MH2100 Calculus III

Midterm II (Solutions)

AY 2018/2019, Semester 1

Quantitative Research Society @NTU

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Overview

- **Historical paper:** the original paper was held in November 2018 with **90 minutes** allowed.
- **Original instructions:** answer all three questions; candidates could write anywhere on the paper; one two-sided A4 page of handwritten notes was permitted.
- **2026 scope note:** the current midterm scope ends at double integrals, including polar coordinates. General Jacobian substitutions, triple integrals, cylindrical and spherical coordinates, and line integrals are beyond that scope. They are retained and solved here because they appear in the historical paper.
- **Question themes:** Q1 eight short integrals in Cartesian, polar, cylindrical and spherical coordinates and a Jacobian; Q2 reversing an order of integration, a linear change of variables, and a triple integral; Q3 work, a scalar line integral, and a potential function.

Instructions to candidates (historical)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. A two-sided A4 piece of paper with handwritten notes is permitted.

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Question 1 (40 Marks)

(beyond 2026 midterm scope): Parts (iv)–(vii) use triple integrals, cylindrical/spherical coordinates, or a general Jacobian substitution.

Circle (only) the correct answer for each question. Each question is worth 5 marks.

(i) Calculate the double integral $\iint_R (6x^2y^3 - 10x^4)\,\mathrm{d}A$ where $R = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 4\}$.

(a) 190

(b) 170

(c) 130

(d) 150

(e) 120

(ii) Use polar coordinates to find the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the disk $x^2 + y^2 \leq 9$ (in the plane $z = 0$).

(a) 81π

(b) 162π

(c) 324π

(d) 27π

(e) None of the above

(iii) Evaluate the iterated integral by converting to polar coordinates:

$$\int_{-5}^{5} \int_{0}^{\sqrt{25-y^2}} (x^2 + y^2)^{3/2}\,\mathrm{d}x\,\mathrm{d}y.$$

(a) 625π

(b) 1250π

(c) $\dfrac{625\pi}{2}$

(d) $\dfrac{625\pi}{4}$

(e) None of the above

(iv) Evaluate the iterated integral:

$$\int_{0}^{\pi} \int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} 4y \sin x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$$

(a) $\dfrac{8}{3}$

(b) 4

(c) $\dfrac{4}{3}$

(d) $\dfrac{2}{3}$

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(e) None of the above

(v) Use cylindrical coordinates to evaluate $\iiint_T \sqrt{x^2+y^2}\,\mathrm{d}V$, where T is the solid bounded by the cylinder $x^2+y^2=1$ and the planes $z=4$ and $z=8$.

(a) $\dfrac{8\pi}{3}$

(b) 4π

(c) $\dfrac{4\pi}{3}$

(d) $\dfrac{2\pi}{3}$

(e) None of the above

(vi) Use spherical coordinates to evaluate $\iiint_B (x^2+y^2+z^2)^2\,\mathrm{d}V$ where B is the ball with center the origin and radius 4.

(a) $\dfrac{65536\pi}{7}$

(b) $\dfrac{4096\pi}{5}$

(c) $\dfrac{8192\pi}{3}$

(d) $\dfrac{32768\pi}{7}$

(e) None of the above

(vii) Find the Jacobian of the plane transformation given by $x=5u\sin v$ and $y=4u\cos v$.

(a) $\dfrac{\partial(x,y)}{\partial(u,v)}=9u$

(b) $\dfrac{\partial(x,y)}{\partial(u,v)}=36u$

(c) $\dfrac{\partial(x,y)}{\partial(u,v)}=-20u$

(d) $\dfrac{\partial(x,y)}{\partial(u,v)}=-u$

(e) None of the above

(viii) Evaluate the iterated integral:

$$\int_1^3 \int_y^3 12xy\,\mathrm{d}x\,\mathrm{d}y.$$

(a) 96

(b) 116

(c) 106

(d) 126

(e) None of the above

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Solution

Method 1: Iterated integration in the stated coordinates

(i) The integrand $6x^2y^3 - 10x^4$ is a polynomial, hence continuous on the rectangle $R = [0,1] \times [0,4]$. By Fubini's Theorem the double integral equals the iterated integral in either order. Integrate first in y:

$$\iint_R (6x^2y^3 - 10x^4)\,\mathrm{d}A = \int_0^1 \int_0^4 (6x^2y^3 - 10x^4)\,\mathrm{d}y\,\mathrm{d}x.$$

With x held fixed, the inner integral is

$$\begin{aligned}\int_0^4 (6x^2y^3 - 10x^4)\,\mathrm{d}y &= \left[\frac{6x^2y^4}{4} - 10x^4y\right]_{y=0}^{y=4} \\ &= \frac{6x^2 \cdot 256}{4} - 10x^4 \cdot 4 \\ &= 384x^2 - 40x^4.\end{aligned}$$

