MH2100 Calculus III

Midterm Practice 1 (Solutions)

AY 2026/2027, Semester 1

Quantitative Research Society @NTU

Practice material

Overview

- **Coverage:** curves and tangents; limits, continuity, partial derivatives and total differentiability; chain rule, directional derivatives, tangent lines, linear approximation and the Implicit Function Theorem; extrema and Lagrange multipliers; order reversal, polar coordinates and volume by a height double integral.
- **Practice design:** four original problems, 100 marks in total; intended for 110 minutes. This practice design is not a confirmed 2026 examination format.
- **Current administration:** Tuesday 6 October 2026, 16:30–18:20, venue TBA. One double-sided typed A4 sheet is permitted; attachments and sticky notes are not. No calculator policy is stated here.

Unofficial practice material

Instructions

Answer **all** questions. Show all working clearly; answers without justification may receive zero marks.

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Unofficial QRS Solutions

Question 1 (25 Marks)

PRACTICE PROBLEM 1.

(a) The curve C has parametrization $\mathbf{r}(t) = (t^2, t^3, 2t)$, $t \in \mathbb{R}$. Find the point on C at $t = 1$, a unit tangent vector there, and an equation of the tangent line. **(8 marks)**

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Determine whether f is continuous at $(0,0)$. Compute $f_x(0,0)$ and $f_y(0,0)$, and decide whether f is totally differentiable at $(0,0)$. Justify every conclusion. **(17 marks)**

Solution

Method 1: Differentiate the parametrisation; path test and the theorem that differentiability implies continuity

(a) Substitute $t = 1$ into $\mathbf{r}(t) = (t^2, t^3, 2t)$:

$$\mathbf{r}(1) = \big(1^2,\, 1^3,\, 2 \cdot 1\big) = (1, 1, 2).$$

Differentiate each component with respect to t:

$$\mathbf{r}'(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}t^2,\ \frac{\mathrm{d}}{\mathrm{d}t}t^3,\ \frac{\mathrm{d}}{\mathrm{d}t}2t\right) = (2t,\, 3t^2,\, 2).$$

At $t = 1$,

$$\mathbf{r}'(1) = (2, 3, 2) \neq \mathbf{0},$$

so the curve is regular at $t = 1$ and $\mathbf{r}'(1)$ is a tangent vector there. Its length is

$$\|\mathbf{r}'(1)\| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{4 + 9 + 4} = \sqrt{17}.$$

Dividing by the length gives the unit tangent vector

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{1}{\sqrt{17}}(2, 3, 2).$$

The tangent line passes through $\mathbf{r}(1) = (1, 1, 2)$ in the direction $\mathbf{r}'(1) = (2, 3, 2)$:

$$(x, y, z) = \mathbf{r}(1) + s\,\mathbf{r}'(1) = (1, 1, 2) + s(2, 3, 2), \qquad s \in \mathbb{R}.$$

Hence the point is $(1, 1, 2)$ and

$$\boxed{\mathbf{T}(1) = \frac{1}{\sqrt{17}}(2, 3, 2), \qquad (x, y, z) = (1, 1, 2) + s(2, 3, 2),\ s \in \mathbb{R}.}$$

(b) **Continuity.** Approach $(0,0)$ along two paths.

Along the x-axis $y = 0$, $x \neq 0$:

$$f(x, 0) = \frac{x^2 \cdot 0}{x^4 + 0^2} = 0 \longrightarrow 0 \qquad (x \to 0).$$

Along the parabola $y = x^2$, $x \neq 0$:

$$f(x, x^2) = \frac{x^2 \cdot x^2}{x^4 + (x^2)^2} = \frac{x^4}{2x^4} = \frac{1}{2} \longrightarrow \frac{1}{2} \qquad (x \to 0).$$

Both paths end at $(0, 0)$, but the two path limits are 0 and $\frac{1}{2}$. Hence $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist, and in particular it is not equal to $f(0, 0) = 0$. Therefore f is not continuous at $(0, 0)$.

Partial derivatives. For $h \neq 0$, $f(h, 0) = \dfrac{h^2 \cdot 0}{h^4} = 0$, so

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{0 - 0}{h} = 0.$$

For $k \neq 0$, $f(0, k) = \dfrac{0 \cdot k}{0 + k^2} = 0$, so

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k\to 0} \frac{0 - 0}{k} = 0.$$

Thus both partial derivatives exist and $f_x(0, 0) = f_y(0, 0) = 0$.

Total differentiability. Theorem: if f is totally differentiable at a point, then f is continuous at that point. Its contrapositive says that a function which is not continuous at a point is not totally differentiable there. Since f is not continuous at $(0, 0)$, f is not totally differentiable at $(0, 0)$, even though both partial derivatives exist.

$$\boxed{f \text{ is not continuous and is not totally differentiable at } (0, 0).}$$

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Method 2: Tangent line as an intersection of tangent planes; parabola family and the definition of differentiability

(a) The point is $\mathbf{r}(1) = (1, 1, 2)$ as before. Write $x = t^2$, $y = t^3$, $z = 2t$. Then $t = \frac{z}{2}$, so every point of C satisfies

$$x = \left(\frac{z}{2}\right)^2 \iff z^2 - 4x = 0, \qquad y = \left(\frac{z}{2}\right)^3 \iff z^3 - 8y = 0.$$

Let $F_1(x, y, z) = z^2 - 4x$ and $F_2(x, y, z) = z^3 - 8y$. The curve lies on both level surfaces $F_1 = 0$ and $F_2 = 0$, so its tangent line lies in both tangent planes at $(1, 1, 2)$. Compute the gradients:

$$\nabla F_1 = (-4,\ 0,\ 2z), \qquad \nabla F_2 = (0,\ -8,\ 3z^2).$$

At $(1, 1, 2)$:

$$\nabla F_1(1, 1, 2) = (-4, 0, 4), \qquad \nabla F_2(1, 1, 2) = (0, -8, 12).$$

The tangent line is perpendicular to both normals, so its direction is their cross product:

$$\begin{aligned}(-4, 0, 4) \times (0, -8, 12) &= \big(0 \cdot 12 - 4 \cdot (-8),\ 4 \cdot 0 - (-4) \cdot 12,\ (-4)(-8) - 0 \cdot 0\big) \\ &= (32, 48, 32) = 16(2, 3, 2).\end{aligned}$$

A unit direction is

$$\frac{(2, 3, 2)}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{1}{\sqrt{17}}(2, 3, 2),$$

and it points in the direction of increasing t because its z-component is positive and $z = 2t$ increases with t. The tangent line through $(1, 1, 2)$ is therefore

$$\boxed{\mathbf{T}(1) = \frac{1}{\sqrt{17}}(2, 3, 2), \qquad (x, y, z) = (1, 1, 2) + s(2, 3, 2),\ s \in \mathbb{R}.}$$

(b) **Continuity.** Use the family of parabolas $y = mx^2$, $m \in \mathbb{R}$, $x \neq 0$:

$$f(x, mx^2) = \frac{x^2 \cdot mx^2}{x^4 + m^2x^4} = \frac{mx^4}{(1 + m^2)x^4} = \frac{m}{1 + m^2}.$$

The value is constant along each parabola, so the limit along $y = mx^2$ is $\dfrac{m}{1 + m^2}$. For $m = 0$ this is 0; for $m = 1$ it is $\frac{1}{2}$. Different parabolas through $(0, 0)$ give different limits, so $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist and f is not continuous at $(0, 0)$.

