connected.

Thus

$$\boxed{\text{(b) is not strongly connected, but is weakly connected.}}$$

(c) In graph (c), the vertices e, b, f form a directed cycle

$$e \to b \to f \to e,$$

so $\{e, b, f\}$ is one strongly connected component. In the other part of the graph, there are directed edges

$$a \to g, \qquad g \to d, \qquad d \to c, \qquad a \to c,$$

but no directed cycles, so each of a, g, d, c forms its own strongly connected component. Moreover, there is no edge at all between the set $\{a, c, d, g\}$ and the set $\{b, e, f\}$ in the underlying undirected graph. So the digraph is not weakly connected.

Hence

$$\boxed{\text{(c) is neither strongly connected nor weakly connected.}}$$

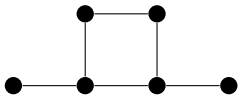
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Question 2 (Degree sequences and isomorphism)

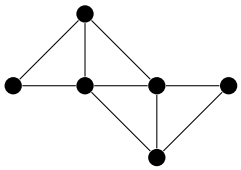
Problem

The degree sequence of a graph $G = (V, E)$ is the sequence of length $n = |V|$ that contains $\deg(v)$ for all $v \in V$. Usually it is sorted non-decreasingly.

(a) State the degree sequence of the following graphs:



Graph 1



Graph 2

(b) Prove that two graphs that do not have the same degree sequence cannot be isomorphic.

Solution

Method 1: Compute degrees directly and use degree preservation under isomorphism

(a) For Graph 1, the four bottom vertices have degrees

$$1,\ 3,\ 3,\ 1$$

from left to right, and the two top vertices each have degree 2. So the degree sequence, sorted non-decreasingly, is

$$(1, 1, 2, 2, 3, 3).$$

Hence

$$\boxed{\text{Graph 1 has degree sequence } (1, 1, 2, 2, 3, 3).}$$

For Graph 2, the leftmost and rightmost vertices each have degree 2, the top and bottom central vertices each have degree 3, and the two middle horizontal vertices each have degree 4. So the degree sequence is

$$(2, 2, 3, 3, 4, 4).$$

Thus

$$\boxed{\text{Graph 2 has degree sequence } (2, 2, 3, 3, 4, 4).}$$

(b) Let G and H be isomorphic graphs, and let

$$f : V(G) \to V(H)$$

be an isomorphism. By definition, f is a bijection preserving adjacency:

$$uv \in E(G) \iff f(u)f(v) \in E(H).$$

Fix a vertex $u \in V(G)$. The neighbors of u in G are carried bijectively by f onto the neighbors of $f(u)$ in H. Hence

$$\deg_G(u) = \deg_H(f(u)).$$

Since f is a bijection on the vertices, the multiset of vertex degrees of G is exactly the same as the multiset of vertex degrees of H. Therefore, after sorting, the degree sequences must be identical.

Consequently, if two graphs do *not* have the same degree sequence, they cannot be isomorphic. Hence

$$\boxed{\text{Different degree sequences imply the graphs are not isomorphic.}}$$

Method 2: Adjacency-matrix row sums

(a) The degree sequences are the same as in Method 1:

$$\boxed{(1,1,2,2,3,3) \text{ for Graph 1,} \qquad (2,2,3,3,4,4) \text{ for Graph 2.}}$$

(b) Let A_G be the adjacency matrix of G, and A_H the adjacency matrix of H. If $G \cong H$, then there exists a permutation matrix P such that

$$A_H = P^T A_G P.$$

The degree of a vertex is the sum of the entries in the corresponding row of the adjacency matrix. Multiplying by a permutation matrix on the left and right merely reorders the rows and columns, so the list of row sums of A_H is just a permutation of the list of row sums of A_G.

Therefore isomorphic graphs must have the same multiset of degrees, and hence the same sorted degree sequence. So if two graphs have different degree sequences, they cannot be isomorphic.

Thus

$$\boxed{\text{If the degree sequences differ, the graphs are not isomorphic.}}$$

Question 3 (Longest simple paths in a connected graph)

Problem

If a graph G is connected and has no simple path with length greater than k, prove that every two simple paths in G of length k have at least one vertex in common.

