MH2100 Calculus III

Midterm Practice 2 (Solutions)

AY 2026/2027, Semester 1

Quantitative Research Society @NTU

Practice material

Overview

- **Coverage:** curves and tangent lines; limits, continuity, partial derivatives and total differentiability; chain rule, directional derivatives, tangent lines, linear approximation and the Implicit Function Theorem; extrema and Lagrange multipliers; order reversal, polar coordinates and volume by a height double integral.
- **Practice design:** four original problems, 100 marks in total; intended for 110 minutes. This practice design is not a confirmed 2026 examination format.
- **Current administration:** Tuesday 6 October 2026, 16:30–18:20, venue TBA. One double-sided typed A4 sheet is permitted; no attachments or sticky notes are permitted. No calculator policy is stated here.

Instructions

Answer **all** questions. Show all working clearly; answers without justification may receive zero marks.

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UNOFFICIAL QRS MARKING SCHEME

Question 1 (25 Marks)

PRACTICE PROBLEM 1.

(a) Let $\mathbf{r}(t) = (\mathrm{e}^t, \sin t, t^2)$. Find the point at $t = 0$, a unit tangent vector there, and an equation of the tangent line. **(8 marks)**

(b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(0,0) = 0$ and $f(x,y) = x^2y/(x^2+y^2)$ otherwise. Determine continuity at the origin, compute both coordinate partial derivatives there, and decide total differentiability there. Justify each conclusion. **(17 marks)**

Solution

Method 1: Componentwise differentiation, Squeeze Theorem and two-path remainder test

(a) The point at $t = 0$ is

$$\mathbf{r}(0) = (\mathrm{e}^0, \sin 0, 0^2) = (1, 0, 0).$$

Differentiate each component:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^t = \mathrm{e}^t, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\sin t = \cos t, \qquad \frac{\mathrm{d}}{\mathrm{d}t}t^2 = 2t.$$

Hence

$$\mathbf{r}'(t) = (\mathrm{e}^t, \cos t, 2t), \qquad \mathbf{r}'(0) = (\mathrm{e}^0, \cos 0, 2\cdot 0) = (1, 1, 0).$$

Since $\mathbf{r}'(0) \neq \mathbf{0}$, the curve has a tangent direction at $t = 0$. Its length is

$$|\mathbf{r}'(0)| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2},$$

so the unit tangent vector is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{2}}(1, 1, 0).$$

The tangent line passes through $\mathbf{r}(0)$ in the direction $\mathbf{r}'(0)$:

$$(x, y, z) = \mathbf{r}(0) + s\,\mathbf{r}'(0) = (1, 0, 0) + s(1, 1, 0), \qquad s \in \mathbb{R}.$$

Therefore

$$\boxed{\mathbf{T}(0) = \tfrac{1}{\sqrt{2}}(1, 1, 0), \qquad (x, y, z) = (1, 0, 0) + s(1, 1, 0)}$$

(b) *Continuity.* For $(x, y) \neq (0, 0)$ we have $x^2 \leq x^2 + y^2$, so $\dfrac{x^2}{x^2+y^2} \leq 1$. Hence

$$|f(x,y) - f(0,0)| = \frac{x^2}{x^2+y^2}\,|y| \leq |y| \leq \sqrt{x^2+y^2}.$$

As $(x, y) \to (0, 0)$, $\sqrt{x^2+y^2} \to 0$, so by the Squeeze Theorem

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0),$$

and f is continuous at $(0, 0)$.

Partial derivatives. For $h \neq 0$, $f(h,0) = \dfrac{h^2 \cdot 0}{h^2 + 0^2} = 0$, so

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

For $k \neq 0$, $f(0,k) = \dfrac{0^2 \cdot k}{0^2 + k^2} = 0$, so

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0.$$

Total differentiability. If f were differentiable at $(0,0)$, its derivative would be $Df(0,0) = (f_x(0,0), f_y(0,0)) = (0,0)$, and the remainder

$$E(h,k) = f(h,k) - f(0,0) - f_x(0,0)\,h - f_y(0,0)\,k = f(h,k)$$

would satisfy

$$\lim_{(h,k)\to(0,0)} \frac{E(h,k)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{h^2 k}{(h^2+k^2)^{3/2}} = 0.$$

Test this limit along two paths. Along $k = 0$, $h \to 0$:

$$\frac{h^2 \cdot 0}{(h^2 + 0^2)^{3/2}} = 0 \;\to\; 0.$$

Along $h = k = t$, $t \to 0^+$:

$$\frac{t^2 \cdot t}{(t^2+t^2)^{3/2}} = \frac{t^3}{(2t^2)^{3/2}} = \frac{t^3}{2\sqrt{2}\,t^3} = \frac{1}{2\sqrt{2}} \;\to\; \frac{1}{2\sqrt{2}}.$$

The two path limits 0 and $\frac{1}{2\sqrt{2}}$ differ, so the limit of $E(h,k)/\sqrt{h^2+k^2}$ does not exist. In particular it is not 0, and f is not differentiable at $(0,0)$. Therefore

$$\boxed{f \text{ is continuous but not totally differentiable at } (0,0)}$$

Method 2: Difference-quotient tangent vector and polar coordinates

(a) The point is $\mathbf{r}(0) = (e^0, \sin 0, 0^2) = (1, 0, 0)$. By the definition of the derivative of a vector function,

$$\mathbf{r}'(0) = \lim_{k\to 0} \frac{\mathbf{r}(k) - \mathbf{r}(0)}{k} = \lim_{k\to 0} \left(\frac{e^k - 1}{k}, \frac{\sin k}{k}, \frac{k^2}{k} \right).$$

Compute each component limit. The two standard limits give

$$\lim_{k\to 0} \frac{e^k - 1}{k} = 1, \qquad \lim_{k\to 0} \frac{\sin k}{k} = 1,$$

and

$$\lim_{k\to 0} \frac{k^2}{k} = \lim_{k\to 0} k = 0.$$

Hence $\mathbf{r}'(0) = (1, 1, 0)$, $|\mathbf{r}'(0)| = \sqrt{1+1+0} = \sqrt{2}$ and

$$\mathbf{T}(0) = \frac{1}{\sqrt{2}}(1, 1, 0).$$

Write the line through $(1, 0, 0)$ with direction $(1, 1, 0)$ in coordinates:

$$x = 1 + s, \qquad y = s, \qquad z = 0.$$

Eliminating $s = y$ gives the equivalent description $x - 1 = y$, $z = 0$. Therefore

$$\boxed{\mathbf{T}(0) = \tfrac{1}{\sqrt{2}}(1, 1, 0), \qquad (x, y, z) = (1, 0, 0) + s(1, 1, 0)}$$

(b) Let $x = r\cos\theta$, $y = r\sin\theta$ with $r > 0$. Then

$$f(r\cos\theta, r\sin\theta) = \frac{r^2\cos^2\theta \cdot r\sin\theta}{r^2(\cos^2\theta + \sin^2\theta)} = r\cos^2\theta\sin\theta.$$

Continuity. Since $|\cos^2\theta\sin\theta| \le 1$ for every θ,

$$|f(x, y) - f(0, 0)| = r\,|\cos^2\theta\sin\theta| \le r = \sqrt{x^2 + y^2} \;\to\; 0,$$

with a bound independent of θ. Hence $\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$ and f is continuous at $(0, 0)$.