Partial derivatives. As computed from the difference quotients, $f(h, 0) = 0$ and $f(0, k) = 0$ for $h, k \neq 0$, so

$$f_x(0, 0) = \lim_{h\to 0} \frac{0 - 0}{h} = 0, \qquad f_y(0, 0) = \lim_{k\to 0} \frac{0 - 0}{k} = 0.$$

Total differentiability from the definition. If f were totally differentiable at $(0, 0)$, its linear map would be $(h, k) \mapsto f_x(0, 0)h + f_y(0, 0)k = 0$, and we would need

$$\lim_{(h,k)\to(0,0)} \frac{f(h, k) - f(0, 0) - 0 \cdot h - 0 \cdot k}{\sqrt{h^2 + k^2}} = \lim_{(h,k)\to(0,0)} \frac{f(h, k)}{\sqrt{h^2 + k^2}} = 0.$$

Along $(h, k) = (t, t^2)$, $t \neq 0$, we have $f(t, t^2) = \frac{1}{2}$, so

$$\frac{|f(t, t^2)|}{\sqrt{t^2 + t^4}} = \frac{1/2}{|t|\sqrt{1 + t^2}} = \frac{1}{2|t|\sqrt{1 + t^2}} \longrightarrow \infty \qquad (t \to 0).$$

The error quotient does not tend to 0, so f is not totally differentiable at $(0, 0)$.

$$\boxed{f \text{ is not continuous and is not totally differentiable at } (0, 0).}$$

Method 3: Polar form with an ε–δ argument; directional derivatives against the gradient formula

This method applies to part (b).

Continuity. Write $x = r\cos\theta$, $y = r\sin\theta$ with $r > 0$. Then

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \frac{r\cos^2\theta\sin\theta}{r^2\cos^4\theta + \sin^2\theta}.$$

For a fixed θ with $\sin\theta \neq 0$, the denominator tends to $\sin^2\theta > 0$ and the numerator tends to 0, so the value tends to 0 as $r \to 0$; for $\sin\theta = 0$ the value is 0. Hence every straight line through the origin gives the limit 0, and the straight-line test alone cannot decide continuity. The polar form has no bound of the form $|f| \leq g(r)$ with $g(r) \to 0$, because the denominator can be small when $\sin\theta$ is of size r. To prove discontinuity, use the definition with $\varepsilon = \frac{1}{2}$.

Let $\delta > 0$. Choose t with $0 < t < 1$ and $t\sqrt{2} < \delta$. The point (t, t^2) satisfies

$$\|(t, t^2)\| = \sqrt{t^2 + t^4} \leq \sqrt{t^2 + t^2} = t\sqrt{2} < \delta,$$

and

$$|f(t, t^2) - f(0,0)| = \left|\frac{t^4}{t^4 + t^4} - 0\right| = \frac{1}{2} \geq \varepsilon.$$

So for $\varepsilon = \frac{1}{2}$ no $\delta > 0$ works, and f is not continuous at $(0,0)$.

Partial derivatives. These are the directional derivatives in the directions $(1,0)$ and $(0,1)$. For a unit vector $\mathbf{u} = (a,b)$ and $t \neq 0$,

$$\frac{f(ta, tb) - f(0,0)}{t} = \frac{1}{t} \cdot \frac{t^3a^2b}{t^4a^4 + t^2b^2} = \frac{a^2b}{t^2a^4 + b^2}.$$

If $b \neq 0$, letting $t \to 0$ gives

$$D_{\mathbf{u}}f(0,0) = \frac{a^2b}{b^2} = \frac{a^2}{b}.$$

If $b = 0$, the quotient is 0 for all $t \neq 0$, so $D_{\mathbf{u}}f(0,0) = 0$. With $\mathbf{u} = (1,0)$ (so $b = 0$) we get $f_x(0,0) = 0$; with $\mathbf{u} = (0,1)$ we get $f_y(0,0) = \frac{0^2}{1} = 0$.

Total differentiability. If f were totally differentiable at $(0,0)$, every directional derivative would be given by the gradient formula

$$D_{\mathbf{u}}f(0,0) = \nabla f(0,0) \cdot \mathbf{u} = (0,0) \cdot (a,b) = 0.$$

Take $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The formula above gives

$$D_{\mathbf{u}}f(0,0) = \frac{a^2}{b} = \frac{1/2}{1/\sqrt{2}} = \frac{\sqrt{2}}{2} \neq 0.$$

This contradicts the gradient formula, so f is not totally differentiable at $(0,0)$.

$\boxed{f \text{ is not continuous and is not totally differentiable at } (0,0).}$

UNOFFICIAL QRS marking scheme (25 marks)

(a) **(8 marks)** (i) **(2 marks)** point; (ii) **(3 marks)** derivative and normalization; (iii) **(3 marks)** tangent line.

(b) **(17 marks)** (i) **(5 marks)** valid path and continuity conclusion; (ii) **(4 marks)** both partial derivatives; (iii) **(5 marks)** valid proof of non-differentiability, including the theorem that differentiability implies continuity; (iv) **(3 marks)** conclusion.

Question 2 (25 Marks)

PRACTICE PROBLEM 2.

(a) Let h be C^2 on a neighbourhood of $(1,2)$, with $h(1,2) = 3$ and $\nabla h(1,2) = (4,-1)$. Define $G(x,y) = h(x^2+y, xy+2)$. At $P=(1,0)$, find the directional derivative of G in the unit direction from P toward $(2,2)$, the tangent line to $G(x,y)=3$, and the linear approximation of $G(1.02,-0.03)$. **(15 marks)**

(b) Let $F(x,y,z) = x^2 + ye^z + z^3 - 2$. Use the Implicit Function Theorem near $(1,0,1)$ to write $z = p(x,y)$. Find $p_x(1,0)$ and $p_y(1,0)$, then obtain the linear approximation of $p(1.04,-0.02)$. **(10 marks)**

Solution

Method 1: Multivariable Chain Rule and the implicit-derivative formulas

(a) Let $u = x^2 + y$ and $v = xy + 2$, so $G(x,y) = h(u,v)$. At $P=(1,0)$,

$$u = 1^2 + 0 = 1, \qquad v = 1\cdot 0 + 2 = 2,$$

so the inner point is $(u,v) = (1,2)$, where $h = 3$ and $(h_u, h_v) = \nabla h(1,2) = (4,-1)$.

