MH2100 Calculus III

Midterm I (Solutions)

AY 2018/2019, Semester 1

Quantitative Research Society @NTU

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Overview

- **Coverage:** parametric curves and vector calculus; partial derivatives, limits, continuity and differentiability; critical points and the second-derivative test.
- **Format:** three questions, 100 marks, 90 minutes (official paper: September 2018).
- **Historical permitted material:** the September 2018 paper permitted a two-sided A4 piece of paper with handwritten notes. Consult current course materials for present assessment rules.

Instructions (verbatim)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. A two-sided A4 piece of paper with handwritten notes is permitted.

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Unofficial QRS@NTU

Question 1 (50 Marks)

Circle (only) the correct answer for each question. Each question is worth 5 marks.

(i) Find parametric equations to represent the line segment from $(-3, 4)$ to $(12, -8)$.

(a) $x = -3 - 15t, \quad y = 4 - 12t, \quad 0 \le t \le 1$

(b) $x = -3 + 15t, \quad y = 4 - 12t, \quad 0 \le t \le 2$

(c) $x = 8 - 15t, \quad y = 4 - 12t, \quad 0 \le t \le 2$

(d) $x = -3 + 15t, \quad y = 4 - 12t, \quad 0 \le t \le 1$

(e) $x = -3 - 15t, \quad y = 4 - 12t, \quad 0 \le t \le 2$

(ii) Find the point(s) on the following curve where the tangent is horizontal.

$$x = t^3 - 3t + 2, \qquad y = t^3 - 3t^2 + 2$$

(a) $(0, 0)$

(b) $(0, 0)$ and $(2, -2)$

(c) $(2, 2)$ and $(-2, -2)$

(d) $(2, 2)$ and $(4, -2)$

(e) None of the above

(iii) Find parametric equations for the tangent line at the point when $t = 1$ to the curve with parametric equations

$$x = 3t, \qquad y = 5t^2, \qquad z = 4t^3.$$

(a) $x = 1 + 3t, \quad y = 1 + 10t, \quad z = 1 + 12t$

(b) $x = 3 + 3t, \quad y = 5 + 10t, \quad z = 4 + 12t$

(c) $x = 1 + 3t, \quad y = 1 + 5t, \quad z = 1 + 4t$

(d) $x = 3 + 3t, \quad y = 5 + 5t, \quad z = 4 + 4t$

(e) None of the above

(iv) Find the unit tangent vector for the curve given by

$$\mathbf{r}(t) = \left\langle \frac{1}{5}t^5, \frac{1}{3}t^3, t \right\rangle.$$

(a) $\dfrac{\langle t^4, t^2, 1\rangle}{\sqrt{t^{10} + t^4}}$

(b) $\dfrac{\langle t^4, t^2, 1\rangle}{\sqrt{4t^8 + 4t^4 + 1}}$

(c) $\dfrac{\langle t^4, t^2, 1\rangle}{\sqrt{t^8 + t^4 + 1}}$

(d) $\dfrac{\langle t^4, t^2, 1\rangle}{\sqrt{t^8 + t^4}}$

(e) None of the above

(v) Find the directional derivative of the function $f(x, y, z) = xyz$ in the direction of the vector $\langle 5, -3, 2\rangle$.

(a) $\frac{1}{\sqrt{38}}(5yz - 3xz + 2xy)$

(b) $\frac{1}{\sqrt{38}}(5xy - 3xz + 2yz)$

(c) $5yz - 3xz + 2xy$

(d) $5xy - 3xz + 2yz$

(e) None of the above

(vi) Find the gradient vector of the function $f(x, y, z) = z^7 e^{2x\sqrt{y}}$.

(a) $e^{x^2\sqrt{y}} \left\langle 7xz\sqrt{y}, \frac{x^2 z}{2\sqrt{y}}, 2 \right\rangle$

(b) $ze^{2x\sqrt{y}} \left\langle z\sqrt{y}, \frac{xz}{2\sqrt{y}}, 7z^5 \right\rangle$

(c) $e^{x^2\sqrt{y}} \left\langle 2xz\sqrt{y}, \frac{x^2 z}{7\sqrt{y}}, 2 \right\rangle$

(d) $ze^{2x\sqrt{y}} \left\langle 2z^6\sqrt{y}, \frac{xz^6}{\sqrt{y}}, 7z^5 \right\rangle$

(vii) Use the Chain Rule to find $\frac{\mathrm{d}w}{\mathrm{d}t}$ for

$$w = 3x^6y^3z, \qquad x = 6t, \qquad y = \cos 2t, \qquad z = t \sin t.$$

(a) $3x^5y^2(36yz - 6xz \sin 2t + xy(\sin t + t \cos t))$

(b) $3x^6y^3(36yz - 6xz \sin 2t + xy(\sin t + t \cos t))$

(c) $648x^5y^2 \sin 2t(\sin t + t \cos t)$

(d) $648x^6y^3 \sin 2t(\sin t + t \cos t)$

(e) None of the above

(viii) Find an equation for the tangent plane to the surface given by the equation $xy+yz+xz = 38$ at the point $(4, 2, 5)$.

(a) $7x + 9y + 6z = 70$

(b) $7x + 9y + 6z = 74$

(c) $7x + 9y + 6z = 76$

(d) $7x + 9y + 6z = 78$

(e) None of the above

(ix) Find an equation for the tangent plane to the surface $z = 4x^3y^2 + 2y$ at the point $(1, -2, 12)$.

(a) $48x + 14y - z = 60$

(b) $48x - 14y - z = 64$

(c) $48x - 14y - z = 60$

(d) $48x + 14y - z = 64$

(e) None of the above

(x) Find and classify the critical points of the function $f(x, y) = \mathrm{e}^{-2x} \sin 4y$ for $x \geq 0$ and $0 \leq y \leq \pi/2$.

(a) Local maximum $f(0, \pi/8) = 1$

(b) Saddle point $f(0, 0) = 0$

(c) Local minimum $f(0, \pi/4) = 0$

(d) None of the above

Solution

Method 1: Direct computation

(i) Let $\mathbf{p} = (-3, 4)$ and $\mathbf{q} = (12, -8)$. By the affine parametrization of a line, the segment from $\mathbf{p}$ to $\mathbf{q}$ is

$$\mathbf{r}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}), \qquad 0 \leq t \leq 1.$$

The displacement is

$$\mathbf{q} - \mathbf{p} = \big(12 - (-3),\ -8 - 4\big) = (15, -12).$$

Hence

$$x = -3 + 15t, \qquad y = 4 - 12t, \qquad 0 \leq t \leq 1.$$

Check the endpoints:

$$t = 0 :\ (x, y) = (-3, 4), \qquad t = 1 :\ (x, y) = (-3 + 15,\ 4 - 12) = (12, -8).$$

Therefore

$$\boxed{\text{(d)} \quad x = -3 + 15t, \quad y = 4 - 12t, \quad 0 \leq t \leq 1.}$$

(ii) Differentiate both coordinates:

$$x'(t) = 3t^2 - 3 = 3(t - 1)(t + 1),$$
$$y'(t) = 3t^2 - 6t = 3t(t - 2).$$

The tangent is horizontal at a parameter with $y'(t) = 0$ and $x'(t) \neq 0$, because then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y'(t)}{x'(t)} = 0$.