Partial derivatives. A point $(h, 0)$ with $h \neq 0$ has $\theta = 0$ or $\theta = \pi$, so $\sin\theta = 0$ and $f(h, 0) = 0$. A point $(0, k)$ with $k \neq 0$ has $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$, so $\cos\theta = 0$ and $f(0, k) = 0$. Hence

$$f_x(0, 0) = \lim_{h\to 0} \frac{0 - 0}{h} = 0, \qquad f_y(0, 0) = \lim_{k\to 0} \frac{0 - 0}{k} = 0.$$

Total differentiability. With $(h, k) = (r\cos\theta, r\sin\theta)$, the remainder quotient is

$$\frac{f(h, k) - 0 - 0\cdot h - 0\cdot k}{\sqrt{h^2 + k^2}} = \frac{r\cos^2\theta\sin\theta}{r} = \cos^2\theta\sin\theta.$$

It does not depend on r, so along the ray of angle θ it stays equal to $\cos^2\theta\sin\theta$ as $r \to 0^+$. On the ray $\theta = 0$ the value is $1 \cdot 0 = 0$; on the ray $\theta = \frac{\pi}{4}$ it is

$$\left(\tfrac{1}{\sqrt{2}}\right)^2 \cdot \tfrac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}.$$

Two rays give different limits, so the remainder quotient has no limit at $(0, 0)$; it does not tend to 0, and f is not differentiable at $(0, 0)$. Therefore

$$\boxed{f \text{ is continuous but not totally differentiable at } (0, 0)}$$

Method 3: Directional derivatives against the gradient

This method applies to part (b).

Continuity. Since $(|x| - |y|)^2 \geq 0$, we have $2|xy| \leq x^2 + y^2$. For $(x, y) \neq (0, 0)$,

$$|f(x, y)| = |x| \cdot \frac{|xy|}{x^2 + y^2} \leq \frac{|x|}{2} \leq \frac{\sqrt{x^2 + y^2}}{2} \;\to\; 0,$$

so by the Squeeze Theorem $\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$, and f is continuous at $(0, 0)$.

All directional derivatives. Let $\mathbf{u} = (a, b)$ with $a^2 + b^2 = 1$. For $t \neq 0$,

$$f(ta, tb) = \frac{t^2a^2 \cdot tb}{t^2(a^2 + b^2)} = t\, a^2b,$$

so

$$D_{\mathbf{u}} f(0, 0) = \lim_{t\to0} \frac{f(ta, tb) - f(0, 0)}{t} = \lim_{t\to0} \frac{t\, a^2b}{t} = a^2b.$$

With $\mathbf{u} = (1, 0)$ this gives $f_x(0, 0) = 1^2 \cdot 0 = 0$; with $\mathbf{u} = (0, 1)$ it gives $f_y(0, 0) = 0^2 \cdot 1 = 0$.

Total differentiability. If f were differentiable at $(0, 0)$, then every directional derivative would be given by the gradient:

$$D_{\mathbf{u}} f(0, 0) = \nabla f(0, 0) \cdot \mathbf{u} = 0 \cdot a + 0 \cdot b = 0 \qquad \text{for every unit } \mathbf{u}.$$

Take $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The direct computation gives

$$D_{\mathbf{u}} f(0, 0) = a^2b = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \neq 0,$$

a contradiction. Hence f is not differentiable at $(0, 0)$, and

$$\boxed{f \text{ is continuous but not totally differentiable at } (0, 0)}$$

UNOFFICIAL QRS marking scheme (25 marks)

(a) **(8 marks)**

- (i) **(2 marks)** point;
- (ii) **(3 marks)** tangent vector and normalization;
- (iii) **(3 marks)** tangent line.

(b) **(17 marks)**

- (i) **(5 marks)** radial bound proving continuity;
- (ii) **(4 marks)** partial derivatives;
- (iii) **(5 marks)** differentiability remainder;
- (iv) **(3 marks)** conclusions.

Question 2 (25 Marks)

PRACTICE PROBLEM 2.

(a) Let h be C^2 near $(0,1)$, with $h(0,1)=2$, $\nabla h(0,1)=(3,2)$, and $h_{uu}(0,1)=2$, $h_{uv}(0,1)=1$, $h_{vv}(0,1)=-3$. Define $G(x,y)=h(x-y,xy)$. At $P=(1,1)$, find $G_{xy}(P)$, the directional derivative in the unit direction from P to $(3,0)$, the tangent line to $G=2$, and the linear approximation of $G(1.01,0.98)$. **(15 marks)**

(b) Let $F(x,y,z)=x+y^2+\mathrm{e}^z-2$. Use the IFT near $(0,1,0)$ to write $z=q(x,y)$. Find $q_x(0,1)$ and $q_y(0,1)$, then linearly approximate $q(0.03,0.99)$. **(10 marks)**

Solution

Method 1: Chain Rule, gradient formulas and the Implicit Function Theorem formula

(a) Let $u=x-y$ and $v=xy$, so $G(x,y)=h(u,v)$. The inner partial derivatives are

$$u_x=1, \qquad u_y=-1, \qquad v_x=y, \qquad v_y=x.$$

At $P=(1,1)$:

$$(u,v)=(1-1,\ 1\cdot 1)=(0,1), \qquad u_x=1, \quad u_y=-1, \quad v_x=1, \quad v_y=1.$$

Chain Rule hypotheses. h is C^2 (in particular differentiable) near $(0,1)$, and u,v are polynomials, so G is differentiable near P and the Chain Rule applies:

$$\begin{aligned} G_x &= h_u\,u_x + h_v\,v_x = h_u + y\,h_v,\\ G_y &= h_u\,u_y + h_v\,v_y = -h_u + x\,h_v, \end{aligned}$$

where h_u, h_v are evaluated at $(x-y,xy)$. At P, with $h_u(0,1)=3$, $h_v(0,1)=2$:

$$\begin{aligned} G_x(P) &= 3+1\cdot 2=5,\\ G_y(P) &= -3+1\cdot 2=-1, \end{aligned}$$

so $\nabla G(P)=(5,-1)$.