Chain Rule hypotheses. h is C^2, hence differentiable, near $(1,2)$; u and v are polynomials, hence differentiable everywhere. So the Chain Rule applies:

$$G_x = h_u\, u_x + h_v\, v_x, \qquad G_y = h_u\, u_y + h_v\, v_y.$$

The partial derivatives of the inner functions are

$$u_x = 2x, \quad u_y = 1, \quad v_x = y, \quad v_y = x,$$

and at $P=(1,0)$:

$$u_x = 2, \quad u_y = 1, \quad v_x = 0, \quad v_y = 1.$$

Therefore

$$G_x(P) = 4\cdot 2 + (-1)\cdot 0 = 8,$$
$$G_y(P) = 4\cdot 1 + (-1)\cdot 1 = 3,$$

so $\nabla G(P) = (8,3)$.

Directional derivative. The vector from P toward $(2,2)$ is $(2,2)-(1,0) = (1,2)$, with length $\sqrt{1^2+2^2} = \sqrt{5}$. The unit direction is

$$\mathbf{u} = \frac{(1,2)}{\sqrt{5}}.$$

Since G is differentiable at P (composition of differentiable maps), $D_{\mathbf{u}}G(P) = \nabla G(P)\cdot\mathbf{u}$:

$$\boxed{D_{\mathbf{u}}G(P) = \frac{(8,3)\cdot(1,2)}{\sqrt{5}} = \frac{14}{\sqrt{5}}.}$$

Here $(8,3)\cdot(1,2) = 8+6 = 14$.

Tangent line. $G(P) = h(1,2) = 3$, so P lies on the level curve $G(x,y) = 3$. G is C^1 near P and $\nabla G(P) = (8,3) \neq \mathbf{0}$, so by the Implicit Function Theorem the level curve is a smooth curve near P with normal $\nabla G(P)$. Its tangent line is $\nabla G(P) \cdot \big((x,y) - (1,0)\big) = 0$:

$$8(x-1) + 3(y-0) = 0, \qquad \text{that is,} \qquad \boxed{8(x-1) + 3y = 0}.$$

Linear approximation. The linearisation of G at P is

$$L(x,y) = G(P) + G_x(P)(x-1) + G_y(P)(y-0) = 3 + 8(x-1) + 3y.$$

At $(1.02, -0.03)$, $x - 1 = 0.02$ and $y = -0.03$, so

$$L(1.02, -0.03) = 3 + 8(0.02) + 3(-0.03) = 3 + 0.16 - 0.09 = 3.07.$$

Hence

$$\boxed{G(1.02, -0.03) \approx L(1.02, -0.03) = 3.07.}$$

(b) Let $F(x,y,z) = x^2 + y\mathrm{e}^z + z^3 - 2$. Its partial derivatives are

$$F_x = 2x, \qquad F_y = \mathrm{e}^z, \qquad F_z = y\mathrm{e}^z + 3z^2.$$

Implicit Function Theorem hypotheses.

- F is C^1 on $\mathbb{R}^3$: F is built from polynomials and e^z, and F_x, F_y, F_z above are continuous.
- $F(1,0,1) = 1^2 + 0 \cdot \mathrm{e}^1 + 1^3 - 2 = 1 + 0 + 1 - 2 = 0$.
- $F_z(1,0,1) = 0 \cdot \mathrm{e}^1 + 3 \cdot 1^2 = 3 \neq 0$.

Hence there is a C^1 function $z = p(x,y)$ near $(1,0)$ with $p(1,0) = 1$ and $F(x,y,p(x,y)) = 0$. Its partial derivatives are

$$p_x = -\frac{F_x}{F_z}, \qquad p_y = -\frac{F_y}{F_z}.$$

At $(1,0,1)$: $F_x = 2 \cdot 1 = 2$, $F_y = \mathrm{e}^1 = \mathrm{e}$, $F_z = 3$. Therefore

$$p_x(1,0) = -\frac{2}{3}, \qquad p_y(1,0) = -\frac{\mathrm{e}}{3}.$$

The linearisation of p at $(1,0)$ is

$$p(x,y) \approx p(1,0) + p_x(1,0)(x-1) + p_y(1,0)(y-0) = 1 - \frac{2}{3}(x-1) - \frac{\mathrm{e}}{3}y.$$

At $(1.04, -0.02)$, $x - 1 = 0.04$ and $y = -0.02$. Since $\frac{2}{3}(0.04) = \frac{0.08}{3} = \frac{2}{75}$ and $\frac{\mathrm{e}}{3}(0.02) = \frac{\mathrm{e}}{150}$,

$$\boxed{\begin{aligned} p(1.04, -0.02) &\approx 1 - \frac{2}{3}(0.04) - \frac{\mathrm{e}}{3}(-0.02) \\ &= 1 - \frac{2}{75} + \frac{\mathrm{e}}{150} \approx 0.991455. \end{aligned}}$$

Method 2: Composition of linearisations and the tangent plane to the level surface

(a) Since h is C^2, it is differentiable at $(1,2)$, so with $h(1,2)=3$ and $\nabla h(1,2)=(4,-1)$,

$$h(u,v)=3+4(u-1)-(v-2)+E_h(u,v), \qquad \frac{E_h(u,v)}{\|(u-1,v-2)\|}\to 0.$$

Expand the inner functions exactly about $P=(1,0)$. Using $x^2=1+2(x-1)+(x-1)^2$ and $x=1+(x-1)$,

$$\begin{aligned} u-1&=x^2+y-1=2(x-1)+y+(x-1)^2,\\ v-2&=xy=y+(x-1)y. \end{aligned}$$

Substituting into the expansion of h,

$$\begin{aligned} G(x,y)&=3+4\big(2(x-1)+y+(x-1)^2\big)-\big(y+(x-1)y\big)+E_h\\ &=3+8(x-1)+3y+\Big[4(x-1)^2-(x-1)y+E_h\Big]. \end{aligned}$$

The bracket is $o\big(\|(x-1,y)\|\big)$: the two quadratic terms are bounded by a constant times $\|(x-1,y)\|^2$, and $\|(u-1,v-2)\|$ is bounded by a constant times $\|(x-1,y)\|$ near P. By uniqueness of the linearisation,

$$L(x,y)=3+8(x-1)+3y, \qquad \nabla G(P)=(8,3).$$

Directional derivative. With $\mathbf{u}=\frac{(1,2)}{\sqrt{5}}$ (the unit vector from P toward $(2,2)$), $P+t\mathbf{u}=\left(1+\frac{t}{\sqrt{5}},\frac{2t}{\sqrt{5}}\right)$ and

$$G(P+t\mathbf{u})=3+8\cdot\frac{t}{\sqrt{5}}+3\cdot\frac{2t}{\sqrt{5}}+o(t)=3+\frac{14t}{\sqrt{5}}+o(t).$$