$$y'(t) = 0 \iff t = 0 \text{ or } t = 2.$$

At these parameters

$$x'(0) = 3(-1)(1) = -3 \neq 0, \qquad x'(2) = 3(1)(3) = 9 \neq 0.$$

The corresponding points are

$$t = 0 :\quad x = 0 - 0 + 2 = 2, \quad y = 0 - 0 + 2 = 2,$$
$$t = 2 :\quad x = 8 - 6 + 2 = 4, \quad y = 8 - 12 + 2 = -2.$$

Therefore

$$\boxed{\text{(d)} \quad (2, 2) \text{ and } (4, -2).}$$

(iii) Let $\mathbf{r}(t) = \langle 3t, 5t^2, 4t^3 \rangle$. Then

$$\mathbf{r}'(t) = \langle 3, 10t, 12t^2 \rangle.$$

At $t = 1$,

$$\mathbf{r}(1) = \langle 3, 5, 4 \rangle, \qquad \mathbf{r}'(1) = \langle 3, 10, 12 \rangle \neq \mathbf{0},$$

so $t = 1$ is a regular parameter and the tangent line is $\mathbf{r}(1) + t\,\mathbf{r}'(1)$:

$$\langle x, y, z \rangle = \langle 3, 5, 4 \rangle + t\langle 3, 10, 12 \rangle.$$

Therefore

$$\boxed{\text{(b)} \quad x = 3 + 3t, \quad y = 5 + 10t, \quad z = 4 + 12t.}$$

(iv) Differentiate each component:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{5}t^5\right) = t^4, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}t^3\right) = t^2, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(t) = 1,$$

so

$$\mathbf{r}'(t) = \langle t^4, t^2, 1 \rangle.$$

Its length is

$$\|\mathbf{r}'(t)\| = \sqrt{(t^4)^2 + (t^2)^2 + 1^2} = \sqrt{t^8 + t^4 + 1} \ \geq 1 > 0,$$

so every t is regular and $\mathbf{T}(t) = \mathbf{r}'(t)/\|\mathbf{r}'(t)\|$:

$$\boxed{\text{(c)} \quad \mathbf{T}(t) = \frac{\langle t^4, t^2, 1 \rangle}{\sqrt{t^8 + t^4 + 1}}.}$$

(v) Let $f(x, y, z) = xyz$. The partial derivatives are

$$f_x = yz, \qquad f_y = xz, \qquad f_z = xy.$$

They are polynomials, hence continuous, so f is differentiable and $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ for every unit vector $\mathbf{u}$. The given vector has length

$$\|\langle 5, -3, 2 \rangle\| = \sqrt{25 + 9 + 4} = \sqrt{38}, \qquad \mathbf{u} = \frac{\langle 5, -3, 2 \rangle}{\sqrt{38}}.$$

Hence

$$D_{\mathbf{u}} f = \langle yz, xz, xy \rangle \cdot \frac{\langle 5, -3, 2 \rangle}{\sqrt{38}} = \frac{5yz - 3xz + 2xy}{\sqrt{38}}.$$

Therefore

$$\boxed{\text{(a)} \quad D_{\mathbf{u}} f = \frac{5yz - 3xz + 2xy}{\sqrt{38}}.}$$

(vi) The factor $\sqrt{y}$ is differentiable for $y > 0$, so work on $y > 0$. Let $f = z^7 \mathrm{e}^{2x\sqrt{y}}$. By the Chain Rule for the exponential,

$$\begin{aligned}
f_x &= z^7 \mathrm{e}^{2x\sqrt{y}} \cdot \frac{\partial}{\partial x}\left(2x\sqrt{y}\right) = z^7 \mathrm{e}^{2x\sqrt{y}} \cdot 2\sqrt{y}, \\
f_y &= z^7 \mathrm{e}^{2x\sqrt{y}} \cdot \frac{\partial}{\partial y}\left(2x\sqrt{y}\right) = z^7 \mathrm{e}^{2x\sqrt{y}} \cdot 2x \cdot \frac{1}{2\sqrt{y}} = z^7 \mathrm{e}^{2x\sqrt{y}} \cdot \frac{x}{\sqrt{y}}, \\
f_z &= 7z^6 \mathrm{e}^{2x\sqrt{y}}.
\end{aligned}$$

Collect the components and factor out $ze^{2x\sqrt{y}}$:

$$\nabla f = e^{2x\sqrt{y}}\left\langle 2z^7\sqrt{y}, \frac{xz^7}{\sqrt{y}}, 7z^6\right\rangle = ze^{2x\sqrt{y}}\left\langle 2z^6\sqrt{y}, \frac{xz^6}{\sqrt{y}}, 7z^5\right\rangle.$$

Therefore

$$\boxed{\text{(d).}}$$

(vii) The function $w = 3x^6y^3z$ is a polynomial, hence differentiable, and x, y, z are differentiable functions of t, so the Chain Rule gives

$$\frac{dw}{dt} = w_x\frac{dx}{dt} + w_y\frac{dy}{dt} + w_z\frac{dz}{dt}.$$

The partial derivatives are

$$w_x = 18x^5y^3z, \qquad w_y = 9x^6y^2z, \qquad w_z = 3x^6y^3.$$

The t-derivatives are (product rule for z)

$$\frac{dx}{dt} = 6, \qquad \frac{dy}{dt} = -2\sin 2t, \qquad \frac{dz}{dt} = \sin t + t\cos t.$$

Substitute:

$$\begin{aligned}\frac{dw}{dt} &= 6\cdot 18x^5y^3z + (-2\sin 2t)\cdot 9x^6y^2z + (\sin t + t\cos t)\cdot 3x^6y^3\\ &= 108x^5y^3z - 18x^6y^2z\sin 2t + 3x^6y^3(\sin t + t\cos t).\end{aligned}$$

Factor out $3x^5y^2$, using $108 = 3\cdot 36$ and $18 = 3\cdot 6$:

$$\frac{dw}{dt} = 3x^5y^2\big(36yz - 6xz\sin 2t + xy(\sin t + t\cos t)\big).$$

Therefore

$$\boxed{\text{(a).}}$$

(viii) Let $F(x, y, z) = xy + yz + xz$. The point lies on the surface:

$$F(4, 2, 5) = 8 + 10 + 20 = 38.$$

The partial derivatives are

$$F_x = y + z, \qquad F_y = x + z, \qquad F_z = x + y,$$

so

$$\nabla F(4, 2, 5) = \langle 2 + 5,\ 4 + 5,\ 4 + 2\rangle = \langle 7, 9, 6\rangle \neq \mathbf{0}.$$

Since $\nabla F \neq \mathbf{0}$, the gradient is normal to the level surface $F = 38$ at the point, and the tangent plane is

$$7(x - 4) + 9(y - 2) + 6(z - 5) = 0.$$

Expand: $7x + 9y + 6z = 28 + 18 + 30 = 76$. Therefore

$$\boxed{\text{(c)} \quad 7x + 9y + 6z = 76.}$$

(ix) Let $g(x,y) = 4x^3y^2 + 2y$. The point lies on the graph:

$$g(1,-2) = 4 \cdot 1 \cdot 4 + 2(-2) = 16 - 4 = 12.$$

The partial derivatives and their values at $(1,-2)$ are

$$\begin{aligned} g_x &= 12x^2y^2, & g_x(1,-2) &= 12 \cdot 1 \cdot 4 = 48, \\ g_y &= 8x^3y + 2, & g_y(1,-2) &= 8 \cdot 1 \cdot (-2) + 2 = -14. \end{aligned}$$