The outer integral is

$$\begin{aligned}\int_0^1 (384x^2 - 40x^4)\,\mathrm{d}x &= \left[128x^3 - 8x^5\right]_0^1 \\ &= 128 - 8 = 120.\end{aligned}$$

Hence

$$\boxed{\text{Answer (e): } 120.}$$

(ii) The paraboloid $z = x^2 + y^2$ satisfies $z \geq 0$, so it lies above the plane $z = 0$, and the volume of the solid over the disk $D = \{(x,y) : x^2 + y^2 \leq 9\}$ is

$$V = \iint_D (x^2 + y^2)\,\mathrm{d}A.$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ one has

$$x^2 + y^2 = r^2, \qquad \mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta,$$

and the disk is $D = \{(r,\theta) : 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}$. Therefore

$$\begin{aligned}V &= \int_0^{2\pi} \int_0^3 r^2 \cdot r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^3 \mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{81}{4}\,\mathrm{d}\theta = \frac{81}{4} \cdot 2\pi = \frac{81\pi}{2}.\end{aligned}$$

The value $81\pi/2$ is none of 81π, 162π, 324π, 27π, so

$$\boxed{\text{Answer (e): None of the above.}}$$

(iii) For each $y \in [-5,5]$ the inner variable runs over $0 \leq x \leq \sqrt{25 - y^2}$. Squaring the upper limit, the region of integration is

$$D = \{(x,y) : x^2 + y^2 \leq 25,\ x \geq 0\},$$

the right half of the disk of radius 5. In polar coordinates the condition $x = r\cos\theta \geq 0$ gives $-\pi/2 \leq \theta \leq \pi/2$, and $x^2 + y^2 \leq 25$ gives $0 \leq r \leq 5$. The integrand becomes

$$(x^2+y^2)^{3/2} = (r^2)^{3/2} = r^3,$$

and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$. Therefore

$$\begin{aligned}\int_{-5}^{5}\int_{0}^{\sqrt{25-y^2}} (x^2+y^2)^{3/2}\,\mathrm{d}x\,\mathrm{d}y &= \int_{-\pi/2}^{\pi/2}\int_{0}^{5} r^3\cdot r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_{-\pi/2}^{\pi/2}\left[\frac{r^5}{5}\right]_0^5 \mathrm{d}\theta = \int_{-\pi/2}^{\pi/2} 625\,\mathrm{d}\theta \\ &= 625\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 625\pi.\end{aligned}$$

Hence

$$\boxed{\text{Answer (a): } 625\pi.}$$

(iv) The innermost integrand $4y\sin x$ does not depend on z, so

$$\int_0^{\sqrt{1-y^2}} 4y\sin x\,\mathrm{d}z = 4y\sin x\,\big[z\big]_0^{\sqrt{1-y^2}} = 4y\sqrt{1-y^2}\,\sin x.$$

For the middle integral let $w = 1 - y^2$. Then $\mathrm{d}w = -2y\,\mathrm{d}y$, and the limits become $y = 0 \mapsto w = 1$, $y = 1 \mapsto w = 0$:

$$\begin{aligned}\int_0^1 4y\sqrt{1-y^2}\,\mathrm{d}y &= \int_1^0 4\sqrt{w}\left(-\frac{1}{2}\right)\mathrm{d}w = 2\int_0^1 w^{1/2}\,\mathrm{d}w \\ &= 2\left[\frac{2}{3}w^{3/2}\right]_0^1 = \frac{4}{3}.\end{aligned}$$

So the middle integral equals $\frac{4}{3}\sin x$, and the outer integral is

$$\begin{aligned}\int_0^{\pi}\frac{4}{3}\sin x\,\mathrm{d}x &= \frac{4}{3}\big[-\cos x\big]_0^{\pi} = \frac{4}{3}\big(-\cos\pi + \cos 0\big) \\ &= \frac{4}{3}\cdot 2 = \frac{8}{3}.\end{aligned}$$

Hence

$$\boxed{\text{Answer (a): } \frac{8}{3}.}$$

(v) In cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ one has

$$\sqrt{x^2+y^2} = r, \qquad \mathrm{d}V = r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

The solid inside the cylinder $x^2 + y^2 = 1$ and between the planes $z = 4$ and $z = 8$ is

$$T = \{(r,\theta,z) : 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi,\ 4 \leq z \leq 8\}.$$

Therefore

$$\iiint_T \sqrt{x^2+y^2}\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_4^8 r\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\int_0^1 r^2\big[z\big]_4^8\,\mathrm{d}r\,\mathrm{d}\theta$$

$$= \int_0^{2\pi}\int_0^1 4r^2\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi}\left[\frac{4r^3}{3}\right]_0^1 \mathrm{d}\theta$$
$$= \int_0^{2\pi}\frac{4}{3}\,\mathrm{d}\theta = \frac{4}{3}\cdot 2\pi = \frac{8\pi}{3}.$$