Hence

$$\boxed{D_{\mathbf{u}}G(P)=\lim_{t\to 0}\frac{G(P+t\mathbf{u})-3}{t}=\frac{14}{\sqrt{5}}.}$$

Tangent line. The tangent line to the level curve $G=3$ through P is the level set $L(x,y)=3$ of the linearisation (valid because $\nabla G(P)\neq\mathbf{0}$):

$$3+8(x-1)+3y=3 \iff \boxed{8(x-1)+3y=0}.$$

Linear approximation. Substituting $x-1=0.02$, $y=-0.03$ into L:

$$L(1.02,-0.03)=3+0.16-0.09=3.07,$$

so

$$\boxed{G(1.02,-0.03)\approx L(1.02,-0.03)=3.07.}$$

(b) Let $F(x,y,z)=x^2+y\mathrm{e}^z+z^3-2$, with $F_x=2x$, $F_y=\mathrm{e}^z$, $F_z=y\mathrm{e}^z+3z^2$. At $(1,0,1)$, $F=0$ and

$$\nabla F(1,0,1)=(2,\ \mathrm{e},\ 3).$$

F is C^1 and $F_z(1,0,1) = 3 \neq 0$, so by the Implicit Function Theorem the level surface $F = 0$ near $(1,0,1)$ is the graph $z = p(x,y)$. The tangent plane to the level surface at $(1,0,1)$ is the tangent plane to this graph:

$$2(x-1) + \mathrm{e}(y-0) + 3(z-1) = 0.$$

Solve for z:

$$z = 1 - \frac{2}{3}(x-1) - \frac{\mathrm{e}}{3}y.$$

The coefficients of $(x-1)$ and y are $p_x(1,0) = -\frac{2}{3}$ and $p_y(1,0) = -\frac{\mathrm{e}}{3}$. Evaluating the tangent plane at $(1.04, -0.02)$:

$$\boxed{\begin{aligned} p(1.04,-0.02) &\approx 1 - \frac{2}{3}(0.04) - \frac{\mathrm{e}}{3}(-0.02) \\ &= 1 - \frac{2}{75} + \frac{\mathrm{e}}{150} \approx 0.991455. \end{aligned}}$$

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Method 3: Single-variable Chain Rule along lines and implicit differentiation

(a) Throughout, $h_u(1,2) = 4$ and $h_v(1,2) = -1$. The Chain Rule along a curve,

$$\frac{\mathrm{d}}{\mathrm{d}t} h\big(u(t), v(t)\big) = h_u\, u'(t) + h_v\, v'(t),$$

applies because h is differentiable and $u(t)$, $v(t)$ below are polynomials in t.

Directional derivative. Let $\mathbf{u} = \frac{(1,2)}{\sqrt{5}}$ and $\varphi(t) = G\left(1 + \frac{t}{\sqrt{5}}, \frac{2t}{\sqrt{5}}\right)$, so $D_{\mathbf{u}}G(P) = \varphi'(0)$. Along this line,

$$u(t) = \left(1 + \frac{t}{\sqrt{5}}\right)^2 + \frac{2t}{\sqrt{5}}, \qquad u'(t) = \frac{2}{\sqrt{5}}\left(1 + \frac{t}{\sqrt{5}}\right) + \frac{2}{\sqrt{5}},$$
$$v(t) = \left(1 + \frac{t}{\sqrt{5}}\right)\frac{2t}{\sqrt{5}} + 2, \qquad v'(t) = \frac{1}{\sqrt{5}} \cdot \frac{2t}{\sqrt{5}} + \left(1 + \frac{t}{\sqrt{5}}\right)\frac{2}{\sqrt{5}}.$$

At $t = 0$: $(u(0), v(0)) = (1, 2)$, $u'(0) = \frac{4}{\sqrt{5}}$, $v'(0) = \frac{2}{\sqrt{5}}$. Hence

$$\varphi'(0) = 4 \cdot \frac{4}{\sqrt{5}} + (-1) \cdot \frac{2}{\sqrt{5}} = \frac{16 - 2}{\sqrt{5}},$$

so

$$\boxed{D_{\mathbf{u}}G(P) = \frac{14}{\sqrt{5}}.}$$

Tangent line. $G(P) = h(1,2) = 3$. The hypothesis of the Implicit Function Theorem for solving $G(x, y) = 3$ as $y = y(x)$ is $G_y(P) \neq 0$; by the Chain Rule $G_y(P) = h_u \cdot 1 + h_v \cdot x = 4 - 1 = 3 \neq 0$. So near $x = 1$ there is a C^1 function $y(x)$ with $y(1) = 0$ and

$$h\big(x^2 + y(x),\ x\, y(x) + 2\big) = 3.$$

Differentiate with respect to x:

$$h_u\big(2x + y'(x)\big) + h_v\big(y(x) + x\, y'(x)\big) = 0.$$

At $x = 1$, $y = 0$:

$$4\big(2 + y'(1)\big) - \big(0 + y'(1)\big) = 0 \iff 8 + 3y'(1) = 0 \iff y'(1) = -\frac{8}{3}.$$

The tangent line is $y = -\frac{8}{3}(x - 1)$; multiplying by 3 and rearranging,

$$\boxed{8(x - 1) + 3y = 0}.$$

Linear approximation. Let $\psi(t) = G(1 + 0.02t, -0.03t)$, so $\psi(0) = G(P) = 3$ and $\psi(1) = G(1.02, -0.03) \approx \psi(0) + \psi'(0)$. Along this segment,

$$u(t) = (1 + 0.02t)^2 - 0.03t, \qquad u'(0) = 2(0.02) - 0.03 = 0.01,$$
$$v(t) = (1 + 0.02t)(-0.03t) + 2, \qquad v'(0) = -0.03.$$