Mixed partial G_{xy}. Differentiate $G_x=h_u(u,v)+y\,h_v(u,v)$ with respect to y, applying the Chain Rule to h_u and h_v (both C^1 since h is C^2) and the Product Rule to $y\,h_v$:

$$\begin{aligned} \frac{\partial}{\partial y}h_u(u,v) &= h_{uu}\,u_y + h_{uv}\,v_y = -h_{uu} + x\,h_{uv},\\ \frac{\partial}{\partial y}\big(y\,h_v(u,v)\big) &= h_v + y\big(h_{vu}\,u_y + h_{vv}\,v_y\big) = h_v + y\big(-h_{vu} + x\,h_{vv}\big). \end{aligned}$$

Since h is C^2, Clairaut's Theorem gives $h_{vu}=h_{uv}$. Adding,

$$G_{xy} = -h_{uu} + x\,h_{uv} + h_v - y\,h_{uv} + xy\,h_{vv}.$$

At P ($x=y=1$, $(u,v)=(0,1)$):

$$G_{xy}(P) = -2+1\cdot 1+2-1\cdot 1+1\cdot 1\cdot(-3) = -3.$$

Therefore

$$\boxed{G_{xy}(P)=-3}$$

Directional derivative. The vector from P to $(3,0)$ is $(3-1,0-1)=(2,-1)$, with length $\sqrt{4+1}=\sqrt{5}$, so

$$\mathbf{u}=\frac{1}{\sqrt{5}}(2,-1).$$

Since G is differentiable at P, $D_{\mathbf{u}}G(P)=\nabla G(P)\cdot\mathbf{u}$:

$$\boxed{D_{\mathbf{u}}G(P)=\tfrac{1}{\sqrt{5}}(5,-1)\cdot(2,-1)=\tfrac{5\cdot2+(-1)(-1)}{\sqrt{5}}=\tfrac{11}{\sqrt{5}}}$$

Tangent line. $G(P)=h(0,1)=2$, so P lies on the level curve $G=2$. Since G is C^1 and $\nabla G(P)=(5,-1)\neq\mathbf{0}$, the Implicit Function Theorem makes $G=2$ a C^1 curve near P with normal vector $\nabla G(P)$. The tangent line is $\nabla G(P)\cdot(x-1,y-1)=0$:

$$\boxed{5(x-1)-(y-1)=0}$$

Linear approximation. The linearisation of G at P is

$$L(x,y)=G(P)+G_x(P)(x-1)+G_y(P)(y-1)=2+5(x-1)-(y-1).$$

At $(1.01,0.98)$: $x-1=0.01$ and $y-1=-0.02$, so

$$L(1.01,0.98)=2+5(0.01)-(-0.02)=2+0.05+0.02=2.07.$$

Therefore

$$\boxed{G(1.01,0.98)\approx L(1.01,0.98)=2.07}$$

(b) *IFT hypotheses.* $F(x,y,z)=x+y^2+\mathrm{e}^z-2$ is C^1 (a polynomial plus an exponential). Its partial derivatives are

$$F_x=1,\qquad F_y=2y,\qquad F_z=\mathrm{e}^z.$$

At $(0,1,0)$:

$$F(0,1,0)=0+1+\mathrm{e}^0-2=0,\qquad F_x=1,\qquad F_y=2,\qquad F_z=\mathrm{e}^0=1\neq0.$$

By the Implicit Function Theorem there is a C^1 function $z=q(x,y)$ near $(0,1)$ with $q(0,1)=0$ and $F(x,y,q(x,y))=0$.

Partial derivatives. Differentiate the identity $F(x,y,q(x,y))=0$ by the Chain Rule:

$$F_x+F_z\,q_x=0,\qquad F_y+F_z\,q_y=0,$$

so

$$q_x=-\frac{F_x}{F_z},\qquad q_y=-\frac{F_y}{F_z}.$$

At $(0,1)$:

$$q_x(0,1)=-\frac{1}{1}=-1,\qquad q_y(0,1)=-\frac{2}{1}=-2.$$

Linear approximation. With $\Delta x=0.03-0=0.03$ and $\Delta y=0.99-1=-0.01$,

$$q(0.03,0.99)\approx q(0,1)+q_x(0,1)\,\Delta x+q_y(0,1)\,\Delta y.$$

Therefore

$$\boxed{q(0.03,0.99)\approx0-0.03-2(-0.01)=-0.01}$$

Method 2: Restriction to lines, Clairaut for G and the implicit slope

(a) *Gradient by restriction to coordinate lines.* On the horizontal line $y = 1$, $G(x, 1) = h(x - 1, x)$; by the one-variable Chain Rule

$$G_x(P) = \frac{\mathrm{d}}{\mathrm{d}x} h(x-1, x)\Big|_{x=1} = h_u(0,1) \cdot 1 + h_v(0,1) \cdot 1 = 3 + 2 = 5.$$

On the vertical line $x = 1$, $G(1, y) = h(1 - y, y)$, so

$$G_y(P) = \frac{\mathrm{d}}{\mathrm{d}y} h(1-y, y)\Big|_{y=1} = h_u(0,1) \cdot (-1) + h_v(0,1) \cdot 1 = -3 + 2 = -1.$$

Mixed partial in the other order. For general (x, y), $G_y = -h_u(u, v) + x\, h_v(u, v)$ with $u = x - y$, $v = xy$, $u_x = 1$, $v_x = y$. Differentiate with respect to x:

$$\frac{\partial}{\partial x}\big(-h_u(u,v)\big) = -\big(h_{uu}\, u_x + h_{uv}\, v_x\big) = -h_{uu} - y\, h_{uv},$$
$$\frac{\partial}{\partial x}\big(x\, h_v(u,v)\big) = h_v + x\big(h_{vu}\, u_x + h_{vv}\, v_x\big) = h_v + x\, h_{vu} + xy\, h_{vv}.$$