The graph of a differentiable g has tangent plane $z = g(a,b) + g_x(a,b)(x-a) + g_y(a,b)(y-b)$:

$$z = 12 + 48(x-1) - 14(y+2).$$

Expand:

$$z = 48x - 48 - 14y - 28 + 12 = 48x - 14y - 64.$$

Therefore

$$\boxed{\text{(b)} \quad 48x - 14y - z = 64.}$$

(x) The domain is $D = \{(x,y) : x \geq 0,\ 0 \leq y \leq \pi/2\}$; its interior is $x > 0$, $0 < y < \pi/2$. The partial derivatives are

$$f_x = -2\mathrm{e}^{-2x} \sin 4y, \qquad f_y = 4\mathrm{e}^{-2x} \cos 4y.$$

Since $\mathrm{e}^{-2x} > 0$ and $0 < 4y < 2\pi$ in the interior,

$$\begin{aligned} f_x = 0 &\iff \sin 4y = 0 \iff 4y = \pi \iff y = \frac{\pi}{4}, \\ f_y = 0 &\iff \cos 4y = 0 \iff 4y \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\} \iff y \in \left\{\frac{\pi}{8}, \frac{3\pi}{8}\right\}. \end{aligned}$$

No y lies in both lists, so f has no interior critical point. Under the literal definition (a critical point is an interior point with $\nabla f = \mathbf{0}$), none of (a)–(c) lists a critical point, and the answer is (d).

The listed points in (a)–(c) lie on the edge $x = 0$, so the intended reading is extrema relative to D, boundary included. On the edge $x = 0$,

$$f(0,y) = \sin 4y, \qquad f\left(0, \frac{\pi}{8}\right) = \sin \frac{\pi}{2} = 1.$$

For every $(x,y) \in D$, $x \geq 0$ gives $\mathrm{e}^{-2x} \leq \mathrm{e}^0 = 1$, and $|\sin 4y| \leq 1$, so

$$f(x,y) \leq \mathrm{e}^{-2x}|\sin 4y| \leq \mathrm{e}^{-2x} \leq 1 = f\left(0, \frac{\pi}{8}\right).$$

Hence $(0, \pi/8)$ is an absolute, and so a local, maximum of f relative to D.

$$\boxed{\text{intended: (a)} \quad \text{Local maximum } f(0,\pi/8) = 1; \qquad \text{literal: (d).}}$$

Method 2: Parameter elimination, option testing, first principles, substitution, level sets, monotonicity

(i) Eliminate the parameter. The line through $(-3, 4)$ and $(12, -8)$ has slope

$$m = \frac{-8-4}{12-(-3)} = \frac{-12}{15} = -\frac{4}{5},$$

so it is

$$y - 4 = -\frac{4}{5}(x+3), \qquad -3 \le x \le 12 \text{ on the segment.}$$

Put $x = -3 + 15t$. Then $-3 \le x \le 12 \iff 0 \le 15t \le 15 \iff 0 \le t \le 1$, and

$$y = 4 - \frac{4}{5}(15t) = 4 - 12t.$$

The other options fail an endpoint test:

$$\begin{aligned} \text{(a), (e)}: &\quad t = 1 \text{ gives } x = -3 - 15 = -18 \notin [-3, 12], \\ \text{(b)}: &\quad t = 2 \text{ gives } x = -3 + 30 = 27 \notin [-3, 12], \\ \text{(c)}: &\quad t = 0 \text{ gives } (8, 4) \neq (-3, 4). \end{aligned}$$

Therefore

$$\boxed{\text{(d)} \quad x = -3 + 15t, \quad y = 4 - 12t, \quad 0 \le t \le 1.}$$

(ii) Test the listed points. Write

$$x'(t) = 3(t-1)(t+1), \qquad y'(t) = 3t(t-2).$$

The point $(0, 0)$. Solve $x(t) = 0$:

$$t^3 - 3t + 2 = (t-1)^2(t+2) = 0 \iff t = 1 \text{ or } t = -2,$$

with $y(1) = 1 - 3 + 2 = 0$ and $y(-2) = -8 - 12 + 2 = -18$. So $(0, 0)$ occurs only at $t = 1$, where

$$x'(1) = 0, \qquad y'(1) = 3(1)(-1) = -3 \neq 0,$$

a vertical tangent. Options (a) and (b) fail.

The point $(-2, -2)$. Solve $x(t) = -2$, i.e. $p(t) = t^3 - 3t + 4 = 0$. Here $p'(t) = 3(t^2 - 1)$, so p increases on $(-\infty, -1]$, decreases on $[-1, 1]$ and increases on $[1, \infty)$. Since $p(1) = 2 > 0$, $p > 0$ on $[-1, \infty)$; since $p(-2) = -8 + 6 + 4 = 2 > 0$ and p increases on $(-\infty, -1]$, $p > 0$ on $[-2, -1]$. Hence the only real root t_0 satisfies $t_0 < -2$, and then

$$y(t_0) = t_0^3 - 3t_0^2 + 2 < -8 - 12 + 2 = -18 \neq -2.$$

So $(-2, -2)$ is not on the curve, and (c) fails.

The point $(2, 2)$. Solve $x(t) = 2$:

$$t^3 - 3t = t(t^2 - 3) = 0 \iff t \in \{0, \pm\sqrt{3}\},$$

with $y(0) = 2$ and $y(\pm\sqrt{3}) = -7 \pm 3\sqrt{3} \neq 2$. At $t = 0$: $y'(0) = 0$, $x'(0) = -3 \neq 0$, so the tangent is horizontal.

The point $(4, -2)$. Solve $x(t) = 4$:

$$t^3 - 3t - 2 = (t-2)(t+1)^2 = 0 \iff t = 2 \text{ or } t = -1,$$

with $y(2) = -2$ and $y(-1) = -1 - 3 + 2 = -2$. The curve passes through $(4, -2)$ twice. At $t = 2$: $y'(2) = 0$, $x'(2) = 9 \neq 0$, a horizontal tangent. (At $t = -1$: $x'(-1) = 0$, $y'(-1) = 9$, a vertical branch through the same point.)