Hence

$$\psi'(0) = 4(0.01) + (-1)(-0.03) = 0.04 + 0.03 = 0.07.$$

The value $\psi(0) + \psi'(0)$ is the linearisation L of G at P evaluated at $(1.02, -0.03)$, because $\psi'(0)$ is the rate of change of G at P along the increment $(0.02, -0.03)$. So $L(1.02, -0.03) = 3 + 0.07 = 3.07$, and

$$\boxed{G(1.02, -0.03) \approx L(1.02, -0.03) = 3.07.}$$

(b) By the Implicit Function Theorem (hypotheses checked in Method 1: $F \in C^1$, $F(1,0,1) = 0$, $F_z(1,0,1) = 3 \neq 0$), there is a C^1 function p near $(1,0)$ with $p(1,0) = 1$ and

$$x^2 + y\,\mathrm{e}^{p(x,y)} + p(x,y)^3 - 2 = 0.$$

Differentiate this identity with respect to x, holding y fixed:

$$2x + y\,\mathrm{e}^{p}\,p_x + 3p^2\,p_x = 0.$$

At $(x,y) = (1,0)$, $p = 1$:

$$2 + 0 + 3p_x(1,0) = 0 \implies p_x(1,0) = -\frac{2}{3}.$$

Differentiate with respect to y, holding x fixed:

$$\mathrm{e}^{p} + y\,\mathrm{e}^{p}\,p_y + 3p^2\,p_y = 0.$$

At $(1,0)$, $p = 1$:

$$\mathrm{e} + 0 + 3p_y(1,0) = 0 \implies p_y(1,0) = -\frac{\mathrm{e}}{3}.$$

Then the linearisation at $(1,0)$, evaluated at $x - 1 = 0.04$, $y = -0.02$, gives

$$\boxed{\begin{aligned} p(1.04, -0.02) &\approx 1 - \frac{2}{3}(0.04) - \frac{\mathrm{e}}{3}(-0.02) \\ &= 1 - \frac{2}{75} + \frac{\mathrm{e}}{150} \approx 0.991455. \end{aligned}}$$

UNOFFICIAL QRS marking scheme (25 marks)

(a) **(15 marks)**

(i) **(5 marks)** chain-rule gradient;

(ii) **(4 marks)** unit direction and derivative;

(iii) **(3 marks)** tangent line;

(iv) **(3 marks)** linear approximation.

(b) **(10 marks)**

(i) **(3 marks)** verifies IFT hypotheses;

(ii) **(4 marks)** implicit derivatives;

(iii) **(3 marks)** approximation.

Question 3 (25 Marks)

PRACTICE PROBLEM 3. Let $f(x, y) = xy$ on the closed elliptical disk $C = \{(x, y) \in \mathbb{R}^2 : x^2 + 4y^2 \leq 4\}$.

(a) Find and classify every interior critical point of f. **(8 marks)**

(b) Use Lagrange multipliers to find every extremum candidate of f on $x^2 + 4y^2 = 4$. **(12 marks)**

(c) Establish that absolute extrema exist on C, then state the absolute maximum and minimum of f on C, including all points where they occur. **(5 marks)**

Solution

Method 1: Second Derivative Test, Lagrange multipliers and the Extreme Value Theorem

(a) For $f(x, y) = xy$,

$$f_x = y, \qquad f_y = x.$$

Critical points solve $f_x = 0$ and $f_y = 0$, that is $y = 0$ and $x = 0$. The only critical point is $(0, 0)$, and it is an interior point of C because $0^2 + 4 \cdot 0^2 = 0 < 4$.

The second partial derivatives are

$$f_{xx} = \frac{\partial}{\partial x}(y) = 0, \qquad f_{yy} = \frac{\partial}{\partial y}(x) = 0, \qquad f_{xy} = \frac{\partial}{\partial y}(y) = 1.$$

f is a polynomial, so it is C^2 and the Second Derivative Test applies. The discriminant at $(0, 0)$ is

$$D = f_{xx}f_{yy} - f_{xy}^2 = 0 \cdot 0 - 1^2 = -1 < 0.$$

Hence

$$\boxed{(0, 0) \text{ is a saddle point.}}$$

(b) Let $g(x, y) = x^2 + 4y^2$, so the boundary is $g = 4$. Then

$$\nabla f = (y,\, x), \qquad \nabla g = (2x,\, 8y).$$

Lagrange condition. f and g are C^1. $\nabla g = \mathbf{0}$ only at $(0, 0)$, and $g(0, 0) = 0 \neq 4$, so $\nabla g \neq \mathbf{0}$ at every point of the constraint curve. Hence every extremum of f on $g = 4$ satisfies $\nabla f = \lambda \nabla g$ for some λ:

$$y = 2\lambda x, \tag{1}$$
$$x = 8\lambda y, \tag{2}$$
$$x^2 + 4y^2 = 4. \tag{3}$$

Case $x = 0$. Then (1) gives $y = 2\lambda \cdot 0 = 0$, and (3) gives $0 + 0 = 4$, which is false. No solution.

Case $y = 0$. Then (2) gives $x = 8\lambda \cdot 0 = 0$, and (3) gives $0 = 4$, which is false. No solution.

Case $x \neq 0$ and $y \neq 0$. Substitute (1) into (2):

$$x = 8\lambda(2\lambda x) = 16\lambda^2 x.$$

Divide by $x \neq 0$: $16\lambda^2 = 1$, so $\lambda = \frac{1}{4}$ or $\lambda = -\frac{1}{4}$.

- $\lambda = \frac{1}{4}$: (1) gives $y = \frac{x}{2}$. Then (3) gives $x^2 + 4 \cdot \frac{x^2}{4} = 2x^2 = 4$, so $x = \pm\sqrt{2}$ and $y = \pm\frac{\sqrt{2}}{2}$ with the same sign:

$$f\left(\sqrt{2}, \tfrac{\sqrt{2}}{2}\right) = \sqrt{2} \cdot \tfrac{\sqrt{2}}{2} = 1, \qquad f\left(-\sqrt{2}, -\tfrac{\sqrt{2}}{2}\right) = (-\sqrt{2})\left(-\tfrac{\sqrt{2}}{2}\right) = 1.$$

- $\lambda = -\frac{1}{4}$: (1) gives $y = -\frac{x}{2}$. Then (3) gives $x^2 + 4 \cdot \frac{x^2}{4} = 2x^2 = 4$, so $x = \pm\sqrt{2}$ and $y = \mp\frac{\sqrt{2}}{2}$:

$$f\left(\sqrt{2}, -\tfrac{\sqrt{2}}{2}\right) = \sqrt{2}\left(-\tfrac{\sqrt{2}}{2}\right) = -1, \qquad f\left(-\sqrt{2}, \tfrac{\sqrt{2}}{2}\right) = (-\sqrt{2}) \cdot \tfrac{\sqrt{2}}{2} = -1.$$

All cases are closed. The extremum candidates on the ellipse are

$$\boxed{\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right).}$$

(c) **Existence (EVT).** $f(x, y) = xy$ is continuous on $\mathbb{R}^2$. C is closed, since $C = g^{-1}([0, 4])$ with g continuous, and bounded, since $x^2 \le 4$ and $4y^2 \le 4$ give $|x| \le 2$, $|y| \le 1$. So C is compact, and by the Extreme Value Theorem f attains an absolute maximum and an absolute minimum on C.

Comparison. An absolute extremum occurs either at an interior critical point or on the boundary. The interior candidate gives $f(0, 0) = 0$. By the Lagrange method, the boundary extrema occur among the four candidates of (b), with values $1, 1, -1, -1$. Comparing $\{0, 1, -1\}$:

$$\boxed{\max_C f = 1 \text{ at } \left(\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, -\frac{\sqrt{2}}{2}\right),}$$

$$\boxed{\min_C f = -1 \text{ at } \left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right).}$$

Method 2: Parametrising the constraint and elliptical polar coordinates

Note: part (b) asks for Lagrange multipliers; part (b) of this method does not use them and should be used only if the course allows this interpretation.