At P ($x = y = 1$, $(u, v) = (0, 1)$, $h_{vu} = h_{uv} = 1$ by Clairaut for h):

$$G_{yx}(P) = -2 - 1 \cdot 1 + 2 + 1 \cdot 1 + 1 \cdot 1 \cdot (-3) = -3.$$

Each second partial of G is a sum of products of continuous functions (second partials of h composed with polynomials, and polynomials), so G is C^2 near P. By Clairaut's Theorem for G, $G_{xy}(P) = G_{yx}(P)$, and

$$\boxed{G_{xy}(P) = -3}$$

Directional derivative by restriction to the line. Parametrise the line through P in direction $\mathbf{u} = \frac{1}{\sqrt{5}}(2, -1)$ by $x = 1 + \frac{2t}{\sqrt{5}}$, $y = 1 - \frac{t}{\sqrt{5}}$. Along it

$$u(t) = x - y = \frac{3t}{\sqrt{5}},$$
$$v(t) = xy = \Big(1 + \frac{2t}{\sqrt{5}}\Big)\Big(1 - \frac{t}{\sqrt{5}}\Big) = 1 + \frac{t}{\sqrt{5}} - \frac{2t^2}{5},$$

so $u'(0) = \frac{3}{\sqrt{5}}$ and $v'(t) = \frac{1}{\sqrt{5}} - \frac{4t}{5}$, $v'(0) = \frac{1}{\sqrt{5}}$. Let $\varphi(t) = G(P + t\mathbf{u}) = h(u(t), v(t))$. By the definition of the directional derivative and the one-variable Chain Rule,

$$D_{\mathbf{u}} G(P) = \varphi'(0) = h_u(0,1)\, u'(0) + h_v(0,1)\, v'(0) = 3 \cdot \frac{3}{\sqrt{5}} + 2 \cdot \frac{1}{\sqrt{5}}.$$

Therefore

$$\boxed{D_{\mathbf{u}} G(P) = \tfrac{9+2}{\sqrt{5}} = \tfrac{11}{\sqrt{5}}}$$

Tangent line by the implicit slope. $G(P) = h(0, 1) = 2$, G is C^1 and $G_y(P) = -1 \neq 0$, so by the Implicit Function Theorem the level curve $G = 2$ is a graph $y = y(x)$ near P with $y(1) = 1$. Differentiating $G(x, y(x)) = 2$:

$$G_x + G_y\, y'(x) = 0 \implies y'(1) = -\frac{G_x(P)}{G_y(P)} = -\frac{5}{-1} = 5.$$

The tangent line is $y - 1 = 5(x - 1)$, that is,

$$\boxed{5(x-1) - (y-1) = 0}$$

Linear approximation by the total differential. With $\mathrm{d}x = 1.01 - 1 = 0.01$ and $\mathrm{d}y = 0.98 - 1 = -0.02$,

$$\mathrm{d}G = G_x(P)\,\mathrm{d}x + G_y(P)\,\mathrm{d}y = 5(0.01) + (-1)(-0.02) = 0.07,$$

so $G(1.01, 0.98) \approx G(P) + \mathrm{d}G = 2 + 0.07$. Therefore

$$\boxed{G(1.01, 0.98) \approx L(1.01, 0.98) = 2.07}$$

(b) *Solving for z explicitly.* $F(x, y, z) = 0$ is $\mathrm{e}^z = 2 - x - y^2$. Near $(0, 1)$ the right side is positive (it equals $2 - 0 - 1 = 1$ at $(0, 1)$ and is continuous), so

$$q(x, y) = \ln(2 - x - y^2), \qquad q(0, 1) = \ln 1 = 0,$$

which agrees with the IFT solution through $(0, 1, 0)$ (the IFT solution is unique near $(0, 1)$; $F_z = \mathrm{e}^z \neq 0$ there). Differentiate:

$$q_x = \frac{-1}{2 - x - y^2}, \qquad q_y = \frac{-2y}{2 - x - y^2}.$$

At $(0, 1)$, where $2 - x - y^2 = 1$:

$$q_x(0, 1) = \frac{-1}{1} = -1, \qquad q_y(0, 1) = \frac{-2 \cdot 1}{1} = -2.$$

The linear approximation with $\Delta x = 0.03$, $\Delta y = -0.01$ gives

$$\boxed{q(0.03, 0.99) \approx 0 - 0.03 - 2(-0.01) = -0.01}$$

For comparison, the exact value is $q(0.03, 0.99) = \ln(2 - 0.03 - 0.9801) = \ln(0.9899) \approx -0.01015$.

Method 3: Second-order Taylor polynomial of the composition

This method should be used only if the course allows this interpretation.

(a) Write $x = 1 + a$, $y = 1 + b$. Then

$$u = x - y = a - b, \qquad v - 1 = xy - 1 = (1 + a)(1 + b) - 1 = a + b + ab.$$

Since h is C^2 near $(0, 1)$, its second-order Taylor expansion with $U = u - 0$, $W = v - 1$ is

$$\begin{aligned} h(U, 1 + W) &= h(0, 1) + h_u U + h_v W + \tfrac{1}{2}\big(h_{uu}U^2 + 2h_{uv}UW + h_{vv}W^2\big) + o(U^2 + W^2) \\ &= 2 + 3U + 2W + U^2 + UW - \tfrac{3}{2}W^2 + o(U^2 + W^2), \end{aligned}$$

with all coefficients evaluated at $(0, 1)$. Substitute $U = a - b$, $W = a + b + ab$, keeping terms of degree at most 2 in (a, b) (cubic terms and $o(U^2 + W^2)$ are $o(a^2 + b^2)$, because $U^2 + W^2 \le C(a^2 + b^2)$ near 0):

$$\begin{aligned} 3U + 2W &= 3a - 3b + 2a + 2b + 2ab = 5a - b + 2ab, \\ U^2 &= a^2 - 2ab + b^2, \\ UW &= (a - b)(a + b) + (a - b)ab = a^2 - b^2 + (\text{cubic}), \\ -\tfrac{3}{2}W^2 &= -\tfrac{3}{2}(a + b)^2 + (\text{cubic}) = -\tfrac{3}{2}a^2 - 3ab - \tfrac{3}{2}b^2 + (\text{cubic}). \end{aligned}$$

Collect: the a^2 coefficient is $1 + 1 - \frac{3}{2} = \frac{1}{2}$, the ab coefficient is $2 - 2 - 3 = -3$, the b^2 coefficient is $1 - 1 - \frac{3}{2} = -\frac{3}{2}$. Hence

$$G(1 + a, 1 + b) = 2 + 5a - b + \tfrac{1}{2}a^2 - 3ab - \tfrac{3}{2}b^2 + o(a^2 + b^2).$$