Solution

Method 1: Shortest-Connector Contradiction

Let

$$P = u_0u_1 \cdots u_k, \qquad Q = v_0v_1 \cdots v_k$$

be two simple paths of length k. We prove that

$$V(P) \cap V(Q) \neq \varnothing.$$

Suppose, for contradiction, that

$$V(P) \cap V(Q) = \varnothing.$$

Since G is connected, there exists a path joining a vertex of P to a vertex of Q. Among all such paths, choose one of minimum length:

$$R = x_0x_1 \cdots x_t,$$

where

$$x_0 = u_i \in V(P), \qquad x_t = v_j \in V(Q).$$

Since P and Q are vertex-disjoint,

$$t \geq 1.$$

We claim that

$$\{x_1, \ldots, x_{t-1}\} \cap \big(V(P) \cup V(Q)\big) = \varnothing.$$

Indeed, if $x_s \in V(P)$ for some $1 \leq s \leq t-1$, then

$$x_sx_{s+1} \cdots x_t$$

is a shorter path from P to Q, contradiction. Similarly, if $x_s \in V(Q)$, then

$$x_0x_1 \cdots x_s$$

is a shorter path from P to Q, contradiction.

Now u_i divides P into two subpaths of lengths

$$i \qquad \text{and} \qquad k-i.$$

Therefore

$$\max\{i, k-i\} \geq \left\lceil \frac{k}{2} \right\rceil.$$

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Choose the longer subpath of P from u_i to one endpoint of P, and denote it by P'. Then

$$\ell(P') \geq \left\lceil \frac{k}{2} \right\rceil.$$

Similarly, v_j divides Q into two subpaths of lengths

$$j \qquad \text{and} \qquad k-j,$$

so the longer subpath Q' from v_j to one endpoint of Q satisfies

$$\ell(Q') \geq \left\lceil \frac{k}{2} \right\rceil.$$

Since P and Q are vertex-disjoint and the internal vertices of R avoid $P \cup Q$, the concatenation

$$P' \cup R \cup Q'$$

is a simple path. Its length is at least

$$\ell(P') + \ell(R) + \ell(Q') \geq \left\lceil \frac{k}{2} \right\rceil + t + \left\lceil \frac{k}{2} \right\rceil.$$

Since $t \geq 1$,

$$\ell(P' \cup R \cup Q') \geq 2\left\lceil \frac{k}{2} \right\rceil + 1 > k.$$

This contradicts the hypothesis that G has no simple path of length greater than k.

Hence

$$V(P) \cap V(Q) \neq \varnothing.$$

Therefore

Every two simple paths of length k in G have at least one common vertex.

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Method 2: Longest-Path Intersection Theorem

We use the following theorem.

Theorem. In a connected graph, any two longest simple paths have a common vertex.

Proof. Let

$$P = u_0u_1 \cdots u_m, \qquad Q = v_0v_1 \cdots v_m$$

be two longest simple paths in a connected graph G. Suppose

$$V(P) \cap V(Q) = \varnothing.$$

Since G is connected, choose a shortest path

$$R = x_0x_1 \cdots x_t$$

from P to Q, where

$$x_0 = u_i \in V(P), \qquad x_t = v_j \in V(Q).$$

Then $t \geq 1$, and by the minimality of R,

$$\{x_1, \ldots, x_{t-1}\} \cap \big(V(P) \cup V(Q)\big) = \varnothing.$$

From u_i, choose the longer subpath of P to an endpoint. Its length is

$$\max\{i, m-i\} \geq \left\lceil \frac{m}{2} \right\rceil.$$

From v_j, choose the longer subpath of Q to an endpoint. Its length is

$$\max\{j, m-j\} \geq \left\lceil \frac{m}{2} \right\rceil.$$

Together with R, these two subpaths form a simple path of length at least

$$\left\lceil \frac{m}{2} \right\rceil + 1 + \left\lceil \frac{m}{2} \right\rceil > m.$$

This contradicts the maximality of P and Q. Therefore

$$V(P) \cap V(Q) \neq \varnothing.$$

This proves the theorem.

□

Now return to the given problem. Since G has no simple path of length greater than k, every simple path of length k is a longest simple path. Hence, by the theorem, any two simple paths of length k have a common vertex.

Therefore

Every two simple paths of length k in G have at least one common vertex.

Method 3: Contrapositive Path-Attachment Lemma

We prove the contrapositive.