Completeness: $y'(t) = 3t(t-2)$ vanishes only at $t = 0, 2$, so there are no further points, and (e) fails.

$$\boxed{\text{(d)} \quad (2,2) \text{ and } (4,-2).}$$

(iii) Eliminate the parameter. From $x = 3t$, $t = x/3$, so the curve is the pair of graphs

$$y = 5\Big(\frac{x}{3}\Big)^2 = \frac{5x^2}{9}, \qquad z = 4\Big(\frac{x}{3}\Big)^3 = \frac{4x^3}{27}.$$

At $t = 1$, $x = 3$, $y = 5$, $z = 4$. Differentiate in x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{10x}{9}\Big|_{x=3} = \frac{10}{3}, \qquad \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{12x^2}{27} = \frac{4x^2}{9}\Big|_{x=3} = 4.$$

A tangent direction is $\langle 1, \frac{10}{3}, 4\rangle$, or $3\langle 1, \frac{10}{3}, 4\rangle = \langle 3, 10, 12\rangle$. Put $x = 3 + 3t$; then $y = 5 + \frac{10}{3}(3t) = 5 + 10t$ and $z = 4 + 4(3t) = 4 + 12t$. Options (a), (c) pass through $(1, 1, 1) \neq (3, 5, 4)$ at $t = 0$, and (d) has direction $\langle 3, 5, 4\rangle$, which is not a multiple of $\langle 3, 10, 12\rangle$ ($3 = 3c$ forces $c = 1$, but $5 \neq 10$). Option (b) is exactly the line just found, so (e) also fails.

$$\boxed{\text{(b)} \quad x = 3 + 3t, \quad y = 5 + 10t, \quad z = 4 + 12t.}$$

(iv) Test unit length. A unit tangent vector is a positive multiple of $\mathbf{r}'(t) = \langle t^4, t^2, 1\rangle$ with length 1 for every t. All options have numerator $\langle t^4, t^2, 1\rangle$, whose length is

$$\|\langle t^4, t^2, 1\rangle\| = \sqrt{t^8 + t^4 + 1}.$$

$$\begin{aligned}
\text{(a), (d)}: &\quad t = 0 \text{ gives denominator } \sqrt{0} = 0, \text{ undefined;} \\
\text{(b)}: &\quad t = 1 \text{ gives length } \frac{\sqrt{1+1+1}}{\sqrt{4+4+1}} = \frac{\sqrt{3}}{3} \neq 1; \\
\text{(c)}: &\quad \text{length } \frac{\sqrt{t^8+t^4+1}}{\sqrt{t^8+t^4+1}} = 1 \text{ for every } t.
\end{aligned}$$

Option (c) is a positive multiple of $\mathbf{r}'(t)$ with length 1, so (e) also fails.

$$\boxed{\text{(c)} \quad \mathbf{T}(t) = \frac{\langle t^4, t^2, 1\rangle}{\sqrt{t^8 + t^4 + 1}}.}$$

(v) Use the definition $D_{\mathbf{u}}f(P) = \lim_{h\to 0} \dfrac{f(P + h\mathbf{u}) - f(P)}{h}$ with

$$\mathbf{u} = (a, b, c) = \frac{(5, -3, 2)}{\sqrt{38}}, \qquad a^2 + b^2 + c^2 = \frac{25 + 9 + 4}{38} = 1.$$

Expand the product:

$$\begin{aligned}
f(P + h\mathbf{u}) &= (x + ah)(y + bh)(z + ch) \\
&= xyz + h(ayz + bxz + cxy) + h^2(abz + acy + bcx) + h^3abc.
\end{aligned}$$

Hence, for $h \neq 0$,

$$\frac{f(P+h\mathbf{u})-f(P)}{h} = (ayz+bxz+cxy)+h(abz+acy+bcx)+h^2abc \xrightarrow[h\to 0]{} ayz+bxz+cxy.$$

Substitute $a = 5/\sqrt{38}$, $b = -3/\sqrt{38}$, $c = 2/\sqrt{38}$:

$$\boxed{\text{(a)} \quad D_{\mathbf{u}}f = \frac{5yz-3xz+2xy}{\sqrt{38}}.}$$

(vi) Use logarithmic differentiation on $y > 0$, $z \neq 0$:

$$\ln|f| = 7\ln|z| + 2x\sqrt{y}.$$

Differentiate in each variable ($\partial_z \ln|z| = 1/z$):

$$\frac{f_x}{f} = 2\sqrt{y}, \qquad \frac{f_y}{f} = \frac{2x}{2\sqrt{y}} = \frac{x}{\sqrt{y}}, \qquad \frac{f_z}{f} = \frac{7}{z}.$$

Multiply by $f = z^7 \mathrm{e}^{2x\sqrt{y}}$:

$$\nabla f = z^7\mathrm{e}^{2x\sqrt{y}}\left\langle 2\sqrt{y}, \frac{x}{\sqrt{y}}, \frac{7}{z}\right\rangle = z\mathrm{e}^{2x\sqrt{y}}\left\langle 2z^6\sqrt{y}, \frac{xz^6}{\sqrt{y}}, 7z^5\right\rangle.$$

On $z = 0$: $f(x,y,0) = 0$ gives $f_x = f_y = 0$, and $f_z(x,y,0) = \lim_{k\to 0} k^7\mathrm{e}^{2x\sqrt{y}}/k = 0$, which is option (d) at $z = 0$ as well. Options (a), (c) carry $\mathrm{e}^{x^2\sqrt{y}}$, not $\mathrm{e}^{2x\sqrt{y}}$; option (b) has first component $z^2\sqrt{y}\,\mathrm{e}^{2x\sqrt{y}} \neq 2z^7\sqrt{y}\,\mathrm{e}^{2x\sqrt{y}}$.

$$\boxed{\text{(d).}}$$

(vii) Note: the question specifies the Chain Rule; this route is an independent check of the option only.

Substitute first, then differentiate. With $x = 6t$, $y = \cos 2t$, $z = t\sin t$ and $3\cdot 6^6 = 139968$,

$$w(t) = 3(6t)^6\cos^3 2t \cdot t\sin t = 139968\, t^7 \sin t\cos^3 2t.$$

Product rule on the three factors t^7, $\sin t$, $\cos^3 2t$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(t^7\sin t\cos^3 2t\big) &= 7t^6\sin t\cos^3 2t + t^7\cos t\cos^3 2t + t^7\sin t\cdot 3\cos^2 2t\,(-2\sin 2t)\\ &= t^6\cos^2 2t\big(7\sin t\cos 2t + t\cos t\cos 2t - 6t\sin t\sin 2t\big).\end{aligned}$$

Now write option (a) in t. First $3x^5y^2 = 3\cdot 6^5 t^5\cos^2 2t = 23328\, t^5\cos^2 2t$. The bracket is

$$\begin{aligned}&36yz - 6xz\sin 2t + xy(\sin t + t\cos t)\\ &= 36t\sin t\cos 2t - 36t^2\sin t\sin 2t + 6t\cos 2t\sin t + 6t^2\cos t\cos 2t\\ &= 6t\big(7\sin t\cos 2t + t\cos t\cos 2t - 6t\sin t\sin 2t\big).\end{aligned}$$

Since $23328\cdot 6 = 139968$, option (a) equals

$$139968\, t^6\cos^2 2t\big(7\sin t\cos 2t + t\cos t\cos 2t - 6t\sin t\sin 2t\big) = w'(t).$$

$$\boxed{\text{(a).}}$$

(viii) Solve for z. The Implicit Function Theorem applies because $F_z = x + y = 6 \neq 0$ at $(4, 2, 5)$; here the solution is explicit:

$$z(x+y) = 38 - xy \implies z = g(x,y) = \frac{38-xy}{x+y}, \qquad g(4,2) = \frac{30}{6} = 5.$$

Quotient rule:

$$g_x = \frac{(-y)(x+y) - (38-xy)}{(x+y)^2} = -\frac{y^2+38}{(x+y)^2}, \qquad g_x(4,2) = -\frac{42}{36} = -\frac{7}{6},$$
$$g_y = \frac{(-x)(x+y) - (38-xy)}{(x+y)^2} = -\frac{x^2+38}{(x+y)^2}, \qquad g_y(4,2) = -\frac{54}{36} = -\frac{3}{2}.$$