G is C^2 near P, and its own Taylor expansion reads

$$\begin{aligned} G(1 + a, 1 + b) = G(P) &+ G_x(P)a + G_y(P)b \\ &+ \tfrac{1}{2}\big(G_{xx}(P)a^2 + 2G_{xy}(P)ab + G_{yy}(P)b^2\big) + o(a^2 + b^2). \end{aligned}$$

The quadratic Taylor polynomial is unique, so comparing coefficients (the ab coefficient of G's expansion is $\frac{1}{2} \cdot 2G_{xy}(P) = G_{xy}(P)$) gives

$$G(P) = 2, \qquad G_x(P) = 5, \qquad G_y(P) = -1, \qquad G_{xy}(P) = -3.$$

Therefore

$$\boxed{G_{xy}(P) = -3}$$

The linear part $5a - b$ is $\nabla G(P) \cdot (a, b)$ with $\nabla G(P) = (5, -1)$. For $\mathbf{u} = \frac{1}{\sqrt{5}}(2, -1)$ (the unit vector from P towards $(3, 0)$),

$$\boxed{D_{\mathbf{u}}G(P) = \tfrac{1}{\sqrt{5}}(5, -1) \cdot (2, -1) = \tfrac{11}{\sqrt{5}}}$$

The tangent line to $G = G(P) = 2$ is where the linear part vanishes, $5a - b = 0$:

$$\boxed{5(x - 1) - (y - 1) = 0}$$

The linear approximation keeps $2 + 5a - b$; with $a = 0.01$, $b = -0.02$ this is $2 + 0.05 + 0.02$:

$$\boxed{G(1.01, 0.98) \approx L(1.01, 0.98) = 2.07}$$

(b) *Tangent plane to the level surface.* F is C^1, $F(0,1,0) = 0$ and

$$\nabla F(0,1,0) = (F_x, F_y, F_z)\big|_{(0,1,0)} = (1,\ 2\cdot 1,\ \mathrm{e}^0) = (1,2,1) \neq \mathbf{0}.$$

Since $F_z(0,1,0) = 1 \neq 0$, the Implicit Function Theorem makes the surface $F = 0$ the graph $z = q(x,y)$ near $(0,1,0)$, and its tangent plane there is the tangent plane of the level surface:

$$1\cdot(x-0) + 2\cdot(y-1) + 1\cdot(z-0) = 0 \iff z = -x - 2(y-1).$$

The tangent plane of a graph is $z = q(0,1) + q_x(0,1)x + q_y(0,1)(y-1)$, so comparing coefficients gives $q_x(0,1) = -1$ and $q_y(0,1) = -2$. Evaluating the plane at $(0.03, 0.99)$:

$$\boxed{q(0.03, 0.99) \approx 0 - 0.03 - 2(-0.01) = -0.01}$$

UNOFFICIAL QRS marking scheme (25 marks)

(a) **(15 marks)**

(i) **(4 marks)** chain-rule gradient;

(ii) **(3 marks)** unit direction and derivative;

(iii) **(2 marks)** tangent line;

(iv) **(2 marks)** approximation;

(v) **(4 marks)** mixed second derivative.

(b) **(10 marks)**

(i) **(3 marks)** IFT hypotheses;

(ii) **(4 marks)** derivatives;

(iii) **(3 marks)** approximation.

Unofficial QRS@NTU solutions

Question 3 (25 Marks)

PRACTICE PROBLEM 3. Let $f(x,y) = x^2 - y^2$ on $C = \{(x,y) \in \mathbb{R}^2 : x^2 + 9y^2 \leq 9\}$.

(a) Find and classify every interior critical point. **(8 marks)**

(b) Use Lagrange multipliers to find every extremum candidate on $x^2+9y^2 = 9$. **(12 marks)**

(c) Use the Extreme Value Theorem to state all absolute extrema on C. **(5 marks)**

Solution

Method 1: Second Derivative Test, Lagrange multipliers and the Extreme Value Theorem

(a) The first partial derivatives of $f(x,y) = x^2 - y^2$ are

$$f_x = 2x, \qquad f_y = -2y.$$

Setting $\nabla f = \mathbf{0}$:

$$2x = 0, \quad -2y = 0 \iff (x,y) = (0,0).$$

The point $(0,0)$ is interior to C because $0^2+9\cdot 0^2 = 0 < 9$. The second partial derivatives are constant:

$$f_{xx} = 2, \qquad f_{yy} = -2, \qquad f_{xy} = 0.$$

f is C^2 (a polynomial), so the Second Derivative Test applies with

$$D = f_{xx}f_{yy} - f_{xy}^2 = 2\cdot(-2) - 0^2 = -4 < 0.$$

Therefore

$$\boxed{(0,0) \text{ is a saddle point}}$$

(b) *Lagrange condition.* Let $g(x,y) = x^2 + 9y^2$, so the boundary is $g = 9$. f and g are C^1, and

$$\nabla g = (2x, 18y),$$

which is $\mathbf{0}$ only at $(0,0)$; since $g(0,0) = 0 \neq 9$, $\nabla g \neq \mathbf{0}$ at every point of $g = 9$. Hence every extremum of f on $g = 9$ satisfies $\nabla f = \lambda \nabla g$ for some λ:

$$2x = \lambda \cdot 2x, \tag{1}$$
$$-2y = \lambda \cdot 18y, \tag{2}$$
$$x^2 + 9y^2 = 9. \tag{3}$$

From (1), $2x(1-\lambda) = 0$, so $x = 0$ or $\lambda = 1$.

Case $x = 0$. Then (3) gives $9y^2 = 9$, so $y = \pm 1$. Equation (2) becomes $-2(\pm 1) = 18\lambda(\pm 1)$, so $\lambda = -\frac{1}{9}$, which is consistent. The values are

$$f(0,1) = 0^2 - 1^2 = -1, \qquad f(0,-1) = 0^2 - (-1)^2 = -1.$$

Case $\lambda = 1$, $x \neq 0$. Then (2) gives $-2y = 18y$, so $20y = 0$ and $y = 0$. Equation (3) gives $x^2 = 9$, so $x = \pm 3$ (both nonzero, consistent). The values are

$$f(3,0) = 3^2 - 0^2 = 9, \qquad f(-3,0) = (-3)^2 - 0^2 = 9.$$

The two cases exhaust (1), so every candidate on the boundary is listed:

$$\boxed{(\pm 3, 0) \text{ with } f = 9, \qquad (0, \pm 1) \text{ with } f = -1}$$

(c) *EVT hypotheses.* f is continuous (a polynomial). $C = \{g \le 9\}$ is closed, being the preimage of the closed set $(-\infty, 9]$ under the continuous g. C is bounded: $x^2 \le 9$ and $9y^2 \le 9$ give $C \subseteq [-3,3] \times [-1,1]$. By the Extreme Value Theorem, f attains an absolute maximum and minimum on C, each at an interior critical point or on the boundary (f is differentiable everywhere, so there are no singular points).