The desired statement is

$$(\text{no simple path has length greater than } k) \Longrightarrow (\text{any two length-}k \text{ simple paths intersect}).$$

Its contrapositive is

$$(\exists \text{ two vertex-disjoint simple paths of length } k) \Longrightarrow (\exists \text{ a simple path of length greater than } k).$$

Assume that there exist two vertex-disjoint simple paths

$$P = u_0u_1 \cdots u_k, \qquad Q = v_0v_1 \cdots v_k.$$

Since G is connected, choose a shortest connector

$$R = x_0x_1 \cdots x_t,$$

where

$$x_0 = u_i \in V(P), \qquad x_t = v_j \in V(Q).$$

Then

$$t \geq 1$$

and, by minimality,

$$\{x_1, \ldots, x_{t-1}\} \cap \big(V(P) \cup V(Q)\big) = \varnothing.$$

Define

$$\alpha = \max\{i, k-i\}, \qquad \beta = \max\{j, k-j\}.$$

Then

$$\alpha \geq \left\lceil \frac{k}{2} \right\rceil, \qquad \beta \geq \left\lceil \frac{k}{2} \right\rceil.$$

Take the length-α subpath of P from u_i to an endpoint of P, the connector R, and the length-β subpath of Q from v_j to an endpoint of Q. Their concatenation is a simple path of length

$$\alpha + t + \beta.$$

Thus

$$\alpha + t + \beta \geq \left\lceil \frac{k}{2} \right\rceil + 1 + \left\lceil \frac{k}{2} \right\rceil > k.$$

Hence G contains a simple path of length greater than k.

Therefore, by contrapositive, if G has no simple path of length greater than k, then no two length-k simple paths can be vertex-disjoint. Hence every two simple paths of length k have a common vertex.

Every two simple paths of length k in G have at least one common vertex.

Method 4: Extremal Separation Principle

Let

$$\mathcal{P}_k = \{P : \ P \text{ is a simple path of length } k\}.$$

The hypothesis implies that every $P \in \mathcal{P}_k$ is extremal, i.e. longest possible.

Suppose there exist

$$P = u_0u_1 \cdots u_k, \qquad Q = v_0v_1 \cdots v_k$$

such that

$$V(P) \cap V(Q) = \varnothing.$$

Because G is connected,

$$d(V(P), V(Q)) < \infty.$$

Choose $u_i \in V(P)$ and $v_j \in V(Q)$ such that

$$d(u_i, v_j) = d(V(P), V(Q)).$$

Let

$$R$$

be a shortest u_i-v_j path. Then R meets $P \cup Q$ only at its endpoints; otherwise $d(V(P), V(Q))$ would be smaller.

Now

$$\max\{d_P(u_i, u_0), d_P(u_i, u_k)\} = \max\{i, k-i\} \geq \left\lceil \frac{k}{2} \right\rceil,$$

and

$$\max\{d_Q(v_j, v_0), d_Q(v_j, v_k)\} = \max\{j, k-j\} \geq \left\lceil \frac{k}{2} \right\rceil.$$

Thus there exist endpoints $u_r \in \{u_0, u_k\}$ and $v_s \in \{v_0, v_k\}$ such that

$$d_P(u_i, u_r) \geq \left\lceil \frac{k}{2} \right\rceil, \qquad d_Q(v_j, v_s) \geq \left\lceil \frac{k}{2} \right\rceil.$$

Therefore

$$u_r \rightsquigarrow_P u_i \rightsquigarrow_R v_j \rightsquigarrow_Q v_s$$

is a simple path of length at least

$$\left\lceil \frac{k}{2} \right\rceil + 1 + \left\lceil \frac{k}{2} \right\rceil > k,$$

contradicting extremality.

Hence

$$V(P) \cap V(Q) \neq \varnothing.$$

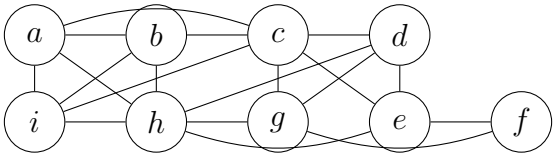
Therefore

Every two simple paths of length k in G have at least one common vertex.

Question 4 (Ex. 10.5.4: Euler circuit in the given graph)

Problem

Does this graph have an Euler circuit? Find one if there is one.



Solution

Method 1: Euler's criterion

A connected graph has an Euler circuit if and only if every vertex has even degree.

From the drawing, the degrees are:

$$\deg(a) = 4, \quad \deg(b) = 4, \quad \deg(c) = 6, \quad \deg(d) = 4, \quad \deg(e) = 4,$$

$$\deg(f) = 2, \quad \deg(g) = 4, \quad \deg(h) = 6, \quad \deg(i) = 4.$$

Every degree is even. The graph is also connected, because each vertex is joined by edges to the rest of the graph.