Tangent plane of the graph:

$$z - 5 = -\frac{7}{6}(x-4) - \frac{3}{2}(y-2).$$

Multiply by 6: $6z - 30 = -7x + 28 - 9y + 18$, so $7x + 9y + 6z = 76$.

$$\boxed{\text{(c)} \quad 7x + 9y + 6z = 76.}$$

(ix) Write the surface as the level set $F(x,y,z) = 4x^3y^2 + 2y - z = 0$. Then

$$F_x = 12x^2y^2, \qquad F_x(1,-2,12) = 48,$$
$$F_y = 8x^3y + 2, \qquad F_y(1,-2,12) = -14,$$
$$F_z = -1, \qquad F_z(1,-2,12) = -1.$$

Since $\nabla F = \langle 48, -14, -1\rangle \neq \mathbf{0}$, it is normal to the surface, and the tangent plane is

$$48(x-1) - 14(y+2) - (z-12) = 0.$$

Expand: $48x - 14y - z = 48 + 28 - 12 = 64$.

$$\boxed{\text{(b)} \quad 48x - 14y - z = 64.}$$

(x) *No interior critical point.* From $f_x = -2\mathrm{e}^{-2x}\sin 4y$ and $f_y = 4\mathrm{e}^{-2x}\cos 4y$,

$$\left(\frac{f_x}{2}\right)^2 + \left(\frac{f_y}{4}\right)^2 = \mathrm{e}^{-4x}\left(\sin^2 4y + \cos^2 4y\right) = \mathrm{e}^{-4x} > 0,$$

so $\nabla f \neq \mathbf{0}$ everywhere: literally the answer is (d).

Monotonicity in x. Fix $y \in [0, \pi/2]$. If $0 < y < \pi/4$, then $\sin 4y > 0$, so $f_x < 0$ and $x \mapsto f(x,y)$ is strictly decreasing on $[0,\infty)$:

$$f(x,y) \leq f(0,y) = \sin 4y.$$

If $y = 0$ or $\pi/4 \leq y \leq \pi/2$, then $\sin 4y \leq 0$, so $f(x,y) \leq 0$. Hence the largest value of f on D is the largest value of $h(y) = \sin 4y$ on $[0, \pi/4]$:

$$h'(y) = 4\cos 4y = 0 \iff y = \frac{\pi}{8}, \qquad h(0) = 0, \quad h\left(\frac{\pi}{8}\right) = 1, \quad h\left(\frac{\pi}{4}\right) = 0.$$

So $f \leq 1 = f(0, \pi/8)$ on D: an absolute, hence local, maximum relative to D.

$$\boxed{\text{intended: (a)} \quad \text{Local maximum } f(0,\pi/8) = 1; \qquad \text{literal: (d).}}$$

Method 3: Logarithmic differentiation, tangent-vector cross products, option-by-option classification

This method applies to parts (vii)–(x).

(vii) Note: the question specifies the multivariable Chain Rule; this route uses only the one-variable Chain Rule on $\ln|w|$ and is an independent check.

Where $x, y, z \neq 0$, take logarithms of $|w|$:

$$\ln|w| = \ln 3 + 6\ln|x| + 3\ln|y| + \ln|z|.$$

Differentiate in t:

$$\frac{w'}{w} = 6\frac{x'}{x} + 3\frac{y'}{y} + \frac{z'}{z} = \frac{36}{x} - \frac{6\sin 2t}{y} + \frac{\sin t + t\cos t}{z}.$$

Multiply by $w = 3x^6y^3z$:

$$\begin{aligned} w' &= 3x^6y^3z \cdot \frac{36}{x} - 3x^6y^3z \cdot \frac{6\sin 2t}{y} + 3x^6y^3z \cdot \frac{\sin t + t\cos t}{z} \\ &= 3x^5y^2\big(36yz - 6xz\sin 2t + xy(\sin t + t\cos t)\big). \end{aligned}$$

Both sides are continuous in t and agree except at the isolated zeros of xyz, so they agree for all t.

$$\boxed{\text{(a).}}$$

(viii) Use two curves on the surface through $(4, 2, 5)$ and implicit differentiation along each.

Curve 1: $y = 2$, $x = 4 + s$, $z = z_1(s)$ with $z_1(0) = 5$. On the surface,

$$2(4+s) + 2z_1 + (4+s)z_1 = 38.$$

Differentiate in s and set $s = 0$:

$$2 + 2z_1' + z_1 + (4+s)z_1' = 0 \implies 2 + 2z_1'(0) + 5 + 4z_1'(0) = 0 \implies z_1'(0) = -\frac{7}{6}.$$

Curve 2: $x = 4$, $y = 2 + s$, $z = z_2(s)$ with $z_2(0) = 5$. On the surface,

$$4(2+s) + (2+s)z_2 + 4z_2 = 38.$$

Differentiate in s and set $s = 0$:

$$4 + z_2 + (2+s)z_2' + 4z_2' = 0 \implies 4 + 5 + 2z_2'(0) + 4z_2'(0) = 0 \implies z_2'(0) = -\frac{3}{2}.$$

Tangent vectors (scaled by 6 and 2): $\mathbf{a} = \langle 6, 0, -7\rangle$, $\mathbf{b} = \langle 0, 2, -3\rangle$. Their cross product is a normal:

$$\mathbf{a} \times \mathbf{b} = \big\langle 0\cdot(-3) - (-7)\cdot 2,\ (-7)\cdot 0 - 6\cdot(-3),\ 6\cdot 2 - 0\cdot 0\big\rangle = \langle 14, 18, 12\rangle = 2\langle 7, 9, 6\rangle.$$

Plane: $7(x-4) + 9(y-2) + 6(z-5) = 0$, i.e. $7x + 9y + 6z = 76$.

$$\boxed{\text{(c)} \quad 7x + 9y + 6z = 76.}$$

(ix) Use the coordinate slices of the graph $z = g(x,y) = 4x^3y^2 + 2y$ through $(1,-2,12)$.

$$y=-2: \quad z = 16x^3 - 4, \quad \frac{\mathrm{d}z}{\mathrm{d}x} = 48x^2\Big|_{x=1} = 48,$$
$$x=1: \quad z = 4y^2 + 2y, \quad \frac{\mathrm{d}z}{\mathrm{d}y} = 8y + 2\Big|_{y=-2} = -14.$$

The tangent vectors $\langle 1,0,48\rangle$ and $\langle 0,1,-14\rangle$ span the tangent plane; their cross product is

$$\langle 1,0,48\rangle \times \langle 0,1,-14\rangle = \big\langle 0\cdot(-14) - 48\cdot 1,\ 48\cdot 0 - 1\cdot(-14),\ 1\cdot 1 - 0\cdot 0\big\rangle = \langle -48,14,1\rangle.$$

Plane: $-48(x-1)+14(y+2)+(z-12)=0$, i.e. $-48x+14y+z = -48-28+12 = -64$, so $48x - 14y - z = 64$.

$$\boxed{\text{(b)} \quad 48x - 14y - z = 64.}$$

(x) Classify each listed point relative to $D = \{x \geq 0,\ 0 \leq y \leq \pi/2\}$.

(a) $(0,\pi/8)$. $f(0,\pi/8) = \sin(\pi/2) = 1$, and on D, $f(x,y) \leq \mathrm{e}^{-2x}|\sin 4y| \leq 1$ because $x \geq 0$. So it is a local maximum relative to D: true.