Compare candidates.

$$f(0,0) = 0, \qquad f(\pm 3, 0) = 9, \qquad f(0, \pm 1) = -1.$$

The largest value is 9 and the smallest is -1. Therefore

$$\boxed{\max_C f = 9 \text{ at } (\pm 3, 0), \qquad \min_C f = -1 \text{ at } (0, \pm 1)}$$

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Method 2: Restriction to the axes and parametrising the constraint

(a) As in Method 1, $f_x = 2x$ and $f_y = -2y$ vanish together only at $(0,0)$, an interior point of C, and $f(0,0) = 0$. Restrict f to the two axes:

$$f(x,0) = x^2 > 0 \quad (x \neq 0), \qquad f(0,y) = -y^2 < 0 \quad (y \neq 0).$$

Every disc centred at $(0,0)$ contains points $(x,0)$, $x \neq 0$, where $f > f(0,0)$, and points $(0,y)$, $y \neq 0$, where $f < f(0,0)$. So $(0,0)$ is neither a local maximum nor a local minimum:

$$\boxed{(0,0) \text{ is a saddle point}}$$

(b) The ellipse $x^2 + 9y^2 = 9$ is parametrised by

$$x = 3\cos t, \qquad y = \sin t, \qquad 0 \leq t < 2\pi,$$

since $(3\cos t)^2 + 9\sin^2 t = 9(\cos^2 t + \sin^2 t) = 9$. Along it,

$$\varphi(t) = f(3\cos t, \sin t) = 9\cos^2 t - \sin^2 t = 9(1 - \sin^2 t) - \sin^2 t = 9 - 10\sin^2 t.$$

Differentiate:

$$\varphi'(t) = -20\sin t \cos t = -10\sin 2t.$$

The curve is closed and φ is 2π-periodic, so there are no endpoints; the candidates are the zeros of φ':

$$\sin 2t = 0 \iff t \in \left\{0,\ \tfrac{\pi}{2},\ \pi,\ \tfrac{3\pi}{2}\right\}.$$

Evaluate each:

$$\begin{aligned} t = 0 &: \ (3,0), \quad \varphi = 9 - 10 \cdot 0 = 9, \\ t = \tfrac{\pi}{2} &: \ (0,1), \quad \varphi = 9 - 10 \cdot 1 = -1, \\ t = \pi &: \ (-3,0), \quad \varphi = 9 - 10 \cdot 0 = 9, \\ t = \tfrac{3\pi}{2} &: \ (0,-1), \quad \varphi = 9 - 10 \cdot 1 = -1. \end{aligned}$$

Therefore

$$\boxed{(\pm 3, 0) \text{ with } f = 9, \qquad (0, \pm 1) \text{ with } f = -1}$$

(c) f is continuous and C is closed and bounded (as in Method 1), so by the Extreme Value Theorem the absolute extrema exist and occur among the interior critical point and the boundary candidates. Their values are $f(0,0) = 0$, $f(\pm 3, 0) = 9$, $f(0, \pm 1) = -1$. Therefore

$$\boxed{\max_C f = 9 \text{ at } (\pm 3, 0), \qquad \min_C f = -1 \text{ at } (0, \pm 1)}$$

Method 3: Eliminating x on the boundary

This method applies to parts (b) and (c).

On $x^2 + 9y^2 = 9$ we have $x^2 = 9 - 9y^2$, and $x^2 \geq 0$ forces $-1 \leq y \leq 1$. Substituting,

$$f(x, y) = (9 - 9y^2) - y^2 = 9 - 10y^2 =: k(y), \qquad -1 \leq y \leq 1.$$

Each $y \in (-1, 1)$ corresponds to the two boundary points $x = \pm\sqrt{9 - 9y^2}$, and $y = \pm 1$ to $x = 0$. k is continuous on the closed interval $[-1, 1]$, so its extrema occur at critical points or endpoints. Since

$$k'(y) = -20y = 0 \iff y = 0,$$

the candidates are:

$$\begin{aligned} y = 0 &: \ x^2 = 9, \ x = \pm 3, \quad k(0) = 9, \\ y = \pm 1 &: \ x^2 = 0, \ x = 0, \quad k(\pm 1) = 9 - 10 = -1. \end{aligned}$$

Hence

$$\boxed{(\pm 3, 0) \text{ with } f = 9, \qquad (0, \pm 1) \text{ with } f = -1}$$

For (c), the Extreme Value Theorem applies as in Method 1; comparing these with the interior value $f(0, 0) = 0$:

$$\boxed{\max_C f = 9 \text{ at } (\pm 3, 0), \qquad \min_C f = -1 \text{ at } (0, \pm 1)}$$

Method 4: Direct bounds on C

This method applies to part (c).

Let $(x, y) \in C$, so $x^2 + 9y^2 \leq 9$.

Upper bound. From $x^2 \leq 9 - 9y^2$ and $y^2 \geq 0$,

$$f(x, y) = x^2 - y^2 \leq 9 - 9y^2 - y^2 = 9 - 10y^2 \leq 9.$$

Equality needs $y^2 = 0$ and $x^2 = 9 - 9y^2 = 9$, that is $(x, y) = (\pm 3, 0)$, where $f = 9$.

Lower bound. From $x^2 \geq 0$, and $9y^2 \leq 9 - x^2 \leq 9$ so $y^2 \leq 1$,

$$f(x, y) = x^2 - y^2 \geq -y^2 \geq -1.$$

Equality needs $x^2 = 0$ and $y^2 = 1$, that is $(x, y) = (0, \pm 1)$, where $f = -1$. Both bounds are attained, so

$$\boxed{\max_C f = 9 \text{ at } (\pm 3, 0), \qquad \min_C f = -1 \text{ at } (0, \pm 1)}$$

UNOFFICIAL QRS marking scheme (25 marks)

(a) **(8 marks)**

- (i) **(3 marks)** critical point;
- (ii) **(3 marks)** discriminant;
- (iii) **(2 marks)** classification.