Therefore, by Euler's criterion, the graph has an Euler circuit. Hence

$$\boxed{\text{Yes, the graph has an Euler circuit.}}$$

Method 2: Exhibit an explicit Euler circuit

One Euler circuit is

$$a \to h \to e \to f \to g \to h \to i \to c \to e \to d \to g \to c \to d \to h \to b \to i \to a \to c \to b \to a.$$

We verify:

- it starts and ends at a, so it is a circuit;
- every consecutive pair of vertices is joined by an edge in the graph;
- each edge of the graph is used exactly once.

Hence it is an Euler circuit.

Therefore

$$\boxed{a \to h \to e \to f \to g \to h \to i \to c \to e \to d \to g \to c \to d \to h \to b \to i \to a \to c \to b \to a}$$

is an Euler circuit.

Question 5 (Can one always add edges to obtain an Euler circuit?)

Problem

Is it true that by adding edges to a simple graph $G = (V, E)$ (without introducing multiple edges between any pair of vertices) one can always create a graph that has an Euler circuit? Justify the answer.

Solution

Method 1: A smallest counterexample

The statement is *false*.

Take the simple graph $G = K_2$, the complete graph on two vertices. It has one edge, and both vertices have degree 1, so it does not have an Euler circuit.

Because K_2 is already complete on its vertex set, there is no additional edge that can be added without creating a multiple edge. Therefore it is impossible to modify K_2 by adding edges alone so as to obtain an Eulerian graph.

Hence

$$\boxed{\text{No, this is not always possible.}}$$

Method 2: A general family of counterexamples

More generally, let $G = K_{2m}$ for any integer $m \geq 1$. This graph is complete, so no new edges can be added. Every vertex has degree

$$2m - 1,$$

which is odd. A connected graph can have an Euler circuit only if every vertex has even degree. Therefore K_{2m} is not Eulerian.

Since no edges can be added to a complete graph, K_{2m} can never be turned into a graph with an Euler circuit by adding edges only.

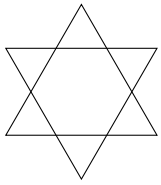
Thus

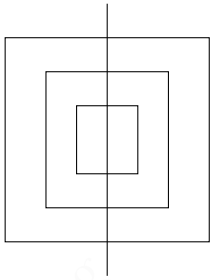
$$\boxed{\text{The claim is false; complete graphs } K_{2m} \text{ provide counterexamples.}}$$

Question 6 (Ex. 10.5.7, 10.5.8: One-stroke drawings)

Problem

Determine whether each of the pictures shown below can be drawn with a pencil in a continuous motion without lifting the pencil or retracing part of the picture.





Solution

Method 1: Use the Euler-trail criterion

A connected graph can be drawn in one continuous stroke without retracing if and only if it has either:

- no odd-degree vertices (Euler circuit), or
- exactly two odd-degree vertices (Euler trail).

(a) For the six-pointed star, the six outer tips each have degree 2, and the six intersection points each have degree 4. Thus every vertex has even degree, and the figure is connected. Therefore it has an Euler circuit, so it can be drawn in one stroke.

Hence

$$\boxed{\text{Yes, the left picture can be drawn in one stroke.}}$$

(b) For the nested-square picture with the vertical line, every intersection of the vertical line with a square contributes degree 4. The only odd-degree vertices are the two endpoints of the vertical line, each of degree 1. The figure is connected, and it has exactly two odd vertices, so it has an Euler trail (but not an Euler circuit).

Therefore

$$\boxed{\text{Yes, the right picture can be drawn in one stroke.}}$$

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Method 2: Start-end reasoning

(a) In the star figure, every time one enters an intersection or tip, there is a matching unused edge to leave by; no dead end is forced. Since there is no special starting or ending point forced by parity, one may begin at any point and return to the starting point after traversing all edges. So the figure is drawable in one continuous closed stroke.

Thus

$$\boxed{\text{Yes.}}$$

(b) In the nested-square figure, the only natural starting and ending points are the two ends of the vertical segment, because those are the only places where the drawing has degree 1. One can start at the top endpoint, move through all square intersections, and finish at the bottom endpoint without repeating an edge. Thus the figure has an open one-stroke drawing.

Hence

$$\boxed{\text{Yes, beginning at one endpoint of the vertical line and ending at the other.}}$$

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Question 7 (Ex. 10.5.16: When do K_n and C_n have Euler circuits?)