(a) As in Method 1, $f_x = y$ and $f_y = x$ vanish together only at $(0,0)$, which is interior to C. Classify it by restricting f to two lines through the origin. For $t \neq 0$,

$$f(t,t) = t^2 > 0 = f(0,0), \qquad f(t,-t) = -t^2 < 0 = f(0,0).$$

Every disc around $(0,0)$ contains points of both lines, so it contains points where $f > f(0,0)$ and points where $f < f(0,0)$. Hence $(0,0)$ is neither a local maximum nor a local minimum:

$$\boxed{(0,0) \text{ is a saddle point.}}$$

(b) Parametrise the ellipse by

$$x = 2\cos t, \qquad y = \sin t, \qquad t \in [0, 2\pi).$$

This covers the whole ellipse once: $x^2 + 4y^2 = 4\cos^2 t + 4\sin^2 t = 4$, and conversely, if $x^2 + 4y^2 = 4$ then $\left(\frac{x}{2}\right)^2 + y^2 = 1$, so $\left(\frac{x}{2}, y\right) = (\cos t, \sin t)$ for exactly one $t \in [0, 2\pi)$. On the ellipse,

$$\varphi(t) = f(2\cos t, \sin t) = 2\sin t \cos t = \sin 2t.$$

φ is differentiable and 2π-periodic, so its extrema on the closed curve occur where

$$\varphi'(t) = 2\cos 2t = 0 \iff 2t = \frac{\pi}{2} + k\pi \iff t = \frac{\pi}{4} + \frac{k\pi}{2}, \ k \in \mathbb{Z}.$$

In $[0, 2\pi)$ this gives four values of t:

$$\begin{aligned}
t = \tfrac{\pi}{4}: &\quad (x,y) = \left(\sqrt{2}, \tfrac{\sqrt{2}}{2}\right), && \varphi = \sin\tfrac{\pi}{2} = 1,\\
t = \tfrac{3\pi}{4}: &\quad (x,y) = \left(-\sqrt{2}, \tfrac{\sqrt{2}}{2}\right), && \varphi = \sin\tfrac{3\pi}{2} = -1,\\
t = \tfrac{5\pi}{4}: &\quad (x,y) = \left(-\sqrt{2}, -\tfrac{\sqrt{2}}{2}\right), && \varphi = \sin\tfrac{5\pi}{2} = 1,\\
t = \tfrac{7\pi}{4}: &\quad (x,y) = \left(\sqrt{2}, -\tfrac{\sqrt{2}}{2}\right), && \varphi = \sin\tfrac{7\pi}{2} = -1.
\end{aligned}$$

Here $2\cos\frac{\pi}{4} = \sqrt{2}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, with signs from the quadrant. The extremum candidates on the ellipse are

$$\boxed{\begin{aligned}&\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, -\frac{\sqrt{2}}{2}\right),\\ &\left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right).\end{aligned}}$$

(c) Cover all of C with elliptical polar coordinates

$$x = 2\rho\cos t, \qquad y = \rho\sin t, \qquad 0 \le \rho \le 1, \ t \in [0, 2\pi).$$

Then $x^2 + 4y^2 = 4\rho^2\cos^2 t + 4\rho^2\sin^2 t = 4\rho^2 \le 4$, so these points are exactly C. In these coordinates

$$f = 2\rho^2 \cos t \sin t = \rho^2 \sin 2t.$$

Since $0 \le \rho^2 \le 1$ and $-1 \le \sin 2t \le 1$,

$$-1 \le f \le 1 \quad \text{on } C.$$

$f = 1$ requires $\rho^2 = 1$ and $\sin 2t = 1$, that is $\rho = 1$ and $t \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$. $f = -1$ requires $\rho = 1$ and $\sin 2t = -1$, that is $t \in \{\frac{3\pi}{4}, \frac{7\pi}{4}\}$. Both bounds are attained, so the absolute extrema exist (this also follows from the EVT, since f is continuous and C is compact), and

$$\boxed{\max_C f = 1 \text{ at } \left(\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, -\frac{\sqrt{2}}{2}\right),}$$

$$\boxed{\min_C f = -1 \text{ at } \left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right).}$$

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Method 3: Hessian eigenvalues and completing the square (AM–GM)

Note: part (b) asks for Lagrange multipliers; part (b) of this method does not use them and should be used only if the course allows this interpretation.

(a) The only critical point is $(0,0)$ (from $f_x = y = 0$, $f_y = x = 0$), interior to C. The Hessian matrix, from $f_{xx} = 0$, $f_{xy} = f_{yx} = 1$, $f_{yy} = 0$, is

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \det(H - \mu I) = \mu^2 - 1 = 0 \iff \mu = \pm 1.$$

One eigenvalue is positive and one is negative, so H is indefinite. Since $f(x,y) = \frac{1}{2}(x,y)H(x,y)^T$ exactly, f increases from 0 along the eigenvector $(1,1)$ and decreases along $(1,-1)$:

$$\boxed{(0,0) \text{ is a saddle point.}}$$

(b) On the ellipse $x^2 + 4y^2 = 4$, complete the square in two ways:

$$\begin{aligned} (x-2y)^2 = x^2 - 4xy + 4y^2 = 4 - 4xy \qquad &\implies xy = 1 - \tfrac{1}{4}(x-2y)^2 \le 1, \\ (x+2y)^2 = x^2 + 4xy + 4y^2 = 4 + 4xy \qquad &\implies xy = -1 + \tfrac{1}{4}(x+2y)^2 \ge -1. \end{aligned}$$

$xy = 1$ exactly when $x = 2y$; with $x^2 + 4y^2 = 4$ this gives $8y^2 = 4$, so $y = \pm\frac{\sqrt{2}}{2}$ and $x = 2y = \pm\sqrt{2}$. $xy = -1$ exactly when $x = -2y$; then $8y^2 = 4$, $y = \pm\frac{\sqrt{2}}{2}$ and $x = -2y = \mp\sqrt{2}$. The first inequality shows that 1 is the maximum of f on the ellipse and the second that -1 is the minimum, each attained at exactly two points. These four points are the extremum candidates on the ellipse:

$$\boxed{\left(\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right).}$$

(c) For every $(x,y) \in C$, the squares $(x \mp 2y)^2 \ge 0$ give (the AM–GM inequality for $|x|$ and $|2y|$)

$$4|xy| = 2|x|\,|2y| \le x^2 + 4y^2 \le 4, \qquad \text{so} \qquad -1 \le xy \le 1.$$

Equality in the first inequality needs $|x| = 2|y|$, and in the second needs $x^2 + 4y^2 = 4$. So the bounds ± 1 are attained exactly at the four points of (b): $+1$ when x and y have equal signs, -1 when they have opposite signs. Attained bounds prove that the absolute extrema exist (consistent with the EVT on the compact set C), and

$$\boxed{\max_C f = 1 \text{ at } \left(\sqrt{2}, \frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, -\frac{\sqrt{2}}{2}\right),}$$

$$\boxed{\min_C f = -1 \text{ at } \left(\sqrt{2}, -\frac{\sqrt{2}}{2}\right), \left(-\sqrt{2}, \frac{\sqrt{2}}{2}\right).}$$

UNOFFICIAL QRS marking scheme (25 marks)

(a) **(8 marks)** (i) **(3 marks)** solves $\nabla f = 0$; (ii) **(3 marks)** Hessian/discriminant; (iii) **(2 marks)** saddle conclusion.