(b) **(12 marks)**

- (i) **(4 marks)** Lagrange system;

(ii) **(4 marks)** all branches;

(iii) **(4 marks)** candidates and values.

(c) **(5 marks)**

(i) **(2 marks)** EVT hypotheses;

(ii) **(3 marks)** global extrema.

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Question 4 (25 Marks)

PRACTICE PROBLEM 4.

(a) Reverse the order of integration and evaluate

$$\int_0^1 \left[\int_{\sqrt{y}}^1 (1+x)\,\mathrm{d}x \right] \mathrm{d}y.$$

(10 marks)

(b) Let D be the part of $x^2+y^2 \le 4y$ in the first quadrant. Find the volume above D and below $z = 20-x^2-y^2$, using polar coordinates and a height double integral. **(15 marks)**

Solution

Method 1: Reversal of order (Fubini) and polar coordinates centred at the origin

(a) *Region in the given order.* The integral is over

$$R = \{(x,y) :\ 0 \le y \le 1,\ \sqrt{y} \le x \le 1\}.$$

Region in the reversed order. If $(x,y) \in R$, then $0 \le \sqrt{y} \le x \le 1$, so $0 \le x \le 1$; squaring $\sqrt{y} \le x$ (both sides nonnegative) gives $y \le x^2$, and $y \ge 0$. Conversely, if $0 \le x \le 1$ and $0 \le y \le x^2$, then $y \le x^2 \le 1$ and $\sqrt{y} \le \sqrt{x^2} = x$. Hence

$$R = \{(x,y) :\ 0 \le x \le 1,\ 0 \le y \le x^2\}.$$

The region lies above the x-axis and below the parabola $y = x^2$ (equivalently, to the right of $x = \sqrt{y}$), for $0 \le x \le 1$.

Fubini hypotheses. The integrand $1+x$ is continuous on the closed bounded region R, which is of both types above, so Fubini's Theorem allows the order to be reversed:

$$\int_0^1 \int_{\sqrt{y}}^1 (1+x)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_0^{x^2} (1+x)\,\mathrm{d}y\,\mathrm{d}x.$$

Inner integral (x fixed, $1+x$ constant in y):

$$\int_0^{x^2} (1+x)\,\mathrm{d}y = (1+x)\big[y\big]_0^{x^2} = (1+x)x^2 = x^2+x^3.$$

Outer integral:

$$\int_0^1 (x^2+x^3)\,\mathrm{d}x = \left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{4+3}{12}.$$

Therefore

$$\boxed{\int_0^1 \int_{\sqrt{y}}^1 (1+x)\,\mathrm{d}x\,\mathrm{d}y = \frac{7}{12}}$$

(b) *The region.* Completing the square, $x^2+y^2 \le 4y \iff x^2+(y-2)^2 \le 4$, the closed disc of centre $(0,2)$ and radius 2. On this disc $0 \le y \le 4$, so it already lies in $y \ge 0$; the first-quadrant part D is the right half $x \ge 0$.

Polar description. With $x = r\cos\theta$, $y = r\sin\theta$, $r \geq 0$:

$$x^2 + y^2 \leq 4y \iff r^2 \leq 4r\sin\theta \iff r \leq 4\sin\theta \quad (r > 0).$$

The condition $r \leq 4\sin\theta$ requires $\sin\theta \geq 0$, and $x \geq 0$ requires $\cos\theta \geq 0$, so

$$D = \left\{(r,\theta) :\ 0 \leq \theta \leq \tfrac{\pi}{2},\ 0 \leq r \leq 4\sin\theta\right\}.$$

Jacobian.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r,$$

so $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$.

Height. On D, $x^2 + y^2 \leq 4y \leq 16$, so $20 - x^2 - y^2 \geq 4 > 0$; the solid lies above D and its volume is the height double integral

$$V = \iint_D (20 - x^2 - y^2)\,\mathrm{d}A = \int_0^{\pi/2}\int_0^{4\sin\theta} (20 - r^2)\,r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Inner integral:

$$\int_0^{4\sin\theta} (20r - r^3)\,\mathrm{d}r = \left[10r^2 - \frac{r^4}{4}\right]_0^{4\sin\theta} = 10\cdot 16\sin^2\theta - \frac{256\sin^4\theta}{4} = 160\sin^2\theta - 64\sin^4\theta.$$

Power reduction. Using $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ and $\cos^2 2\theta = \frac{1+\cos 4\theta}{2}$,

$$\sin^4\theta = \frac{(1-\cos 2\theta)^2}{4} = \frac{1 - 2\cos 2\theta + \cos^2 2\theta}{4} = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}.$$

Hence

$$\int_0^{\pi/2} \sin^2\theta\,\mathrm{d}\theta = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{4},$$
$$\int_0^{\pi/2} \sin^4\theta\,\mathrm{d}\theta = \left[\frac{3\theta}{8} - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right]_0^{\pi/2} = \frac{3\pi}{16},$$

since $\sin\pi = \sin 2\pi = 0$. *Outer integral:*

$$V = 160\cdot\frac{\pi}{4} - 64\cdot\frac{3\pi}{16} = 40\pi - 12\pi.$$

Therefore

$$\boxed{V = 28\pi}$$

Method 2: Original order of integration and polar coordinates centred at $(0, 2)$

(a) Evaluate in the given order on $R = \{0 \le y \le 1,\ \sqrt{y} \le x \le 1\}$ (Fubini's Theorem gives the same value as the reversed order, since $1 + x$ is continuous on R). *Inner integral* (y fixed):

$$\int_{\sqrt{y}}^{1} (1 + x)\,\mathrm{d}x = \left[x + \frac{x^2}{2}\right]_{\sqrt{y}}^{1} = \left(1 + \frac{1}{2}\right) - \left(\sqrt{y} + \frac{y}{2}\right) = \frac{3}{2} - y^{1/2} - \frac{y}{2}.$$

Outer integral:

$$\int_0^1 \left(\frac{3}{2} - y^{1/2} - \frac{y}{2}\right) \mathrm{d}y = \left[\frac{3}{2}y - \frac{2}{3}y^{3/2} - \frac{y^2}{4}\right]_0^1 = \frac{3}{2} - \frac{2}{3} - \frac{1}{4} = \frac{18 - 8 - 3}{12}.$$

Therefore

$$\boxed{\int_0^1 \int_{\sqrt{y}}^{1} (1 + x)\,\mathrm{d}x\,\mathrm{d}y = \frac{7}{12}}$$