Problem

For what values of n do these graphs have an Euler circuit?

(a) K_n,

(b) C_n.

Solution

Method 1: Apply Euler's criterion

A connected graph has an Euler circuit if and only if every vertex has even degree.

(a) In K_n, each vertex is adjacent to every other vertex, so each vertex has degree

$$n - 1.$$

Also, K_n is connected for every $n \geq 1$. Therefore K_n has an Euler circuit exactly when

$$n - 1 \text{ is even},$$

that is, exactly when

$$n \text{ is odd}.$$

Hence

$$\boxed{K_n \text{ has an Euler circuit exactly for odd } n.}$$

(b) In C_n, every vertex has degree 2, which is even. The cycle graph C_n is connected (for the usual definition $n \geq 3$). Therefore every cycle graph has an Euler circuit.

Thus

$$\boxed{C_n \text{ has an Euler circuit for every } n \geq 3.}$$

Method 2: Direct structural interpretation

(a) In a complete graph K_n, traversing an Euler circuit would require every time the walk enters a vertex to leave along a different unused edge, so the degree at each vertex must be even. Since the degree is $n - 1$, this forces n to be odd. Conversely, when n is odd, K_n is connected and all degrees are even, so an Euler circuit exists.

Therefore

$$\boxed{K_n \text{ is Eulerian if and only if } n \text{ is odd.}}$$

(b) The graph C_n is itself a single cycle:

$$v_1 \to v_2 \to \cdots \to v_n \to v_1.$$

This closed walk uses every edge exactly once. Hence C_n is automatically an Euler circuit for the graph.

So

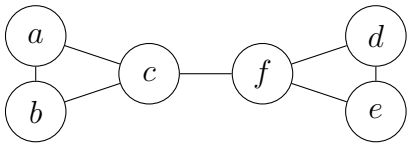
$\boxed{C_n \text{ has an Euler circuit for every } n \geq 3.}$

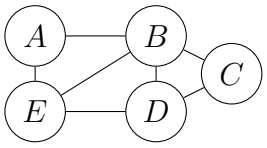
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Question 8 (Ex. 10.5.20, 10.5.21: Hamilton circuits)

Problem

Determine whether each of the given graphs below has a Hamilton circuit. If it does, find such a circuit. If it does not, give an argument to show why no such circuit exists.





Solution

Method 1: Use bridges and explicit construction

(a) The left graph does *not* have a Hamilton circuit.

The edge cf is a bridge: if it is removed, the graph breaks into two components,

$$\{a, b, c\} \qquad \text{and} \qquad \{d, e, f\}.$$

A graph containing a Hamilton circuit cannot have a bridge, because every edge on a Hamilton circuit lies on a cycle, and an edge lying on a cycle cannot be a bridge.

Therefore the left graph has no Hamilton circuit. Hence

$$\boxed{\text{The left graph has no Hamilton circuit.}}$$

(b) The right graph *does* have a Hamilton circuit. One such circuit is

$$A \to B \to C \to D \to E \to A.$$

Each consecutive pair is joined by an edge:

$$AB,\ BC,\ CD,\ DE,\ EA.$$

This closed walk visits each vertex exactly once before returning to the start.

Therefore

$$\boxed{A \to B \to C \to D \to E \to A}$$

is a Hamilton circuit for the right graph.

Method 2: Forced-edge analysis

(a) In the left graph, the vertices a, b, d, e each have degree 2. In any Hamilton circuit, every degree-2 vertex must use both of its incident edges, because the circuit must enter and leave that vertex exactly once.

Thus a Hamilton circuit would have to contain:

$$ac,\ bc,\ df,\ ef.$$

That already forces vertex c to use both ca and cb, leaving no remaining capacity at c to use the bridge edge cf. Similarly, vertex f would be forced to use fd and fe, leaving no capacity to use fc. But then there would be no way to connect the two triangles in one Hamilton cycle.

Hence the left graph has no Hamilton circuit:

$$\boxed{\text{No Hamilton circuit exists in the left graph.}}$$

(b) In the right graph, the vertices A and C have degree 2. Therefore any Hamilton circuit must use both edges incident with A and both edges incident with C. So the circuit must contain

$$AB,\ AE,\ BC,\ CD.$$

Since B already uses AB and BC, it can use no other Hamilton-cycle edges. Since C uses BC and CD, and A uses AB and AE, the only way to close the cycle through all five vertices is to use

$$DE$$

and return via

$$EA.$$

Thus the Hamilton circuit is forced to be

$$A \to B \to C \to D \to E \to A.$$

Therefore

$$\boxed{A \to B \to C \to D \to E \to A}$$

is a Hamilton circuit.