(b) $(0,0)$. $f(0,0) = \sin 0 = 0$. If $(x,y) \in D$ and $x^2 + y^2 < (\pi/4)^2$, then $0 \leq 4y < \pi$, so $\sin 4y \geq 0$ and $f(x,y) \geq 0 = f(0,0)$. So $(0,0)$ is a local minimum relative to D, not a saddle: false.

(c) $(0,\pi/4)$. $f(0,\pi/4) = \sin\pi = 0$. For small $s > 0$,

$$f\Big(0, \frac{\pi}{4} - s\Big) = \sin(\pi - 4s) = \sin 4s > 0, \qquad f\Big(0, \frac{\pi}{4} + s\Big) = \sin(\pi + 4s) = -\sin 4s < 0.$$

Every neighbourhood contains values below 0, so it is not a local minimum: false.

(d) literal reading. If $f_x = -2\mathrm{e}^{-2x}\sin 4y = 0$ then $\sin 4y = 0$, so $\cos 4y = \pm 1$ and $f_y = \pm 4\mathrm{e}^{-2x} \neq 0$. No interior point has $\nabla f = \mathbf{0}$, so literally no listed point is a critical point.

$$\boxed{\text{intended: (a)} \quad \text{Local maximum } f(0,\pi/8) = 1; \qquad \text{literal: (d).}}$$

Unofficial QRS marking scheme (50 marks)

(i) **(5 marks)** 2: correct displacement; 2: correct parameter interval; 1: selects (d).

(ii) **(5 marks)** 2: $y' = 0$; 2: checks $x' \neq 0$ and finds both points; 1: selects (d).

(iii) **(5 marks)** 2: point; 2: tangent vector; 1: selects (b).

(iv) **(5 marks)** 2: derivative; 2: norm; 1: selects (c).

(v) **(5 marks)** 2: gradient; 2: normalized direction; 1: selects (a).

(vi) **(5 marks)** 4: all gradient components; 1: selects (d).

(vii) **(5 marks)** 4: correct chain-rule terms and simplification; 1: selects (a).

(viii) **(5 marks)** 2: normal; 2: plane through the point; 1: selects (c).

(ix) **(5 marks)** 2: partial derivatives; 2: plane; 1: selects (b).

(x) **(5 marks)** 2: identifies the boundary-definition issue; 2: constrained local-maximum argument; 1: states intended (a) and literal (d).

Question 2 (25 Marks)

(i) Evaluate each of the following limits or show that the limit does not exist. Justify your answers.

(a) $\lim\limits_{(x,y)\to(6,6)} \dfrac{5\mathrm{e}^{x-y}}{\cos(\pi(x-y))}$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^{21}y^{21}}{x^{42}+y^{84}}$

(ii) Define

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right); & \text{if } x^2+y^2 \neq 0, \\ 0; & \text{if } x^2+y^2 = 0. \end{cases}$$

(a) Is f continuous at $(0,0)$? Justify your answer.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

(c) Is f differentiable at $(0,0)$? Justify your answer.

Solution

Method 1: Direct limit and derivative definitions

(i) (a) The maps $(x,y) \mapsto \mathrm{e}^{x-y}$ and $(x,y) \mapsto \cos(\pi(x-y))$ are compositions of the polynomial $x-y$ with continuous one-variable functions, so both are continuous on $\mathbb{R}^2$. At $(6,6)$,

$$5\mathrm{e}^{6-6} = 5\mathrm{e}^0 = 5, \qquad \cos\big(\pi(6-6)\big) = \cos 0 = 1 \neq 0.$$

By the continuity laws (a quotient of continuous functions is continuous where the denominator is nonzero), the quotient is continuous at $(6,6)$, so its limit equals its value:

$$\lim_{(x,y)\to(6,6)} \frac{5\mathrm{e}^{x-y}}{\cos(\pi(x-y))} = \frac{5}{1} = 5.$$

$$\boxed{\lim_{(x,y)\to(6,6)} \frac{5\mathrm{e}^{x-y}}{\cos(\pi(x-y))} = 5.}$$

(b) Let $q(x,y) = \dfrac{x^{21}y^{21}}{x^{42}+y^{84}}$. Since $x^{42}+y^{84} = 0$ only at $(0,0)$, q is defined on $\mathbb{R}^2\setminus\{(0,0)\}$, and the origin is a limit point of the domain.

Path 1, $y = 0$, $x \neq 0$:

$$q(x,0) = \frac{x^{21}\cdot 0}{x^{42}+0} = 0 \xrightarrow[x\to 0]{} 0.$$

Path 2, $y = x$, $x \neq 0$:

$$q(x,x) = \frac{x^{21}x^{21}}{x^{42}+x^{84}} = \frac{x^{42}}{x^{42}(1+x^{42})} = \frac{1}{1+x^{42}} \xrightarrow[x\to 0]{} \frac{1}{1+0} = 1.$$

The two path limits differ ($0 \neq 1$), so by uniqueness of limits (two-path test)

$$\boxed{\text{the limit does not exist.}}$$

(ii) (a) For $(x,y) \neq (0,0)$, since $|\sin u| \leq 1$ for every real u,

$$|f(x,y) - f(0,0)| = (x^2+y^2)\left|\sin\Big(\frac{1}{x^2+y^2}\Big)\right| \leq x^2+y^2.$$

As $(x,y) \to (0,0)$, $x^2+y^2 \to 0$. By the Squeeze Theorem, $f(x,y) \to 0 = f(0,0)$.

$$\boxed{f \text{ is continuous at } (0,0).}$$

(b) For $h \neq 0$, $f(h,0) = h^2\sin(1/h^2)$ and $f(0,0) = 0$, so

$$f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{h^2\sin(1/h^2)}{h} = \lim_{h\to 0} h\sin\Big(\frac{1}{h^2}\Big).$$

Since $\left|h\sin(1/h^2)\right| \leq |h| \to 0$, the Squeeze Theorem gives $f_x(0,0) = 0$. Likewise, for $k \neq 0$, $f(0,k) = k^2\sin(1/k^2)$, so

$$f_y(0,0) = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} k\sin\Big(\frac{1}{k^2}\Big) = 0,$$

because $\left|k\sin(1/k^2)\right| \leq |k| \to 0$.

$$\boxed{f_x(0,0) = f_y(0,0) = 0.}$$

(c) First check the sufficient criterion (continuous partials imply total differentiability). For $(x,y) \neq (0,0)$, by the product and Chain Rules,

$$f_x = 2x\sin\Big(\frac{1}{x^2+y^2}\Big) - \frac{2x}{x^2+y^2}\cos\Big(\frac{1}{x^2+y^2}\Big).$$

Along $y = 0$ at $x_n = 1/\sqrt{2\pi n}$ we have $1/x_n^2 = 2\pi n$, $\sin(2\pi n) = 0$, $\cos(2\pi n) = 1$, so

$$f_x(x_n,0) = -\frac{2}{x_n} = -2\sqrt{2\pi n} \to -\infty \neq 0 = f_x(0,0).$$

So f_x is not continuous at $(0,0)$; the criterion does not apply, and we use the definition.