(b) *Shifted polar coordinates.* D is the right half of the disc $x^2 + (y - 2)^2 \le 4$ (see Method 1). Put

$$x = \rho\cos\varphi, \qquad y = 2 + \rho\sin\varphi, \qquad \rho \ge 0.$$

Then $x^2 + (y - 2)^2 = \rho^2 \le 4$ gives $0 \le \rho \le 2$, and $x \ge 0$ gives $\cos\varphi \ge 0$, so $-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$. The condition $y \ge 0$ holds automatically: $y = 2 + \rho\sin\varphi \ge 2 - 2 = 0$. Thus

$$D = \left\{(\rho, \varphi) :\ 0 \le \rho \le 2,\ -\tfrac{\pi}{2} \le \varphi \le \tfrac{\pi}{2}\right\}.$$

Jacobian.

$$\frac{\partial(x, y)}{\partial(\rho, \varphi)} = \det \begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix} = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho.$$

Height in the new variables.

$$x^2 + y^2 = \rho^2\cos^2\varphi + 4 + 4\rho\sin\varphi + \rho^2\sin^2\varphi = \rho^2 + 4\rho\sin\varphi + 4,$$

so $20 - x^2 - y^2 = 16 - \rho^2 - 4\rho\sin\varphi$, which is positive on D (Method 1). Hence

$$V = \int_{-\pi/2}^{\pi/2} \int_0^2 \big(16 - \rho^2 - 4\rho\sin\varphi\big)\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi.$$

Inner integral:

$$\int_0^2 \big(16\rho - \rho^3 - 4\rho^2\sin\varphi\big)\,\mathrm{d}\rho = \left[8\rho^2 - \frac{\rho^4}{4} - \frac{4\rho^3}{3}\sin\varphi\right]_0^2 = 32 - 4 - \frac{32}{3}\sin\varphi.$$

Outer integral:

$$V = \int_{-\pi/2}^{\pi/2} \left(28 - \frac{32}{3}\sin\varphi\right) \mathrm{d}\varphi = \left[28\varphi + \frac{32}{3}\cos\varphi\right]_{-\pi/2}^{\pi/2} = 28\pi + \frac{32}{3}(0 - 0).$$

Therefore

$$\boxed{V = 28\pi}$$

Method 3: Cartesian height integral with a trigonometric substitution

This method applies to part (b).

Region as a type II region. On D, $x^2 \le 4y-y^2$ with $x \ge 0$, and $4y-y^2 \ge 0$ forces $0 \le y \le 4$:

$$D = \left\{(x,y) :\ 0 \le y \le 4,\ 0 \le x \le \sqrt{4y-y^2}\right\}.$$

The height $20 - x^2 - y^2$ is continuous and positive on D (Method 1), so by Fubini's Theorem

$$V = \int_0^4 \int_0^{\sqrt{4y-y^2}} (20 - x^2 - y^2)\,\mathrm{d}x\,\mathrm{d}y.$$

Inner integral. Write $s = \sqrt{4y-y^2}$, so $s^2 = 4y - y^2$:

$$\begin{aligned}\int_0^s (20 - y^2 - x^2)\,\mathrm{d}x &= \left[(20-y^2)x - \frac{x^3}{3}\right]_0^s = (20-y^2)s - \frac{s\cdot s^2}{3} \\ &= s\left(20 - y^2 - \frac{4y-y^2}{3}\right) = \frac{s\,(60-4y-2y^2)}{3}.\end{aligned}$$

Shift. Let $y = 2 + t,\ \mathrm{d}y = \mathrm{d}t$, with new limits $t = -2$ to $t = 2$. Then

$$\begin{aligned}4y - y^2 &= 8 + 4t - (4 + 4t + t^2) = 4 - t^2, \\ 60 - 4y - 2y^2 &= 60 - 8 - 4t - (8 + 8t + 2t^2) = 44 - 12t - 2t^2,\end{aligned}$$

so

$$V = \frac{1}{3}\int_{-2}^{2} \sqrt{4-t^2}\,\left(44 - 12t - 2t^2\right)\mathrm{d}t.$$

The odd term. $\int t\sqrt{4-t^2}\,\mathrm{d}t = -\frac{1}{3}(4-t^2)^{3/2}$, which equals 0 at $t = \pm 2$, so $\int_{-2}^{2} t\sqrt{4-t^2}\,\mathrm{d}t = 0$.

The other two terms. Substitute $t = 2\sin u,\ \mathrm{d}t = 2\cos u\,\mathrm{d}u$, with new limits $u = -\frac{\pi}{2}$ to $u = \frac{\pi}{2}$, where $\cos u \ge 0$ and $\sqrt{4-t^2} = 2\cos u$:

$$\begin{aligned}\int_{-2}^{2} \sqrt{4-t^2}\,\mathrm{d}t &= \int_{-\pi/2}^{\pi/2} 4\cos^2 u\,\mathrm{d}u = \int_{-\pi/2}^{\pi/2} 2(1+\cos 2u)\,\mathrm{d}u = \big[2u + \sin 2u\big]_{-\pi/2}^{\pi/2} = 2\pi, \\ \int_{-2}^{2} t^2\sqrt{4-t^2}\,\mathrm{d}t &= \int_{-\pi/2}^{\pi/2} 16\sin^2 u\cos^2 u\,\mathrm{d}u = \int_{-\pi/2}^{\pi/2} 4\sin^2 2u\,\mathrm{d}u \\ &= \int_{-\pi/2}^{\pi/2} 2(1-\cos 4u)\,\mathrm{d}u = \left[2u - \frac{\sin 4u}{2}\right]_{-\pi/2}^{\pi/2} = 2\pi.\end{aligned}$$

Evaluation.

$$V = \frac{1}{3}\big(44\cdot 2\pi - 12\cdot 0 - 2\cdot 2\pi\big) = \frac{84\pi}{3}.$$

Therefore

$$\boxed{V = 28\pi}$$

UNOFFICIAL QRS marking scheme (25 marks)

(a) **(10 marks)**

(i) **(3 marks)** reversed region;

(ii) **(3 marks)** bounds;

(iii) **(2 marks)** inner integral;

(iv) **(2 marks)** value 7/12.

(b) **(15 marks)**

(i) **(4 marks)** polar geometric reasoning;

(ii) **(3 marks)** height and Jacobian;

(iii) **(5 marks)** double integral;

(iv) **(3 marks)** value 28π. Equivalent correct Cartesian work receives full credit.

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