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Question 9 (Edge bounds for a graph with k connected components)

Problem

Given a simple graph G with n vertices and m edges. If G has exactly k connected components, prove that

$$n - k \leq m \leq \frac{(n-k)(n-k+1)}{2}.$$

Solution

Method 1: Componentwise bounds

Let the connected components of G have vertex counts

$$n_1, n_2, \ldots, n_k,$$

where each $n_i \geq 1$ and

$$n_1 + n_2 + \cdots + n_k = n.$$

Lower bound. Each connected component on n_i vertices has at least

$$n_i - 1$$

edges, because every connected graph contains a spanning tree, and a tree on n_i vertices has exactly $n_i - 1$ edges. Summing over all components,

$$m \geq \sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - k = n - k.$$

So

$$\boxed{m \geq n - k.}$$

Upper bound. Since the graph is simple, a connected component on n_i vertices has at most

$$\binom{n_i}{2}$$

edges, attained when the component is complete. Therefore

$$m \leq \sum_{i=1}^{k} \binom{n_i}{2}. \tag{1}$$

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We now maximize the right-hand side under the constraint $n_1 + \cdots + n_k = n$ with $n_i \geq 1$. The sum is maximized by making one component as large as possible and all others as small as possible. To justify this, suppose $x \geq y \geq 2$. Then

$$\begin{aligned}\binom{x+1}{2} + \binom{y-1}{2} - \left(\binom{x}{2} + \binom{y}{2}\right) &= x - (y-1) \\ &= x - y + 1 \\ &\geq 1.\end{aligned}$$

So moving one vertex from a smaller nontrivial component to a larger component increases the total number of edges.

Repeating this process shows that the maximum occurs when

$$(n_1, n_2, \ldots, n_k) = (n-k+1, 1, 1, \ldots, 1).$$

Then (1) gives

$$m \leq \binom{n-k+1}{2} = \frac{(n-k)(n-k+1)}{2}.$$

Hence

$$\boxed{m \leq \frac{(n-k)(n-k+1)}{2}.}$$

Combining the two bounds,

$$\boxed{n-k \leq m \leq \frac{(n-k)(n-k+1)}{2}.}$$

Method 2: Spanning forest for the lower bound, and a quadratic inequality for the upper bound

Again let the component sizes be $n_1, \ldots, n_k$, with

$$n_i \geq 1, \qquad \sum_{i=1}^{k} n_i = n.$$

Lower bound via spanning forest. Choose a spanning tree in each connected component. The union of these trees is a spanning forest of G. A spanning tree in the ith component uses $n_i - 1$ edges, so the spanning forest uses

$$\sum_{i=1}^{k} (n_i - 1) = n - k$$

edges. Since this forest is a subgraph of G, the graph G has at least that many edges:

$$\boxed{m \geq n - k.}$$

Upper bound via $x_i = n_i - 1$. Set

$$x_i = n_i - 1 \qquad (i = 1, \ldots, k).$$

Then each $x_i \geq 0$, and

$$x_1 + x_2 + \cdots + x_k = n - k.$$

Now

$$\binom{n_i}{2} = \binom{x_i + 1}{2} = \frac{x_i(x_i + 1)}{2}.$$

Hence

$$m \leq \sum_{i=1}^{k} \binom{n_i}{2} = \frac{1}{2} \sum_{i=1}^{k} (x_i^2 + x_i) = \frac{1}{2} \left(\sum_{i=1}^{k} x_i^2 + \sum_{i=1}^{k} x_i \right).$$

Because the x_i are nonnegative,

$$\sum_{i=1}^{k} x_i^2 \leq \left(\sum_{i=1}^{k} x_i \right)^2.$$

Therefore

$$\begin{aligned} m &\leq \frac{1}{2} \left(\left(\sum_{i=1}^{k} x_i \right)^2 + \sum_{i=1}^{k} x_i \right) \\ &= \frac{1}{2} \big((n-k)^2 + (n-k) \big) \\ &= \frac{(n-k)(n-k+1)}{2}. \end{aligned}$$

So

$$\boxed{m \leq \frac{(n-k)(n-k+1)}{2}.}$$

Combining the lower and upper estimates gives

$$\boxed{n - k \leq m \leq \frac{(n-k)(n-k+1)}{2}.}$$

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