If f is totally differentiable at $(0,0)$, its linear map is $A(h,k) = f_x(0,0)h + f_y(0,0)k = 0$ (differentiability determines the partial derivatives). Let $r = \sqrt{h^2+k^2} > 0$. Then $f(h,k) = r^2\sin(1/r^2)$ and

$$\frac{|f(h,k) - f(0,0) - A(h,k)|}{r} = \frac{r^2\left|\sin\left(\frac{1}{r^2}\right)\right|}{r} = r\left|\sin\left(\frac{1}{r^2}\right)\right| \leq r \to 0.$$

By the Squeeze Theorem the quotient tends to 0, so $A = 0$ satisfies the definition.

$$\boxed{f \text{ is differentiable at } (0,0), \quad Df(0,0) = 0.}$$

Method 2: Reduction to one variable (substitution, line family, polar coordinates, symmetry)

(i) (a) Substitute $u = x - y$. Then $|u| \le |x-6| + |y-6|$, so $u \to 0$ as $(x,y) \to (6,6)$, and the expression is $\varphi(u)$ with

$$\varphi(u) = \frac{5\mathrm{e}^u}{\cos(\pi u)}.$$

The one-variable function φ is continuous at $u = 0$ because $\cos 0 = 1 \neq 0$, so

$$\lim_{(x,y)\to(6,6)} \varphi(x-y) = \varphi(0) = \frac{5\mathrm{e}^0}{\cos 0} = \frac{5}{1} = 5.$$

$$\boxed{\lim_{(x,y)\to(6,6)} \frac{5\mathrm{e}^{x-y}}{\cos(\pi(x-y))} = 5.}$$

(b) Approach along the line $y = mx$, $x \neq 0$, for a fixed $m \in \mathbb{R}$:

$$q(x,mx) = \frac{x^{21}(mx)^{21}}{x^{42} + (mx)^{84}} = \frac{m^{21}x^{42}}{x^{42}\big(1 + m^{84}x^{42}\big)} = \frac{m^{21}}{1 + m^{84}x^{42}} \xrightarrow[x\to 0]{} m^{21}.$$

The path limit depends on m: $m = 0$ gives 0, $m = 1$ gives 1, $m = -1$ gives -1. Two paths with different limits exist, so

$$\boxed{\text{the limit does not exist.}}$$

(ii) (a) Use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$, so $x^2 + y^2 = r^2$ and $(x,y) \to (0,0) \iff r \to 0^+$. Then

$$f(r\cos\theta, r\sin\theta) = r^2 \sin\Big(\frac{1}{r^2}\Big),$$

which does not depend on θ. Since $-r^2 \le r^2\sin(1/r^2) \le r^2$ and $r^2 \to 0$,

$$\lim_{r\to 0^+} r^2 \sin\Big(\frac{1}{r^2}\Big) = 0 = f(0,0),$$

uniformly in θ.

$$\boxed{f \text{ is continuous at } (0,0).}$$

(b) Let $g(t) = f(t,0)$, so $g(t) = t^2\sin(1/t^2)$ for $t \neq 0$, $g(0) = 0$, and $f_x(0,0) = g'(0)$. The formula for f depends only on $x^2 + y^2$, so $f(x,y) = f(y,x)$; in particular $f(0,t) = f(t,0) = g(t)$, and $f_y(0,0) = g'(0)$ too. For $t \neq 0$,

$$\left|\frac{g(t) - g(0)}{t}\right| = \frac{t^2|\sin(1/t^2)|}{|t|} \le |t| \xrightarrow[t\to 0]{} 0,$$

so $g'(0) = 0$.

$$\boxed{f_x(0,0) = f_y(0,0) = 0.}$$

(c) The candidate linear map is $A = 0$, from $f_x(0,0) = f_y(0,0) = 0$. In polar coordinates with $(h,k) = (r\cos\theta, r\sin\theta)$, $r > 0$,

$$\frac{f(h,k) - f(0,0) - A(h,k)}{\sqrt{h^2+k^2}} = \frac{r^2\sin(1/r^2)}{r} = r\sin\Big(\frac{1}{r^2}\Big).$$

This depends on r only, and $|r\sin(1/r^2)| \le r \to 0$ as $r \to 0^+$, uniformly in θ.

$$\boxed{f \text{ is differentiable at } (0,0), \quad Df(0,0) = 0.}$$

Method 3: Unbounded curve path and ε–δ verification

This method applies to parts (i)(b) and (ii).

(i) (b) Suppose the limit L existed. Taking $\varepsilon = 1$ there would be $\delta > 0$ with $|q - L| < 1$, hence $|q| < |L| + 1$, for $0 < \|(x, y)\| < \delta$. So q would be bounded near the origin. Along the curve $x = t^2$, $y = t$ $(t \neq 0)$:

$$q(t^2, t) = \frac{(t^2)^{21} t^{21}}{(t^2)^{42} + t^{84}} = \frac{t^{63}}{t^{84} + t^{84}} = \frac{t^{63}}{2t^{84}} = \frac{1}{2t^{21}}.$$

As $t \to 0^+$, $(t^2, t) \to (0, 0)$ while $\dfrac{1}{2t^{21}} \to +\infty$, so q is unbounded near the origin. This contradiction shows

$$\boxed{\text{the limit does not exist.}}$$

(ii) (a) Let $\varepsilon > 0$ and choose $\delta = \sqrt{\varepsilon}$. If $0 < \sqrt{x^2 + y^2} < \delta$, then

$$|f(x, y) - f(0, 0)| \leq x^2 + y^2 < \delta^2 = \varepsilon,$$

using $|\sin u| \leq 1$. This is the ε–δ definition of $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$.

$$\boxed{f \text{ is continuous at } (0, 0).}$$

(b) Test the linear map $A = 0$ directly in the total-differentiability definition. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. If $0 < r = \sqrt{h^2 + k^2} < \delta$, then

$$\frac{|f(h, k) - f(0, 0) - 0|}{r} = \frac{r^2 |\sin(1/r^2)|}{r} \leq r < \varepsilon.$$

So f is totally differentiable at $(0, 0)$ with $A = 0$. Since differentiability determines the partial derivatives,

$$f_x(0, 0) = A(1, 0) = 0, \qquad f_y(0, 0) = A(0, 1) = 0.$$

$$\boxed{f_x(0, 0) = f_y(0, 0) = 0.}$$

(c) The ε–δ statement in (b) is exactly the definition of total differentiability at $(0, 0)$ with linear map $A = 0$, and A is unique (uniqueness of the linear derivative).

$$\boxed{f \text{ is differentiable at } (0, 0), \quad Df(0, 0) = 0.}$$

Unofficial QRS marking scheme (25 marks)

(i) **(10 marks)**

(a) **(5 marks)** 2: recognizes continuity of the quotient; 2: evaluates numerator and denominator; 1: final value 5.

(b) **(5 marks)** 2: path $y = 0$; 2: path $y = x$; 1: concludes nonexistence.

(ii) **(15 marks)**

(a) **(5 marks)** 3: valid squeeze bound; 2: continuity conclusion.

(b) **(5 marks)** 2 each: correct difference quotients; 1: both values equal zero.

(c) **(5 marks)** 2: identifies $L = 0$; 2: normalizes the remainder; 1: differentiability conclusion.

Question 3 (25 Marks)

Let

$$f(x,y) = y^3 - x^2 - 7y^2 + 2xy + 9y.$$

Find and classify all critical points of f. (For each critical point, determine whether or not it is a local max/min or a saddle point.)