(b) **(12 marks)** (i) **(4 marks)** Lagrange system; (ii) **(4 marks)** all branches; (iii) **(4 marks)** candidates and values.

(c) **(5 marks)** (i) **(2 marks)** continuity, compactness and EVT; (ii) **(3 marks)** global comparison.

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Question 4 (25 Marks)

PRACTICE PROBLEM 4.

(a) Reverse the order of integration and evaluate

$$\int_0^1 \left[\int_{x^2}^{x} (1+2y)\,\mathrm{d}y \right] \mathrm{d}x.$$

(10 marks)

(b) Let $D = \{(x,y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0,\ 1 \leq x^2+y^2 \leq 4\}$. Find the volume of the solid above D and below $z = 6 - x^2 - y^2$. Use polar coordinates and a height double integral. **(15 marks)**

Solution

Method 1: Reversal of order by Fubini and the polar height integral

(a) **Region, order** $\mathrm{d}y\,\mathrm{d}x$**.** The given limits describe

$$R = \{(x,y) : 0 \leq x \leq 1,\ x^2 \leq y \leq x\},$$

which is the region between the parabola $y = x^2$ (below) and the line $y = x$ (above); the two curves meet where $x^2 = x$, that is at $x = 0$ and $x = 1$, and $x^2 \leq x$ on $[0,1]$.

Region, order $\mathrm{d}x\,\mathrm{d}y$**.** The values of y on R run from 0 to 1. For fixed $y \in [0,1]$, the condition $y \leq x$ gives the left end $x = y$, and $x^2 \leq y$ with $x \geq 0$ gives the right end $x = \sqrt{y}$. Since $y \leq \sqrt{y}$ on $[0,1]$,

$$R = \{(x,y) : 0 \leq y \leq 1,\ y \leq x \leq \sqrt{y}\}.$$

Fubini. The integrand $1+2y$ is continuous on the closed bounded region R, which is of both types above, so by Fubini's Theorem both iterated integrals equal $\iint_R (1+2y)\,\mathrm{d}A$:

$$\int_0^1 \int_{x^2}^{x} (1+2y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_{y}^{\sqrt{y}} (1+2y)\,\mathrm{d}x\,\mathrm{d}y.$$

Inner integral. The integrand does not depend on x:

$$\int_y^{\sqrt{y}} (1+2y)\,\mathrm{d}x = (1+2y)\big[x\big]_{x=y}^{x=\sqrt{y}} = (1+2y)(\sqrt{y}-y).$$

Outer integral. Expand:

$$(1+2y)(\sqrt{y}-y) = (\sqrt{y}-y) + 2\left(y^{3/2}-y^2\right).$$

Antiderivatives:

$$\int_0^1 (\sqrt{y}-y)\,\mathrm{d}y = \left[\frac{2}{3}y^{3/2} - \frac{1}{2}y^2\right]_0^1 = \frac{2}{3} - \frac{1}{2},$$
$$\int_0^1 \left(y^{3/2}-y^2\right)\,\mathrm{d}y = \left[\frac{2}{5}y^{5/2} - \frac{1}{3}y^3\right]_0^1 = \frac{2}{5} - \frac{1}{3}.$$

Adding, with $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ and $2\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{2}{15}$, so $\frac{1}{6} + \frac{2}{15} = \frac{5+4}{30}$:

$$\boxed{\left(\frac{2}{3} - \frac{1}{2}\right) + 2\left(\frac{2}{5} - \frac{1}{3}\right) = \frac{3}{10}.}$$

(b) **Region in polar coordinates.** Let $x = r\cos\theta$, $y = r\sin\theta$ with $r \geq 0$. The conditions $x \geq 0$, $y \geq 0$ give the first quadrant $0 \leq \theta \leq \frac{\pi}{2}$, and $1 \leq x^2 + y^2 \leq 4$ gives $1 \leq r^2 \leq 4$, that is $1 \leq r \leq 2$:

$$D = \left\{(r,\theta) : 1 \leq r \leq 2,\ 0 \leq \theta \leq \tfrac{\pi}{2}\right\}.$$

Jacobian.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r,$$

so $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$.

Height. On D, $x^2 + y^2 = r^2 \leq 4$, so the height $6 - x^2 - y^2 = 6 - r^2 \geq 2 > 0$. The solid lies between $z = 0$ and $z = 6 - r^2$, and its volume is the double integral of the height:

$$V = \iint_D (6 - x^2 - y^2)\,\mathrm{d}A = \int_0^{\pi/2}\int_1^2 (6 - r^2)\,r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Inner integral.

$$\int_1^2 (6r - r^3)\,\mathrm{d}r = \left[3r^2 - \frac{r^4}{4}\right]_1^2 = (12 - 4) - \left(3 - \frac{1}{4}\right) = 8 - \frac{11}{4} = \frac{21}{4}.$$

Outer integral. The inner value does not depend on θ:

$$V = \int_0^{\pi/2} \frac{21}{4}\,\mathrm{d}\theta = \frac{21}{4}\big[\theta\big]_0^{\pi/2} = \frac{21}{4}\cdot\frac{\pi}{2} = \boxed{\frac{21\pi}{8}}$$

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Method 2: Iterated integral in the given order; Cartesian height integral

(a) Note: part (a) asks for the reversed order; this method evaluates the given iterated integral in its original order $\mathrm{d}y\,\mathrm{d}x$ and should be used only as a check on the reversed computation. The region is $R = \{0 \le x \le 1,\ x^2 \le y \le x\}$.

Inner integral. An antiderivative of $1 + 2y$ in y is $y + y^2$:

$$\int_{x^2}^{x} (1 + 2y)\,\mathrm{d}y = \big[y + y^2\big]_{y=x^2}^{y=x} = (x + x^2) - (x^2 + x^4) = x - x^4.$$

Outer integral.

$$\int_0^1 (x - x^4)\,\mathrm{d}x = \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}.$$

Hence

$$\boxed{\int_0^1 \left[\int_{x^2}^{x} (1 + 2y)\,\mathrm{d}y\right] \mathrm{d}x = \frac{3}{10}.}$$

(b) Note: the marking scheme accepts equivalent correct Cartesian work.