Solution

Method 1: Direct derivative and second-derivative test

The function f is a polynomial on the open set $\mathbb{R}^2$, so its partial derivatives exist everywhere and the critical points are exactly the solutions of $\nabla f = \mathbf{0}$. The first partials are

$$f_x = \frac{\partial}{\partial x}\big(y^3 - x^2 - 7y^2 + 2xy + 9y\big) = -2x + 2y,$$
$$f_y = \frac{\partial}{\partial y}\big(y^3 - x^2 - 7y^2 + 2xy + 9y\big) = 3y^2 - 14y + 2x + 9.$$

From $f_x = 0$,

$$-2x + 2y = 0 \implies x = y.$$

Substitute $x = y$ into $f_y = 0$:

$$3y^2 - 14y + 2y + 9 = 3y^2 - 12y + 9 = 3(y^2 - 4y + 3) = 3(y-1)(y-3) = 0,$$

so $y = 1$ or $y = 3$, and the critical points are $(1,1)$ and $(3,3)$.

The second partials are

$$f_{xx} = -2, \qquad f_{xy} = \frac{\partial}{\partial y}(-2x+2y) = 2, \qquad f_{yx} = \frac{\partial}{\partial x}(3y^2-14y+2x+9) = 2, \qquad f_{yy} = 6y-14.$$

They are continuous (polynomials), so $f_{xy} = f_{yx}$ as Clairaut's Theorem guarantees, and the Hessian discriminant test applies with

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (-2)(6y-14) - 2^2 = -12y + 28 - 4 = 24 - 12y.$$

At $(1,1)$:

$$D(1,1) = 24 - 12 = 12 > 0, \qquad f_{xx} = -2 < 0 \implies \text{strict local maximum},$$

$$f(1,1) = 1 - 1 - 7 + 2 + 9 = 4.$$

At $(3,3)$:

$$D(3,3) = 24 - 36 = -12 < 0 \implies \text{saddle point},$$
$$f(3,3) = 27 - 9 - 63 + 18 + 27 = 0.$$

$$\boxed{(1,1) \text{ is a local maximum; } (3,3) \text{ is a saddle point.}}$$

Method 2: Completing the square

Group the terms containing x and complete the square:

$$-x^2+2xy=-(x^2-2xy)=-\big((x-y)^2-y^2\big)=-(x-y)^2+y^2.$$

Hence

$$f=y^3-7y^2+9y+y^2-(x-y)^2=y^3-6y^2+9y-(x-y)^2=y(y-3)^2-(x-y)^2.$$

Let $u=x-y$ and $h(y)=y(y-3)^2$, and set $F(u,y)=f(u+y,y)=h(y)-u^2$. The change $(x,y)\mapsto(u,y)$ is an invertible linear map, so it carries neighbourhoods to neighbourhoods. By the Chain Rule,

$$F_u=f_x,\qquad F_y=f_x+f_y,$$

so $\nabla F=\mathbf{0}\iff\nabla f=\mathbf{0}$. Now

$$F_u=-2u,\qquad F_y=h'(y)=(y-3)^2+2y(y-3)=(y-3)(3y-3)=3(y-1)(y-3).$$

So $F_u=F_y=0\iff u=0$ and $y\in\{1,3\}$, i.e. $x=y\in\{1,3\}$: the critical points are $(1,1)$ and $(3,3)$.

At $(1,1)$. Let $v=y-1$, so $u=x-y$ and $(x,y)\to(1,1)\iff(u,v)\to(0,0)$. Expand

$$h(1+v)=(1+v)(v-2)^2=(1+v)(v^2-4v+4)=4-3v^2+v^3,$$

so

$$f=4-u^2-3v^2+v^3.$$

For $0<|v|<1$, $v^3\le|v|^3<v^2$, hence $-3v^2+v^3<-2v^2<0$ and $f<4-u^2\le4$. For $v=0$, $u\neq0$, $f=4-u^2<4$. So $f<4=f(1,1)$ for all $(u,v)\neq(0,0)$ with $|v|<1$: a strict local maximum.

At $(3,3)$. Let $v=y-3$. Expand

$$h(3+v)=(3+v)v^2=3v^2+v^3,\qquad\text{so}\qquad f=3v^2+v^3-u^2,\quad f(3,3)=0.$$

On the line $u=0$ (i.e. $x=y$) with $0<|v|<3$: $f=v^2(3+v)>0$. On the line $v=0$ (i.e. $y=3$) with $u\neq0$: $f=-u^2<0$. Every neighbourhood of $(3,3)$ contains points with $f>0$ and points with $f<0$, so $(3,3)$ is neither a local maximum nor a local minimum: a saddle point.

$$\boxed{(1,1)\text{ is a local maximum; }(3,3)\text{ is a saddle point.}}$$

Method 3: Hessian eigenvalues

Note: this method uses eigenvalues of a symmetric matrix, from linear algebra outside the MH2100 slides.

From $f_x = -2x + 2y = 0$ we get $x = y$, and then $f_y = 3y^2 - 12y + 9 = 3(y-1)(y-3) = 0$, so the stationary points are $(1,1)$ and $(3,3)$. The Hessian matrix is

$$H(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & 6y-14 \end{pmatrix}.$$

By the second-order Taylor expansion at a stationary point p (f is a polynomial, so C^2),

$$f(p+(h,k)) = f(p) + \tfrac{1}{2}\,(h,k)\,H(p)\,(h,k)^T + o(h^2+k^2).$$

If both eigenvalues of $H(p)$ are negative, the quadratic term is at most $\frac{1}{2}\lambda_{\max}(h^2+k^2) < 0$, giving a strict local maximum; if the eigenvalues have opposite signs, f increases along one eigenvector and decreases along the other, giving a saddle.

At $(1,1)$:

$$\det\begin{pmatrix} -2-\lambda & 2 \\ 2 & -8-\lambda \end{pmatrix} = (-2-\lambda)(-8-\lambda) - 4 = \lambda^2 + 10\lambda + 12 = 0,$$

$$\lambda = \frac{-10 \pm \sqrt{100-48}}{2} = -5 \pm \sqrt{13}.$$

Since $\sqrt{13} < 5$, both eigenvalues are negative: $(1,1)$ is a strict local maximum.

At $(3,3)$:

$$\det\begin{pmatrix} -2-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = (-2-\lambda)(4-\lambda) - 4 = \lambda^2 - 2\lambda - 12 = 0,$$

$$\lambda = \frac{2 \pm \sqrt{4+48}}{2} = 1 \pm \sqrt{13}.$$

Since $\sqrt{13} > 1$, $1+\sqrt{13} > 0 > 1-\sqrt{13}$: $(3,3)$ is a saddle point.

$$\boxed{(1,1) \text{ is a local maximum; } (3,3) \text{ is a saddle point.}}$$

Unofficial QRS marking scheme (25 marks)

The breakdown follows Method 1; an equivalent rigorous structural proof, such as Method 2, receives the corresponding full credit.

- **8 marks:** computes f_x, f_y and correctly solves the simultaneous equations.
- **5 marks:** identifies both critical points $(1,1)$ and $(3,3)$.
- **6 marks:** computes the Hessian entries and discriminant correctly.
- **3 marks:** applies the test to classify $(1,1)$ as a local maximum.
- **3 marks:** applies the test to classify $(3,3)$ as a saddle point.