For $a > 0$ let $Q_a = \{(x, y) : x \ge 0,\ y \ge 0,\ x^2 + y^2 \le a^2\}$ be the closed quarter disc of radius a. Then $Q_2 = Q_1 \cup D$, and $Q_1 \cap D$ is the arc $x^2 + y^2 = 1$, $x, y \ge 0$, which has zero area. The integrand is continuous, so by additivity of the double integral $I(2) = I(1) + V$, that is

$$V = I(2) - I(1), \qquad I(a) = \iint_{Q_a} (6 - x^2 - y^2)\,\mathrm{d}A.$$

Cartesian description. $Q_a = \{0 \le x \le a,\ 0 \le y \le \sqrt{a^2 - x^2}\}$. Write $s = \sqrt{a^2 - x^2}$.

Inner integral.

$$\int_0^s (6 - x^2 - y^2)\,\mathrm{d}y = \left[(6 - x^2)y - \frac{y^3}{3}\right]_0^s = (6 - x^2)s - \frac{s^3}{3}.$$

Since $s^3 = (a^2 - x^2)s$,

$$(6 - x^2)s - \frac{(a^2 - x^2)s}{3} = s\left(6 - \frac{a^2}{3} - \frac{2}{3}x^2\right).$$

Outer integral. Hence

$$I(a) = \left(6 - \frac{a^2}{3}\right) J_0(a) - \frac{2}{3} J_2(a),$$

where

$$J_0(a) = \int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x, \qquad J_2(a) = \int_0^a x^2 \sqrt{a^2 - x^2}\,\mathrm{d}x.$$

Substitute $x = a\sin\varphi$, $\mathrm{d}x = a\cos\varphi\,\mathrm{d}\varphi$; new limits $x = 0 \mapsto \varphi = 0$, $x = a \mapsto \varphi = \frac{\pi}{2}$; and $\sqrt{a^2 - x^2} = a\cos\varphi$ because $\cos\varphi \ge 0$ on $[0, \frac{\pi}{2}]$. Then

$$\begin{aligned} J_0(a) &= \int_0^{\pi/2} a^2 \cos^2\varphi\,\mathrm{d}\varphi = a^2 \int_0^{\pi/2} \frac{1 + \cos 2\varphi}{2}\,\mathrm{d}\varphi \\ &= a^2 \left[\frac{\varphi}{2} + \frac{\sin 2\varphi}{4}\right]_0^{\pi/2} = \frac{\pi a^2}{4}, \end{aligned}$$

$$\begin{aligned} J_2(a) &= \int_0^{\pi/2} a^4 \sin^2\varphi \cos^2\varphi \,\mathrm{d}\varphi = \frac{a^4}{4}\int_0^{\pi/2} \sin^2 2\varphi \,\mathrm{d}\varphi \\ &= \frac{a^4}{4}\int_0^{\pi/2} \frac{1-\cos 4\varphi}{2}\,\mathrm{d}\varphi = \frac{a^4}{4}\left[\frac{\varphi}{2} - \frac{\sin 4\varphi}{8}\right]_0^{\pi/2} \\ &= \frac{a^4}{4}\cdot\frac{\pi}{4} = \frac{\pi a^4}{16}. \end{aligned}$$

Therefore

$$I(a) = \left(6 - \frac{a^2}{3}\right)\frac{\pi a^2}{4} - \frac{2}{3}\cdot\frac{\pi a^4}{16} = \frac{3\pi a^2}{2} - \frac{\pi a^4}{12} - \frac{\pi a^4}{24} = \frac{3\pi a^2}{2} - \frac{\pi a^4}{8}.$$

Evaluation.

$$I(2) = \frac{3\pi\cdot 4}{2} - \frac{16\pi}{8} = 6\pi - 2\pi = 4\pi, \qquad I(1) = \frac{3\pi}{2} - \frac{\pi}{8} = \frac{11\pi}{8}.$$

Hence

$$V = I(2) - I(1) = 4\pi - \frac{11\pi}{8} = \boxed{\frac{21\pi}{8}}$$

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Method 3: Volume by horizontal cross-sections

This method applies to part (b). This method should be used only if the course allows this interpretation.

The solid is $S = \{(x, y, z) : (x, y) \in D,\ 0 \le z \le 6 - x^2 - y^2\}$. On D the height $6 - r^2$ ranges from $6 - 4 = 2$ (at $r = 2$) to $6 - 1 = 5$ (at $r = 1$), so S lies in $0 \le z \le 5$. For fixed z, the cross-section is

$$A(z) = \{(x, y) \in D :\ x^2 + y^2 \le 6 - z\} = \{x \ge 0,\ y \ge 0,\ 1 \le x^2 + y^2 \le \min(4,\, 6 - z)\}.$$

A quarter annulus $\{x, y \ge 0,\ 1 \le x^2 + y^2 \le \rho^2\}$ has area $\frac{1}{4}\pi(\rho^2 - 1)$.

For $0 \le z \le 2$: $6 - z \ge 4$, so $A(z) = D$ and

$$\operatorname{area} A(z) = \frac{1}{4}\pi(4 - 1) = \frac{3\pi}{4}.$$

For $2 \le z \le 5$: $6 - z \le 4$, so the outer radius squared is $6 - z$ and

$$\operatorname{area} A(z) = \frac{1}{4}\pi\big((6 - z) - 1\big) = \frac{\pi(5 - z)}{4}.$$

Integrate the areas over z (Cavalieri's principle):

$$\begin{aligned} V &= \int_0^2 \frac{3\pi}{4}\,\mathrm{d}z + \int_2^5 \frac{\pi(5 - z)}{4}\,\mathrm{d}z = \frac{3\pi}{4}\cdot 2 + \frac{\pi}{4}\left[5z - \frac{z^2}{2}\right]_2^5 \\ &= \frac{3\pi}{2} + \frac{\pi}{4}\left[\left(25 - \frac{25}{2}\right) - (10 - 2)\right] = \frac{3\pi}{2} + \frac{\pi}{4}\cdot\frac{9}{2} = \frac{12\pi}{8} + \frac{9\pi}{8}. \end{aligned}$$

Hence

$$V = \boxed{\frac{21\pi}{8}}$$

UNOFFICIAL QRS marking scheme (25 marks)

(a) **(10 marks)**

- (i) **(3 marks)** reversed region;
- (ii) **(3 marks)** bounds;
- (iii) **(2 marks)** inner integral;
- (iv) **(2 marks)** value 3/10.

(b) **(15 marks)**

- (i) **(4 marks)** polar region;
- (ii) **(3 marks)** height and Jacobian;
- (iii) **(5 marks)** double integral;
- (iv) **(3 marks)** value $21\pi/8$. Equivalent correct Cartesian work receives full credit.