Hence

$$\boxed{\text{Answer (a): } \frac{8\pi}{3}.}$$

(vi) In spherical coordinates $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$ one has

$$x^2+y^2+z^2 = \rho^2, \qquad \mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta,$$

and the ball of radius 4 is $B = \{0\le\rho\le 4,\ 0\le\phi\le\pi,\ 0\le\theta\le 2\pi\}$. The integrand is $(\rho^2)^2 = \rho^4$, so

$$\iiint_B (x^2+y^2+z^2)^2\,\mathrm{d}V = \int_0^{2\pi}\int_0^{\pi}\int_0^4 \rho^4\cdot\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

The limits are constants and the integrand is the product $\rho^6\cdot\sin\phi$, so the iterated integral is a product of three single integrals:

$$\int_0^{2\pi}\mathrm{d}\theta\cdot\int_0^{\pi}\sin\phi\,\mathrm{d}\phi\cdot\int_0^4\rho^6\,\mathrm{d}\rho = 2\pi\cdot\big[-\cos\phi\big]_0^{\pi}\cdot\left[\frac{\rho^7}{7}\right]_0^4$$
$$= 2\pi\cdot 2\cdot\frac{16384}{7} = \frac{65536\pi}{7}.$$

Hence

$$\boxed{\text{Answer (a): } \frac{65536\pi}{7}.}$$

(vii) The four partial derivatives are

$$\frac{\partial x}{\partial u} = 5\sin v, \qquad \frac{\partial x}{\partial v} = 5u\cos v, \qquad \frac{\partial y}{\partial u} = 4\cos v, \qquad \frac{\partial y}{\partial v} = -4u\sin v.$$

Expanding the determinant along the first row,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 5\sin v & 5u\cos v \\ 4\cos v & -4u\sin v \end{vmatrix} = (5\sin v)(-4u\sin v) - (5u\cos v)(4\cos v)$$
$$= -20u\sin^2 v - 20u\cos^2 v = -20u(\sin^2 v+\cos^2 v) = -20u.$$

Hence

$$\boxed{\text{Answer (c): } -20u.}$$

(viii) With y held fixed, the inner integral is

$$\int_y^3 12xy\,\mathrm{d}x = 12y\left[\frac{x^2}{2}\right]_{x=y}^{x=3} = 6y(9-y^2) = 54y - 6y^3.$$

The outer integral is

$$\int_1^3 (54y-6y^3)\,\mathrm{d}y = \left[27y^2 - \frac{3}{2}y^4\right]_1^3 = \left(243-\frac{243}{2}\right) - \left(27-\frac{3}{2}\right)$$
$$= \frac{243}{2} - \frac{51}{2} = \frac{192}{2} = 96.$$

Hence

$$\boxed{\text{Answer (a): } 96.}$$

Method 2: Fubini in the other order (parts (i), (iv), (viii))

(i) The integrand is continuous on the rectangle $[0,1]\times[0,4]$, so by Fubini's Theorem one may integrate first in x. With y held fixed,

$$\int_0^1 (6x^2y^3 - 10x^4)\,\mathrm{d}x = \left[2x^3y^3 - 2x^5\right]_{x=0}^{x=1} = 2y^3 - 2.$$

Then

$$\int_0^4 (2y^3 - 2)\,\mathrm{d}y = \left[\frac{y^4}{2} - 2y\right]_0^4 = \frac{256}{2} - 8 = 128 - 8 = 120.$$

Hence

$$\boxed{\text{Answer (e): } 120.}$$

(iv) For fixed x, the (y,z)-region of integration is

$$E = \{(y,z) : 0 \le y \le 1,\ 0 \le z \le \sqrt{1-y^2}\} = \{(y,z) : y \ge 0,\ z \ge 0,\ y^2 + z^2 \le 1\},$$

the quarter of the unit disk in the first quadrant of the yz-plane. Described with z outside,

$$E = \{(y,z) : 0 \le z \le 1,\ 0 \le y \le \sqrt{1-z^2}\}.$$

The integrand $4y\sin x$ is continuous on the closed bounded solid $[0,\pi]\times E$, so by Fubini's Theorem

$$\int_0^\pi \int_0^1 \int_0^{\sqrt{1-y^2}} 4y\sin x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = \int_0^\pi \int_0^1 \int_0^{\sqrt{1-z^2}} 4y\sin x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x.$$

The inner integral is

$$\int_0^{\sqrt{1-z^2}} 4y\sin x\,\mathrm{d}y = 4\sin x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-z^2}} = 2(1-z^2)\sin x.$$

The middle integral is

$$\int_0^1 2(1-z^2)\sin x\,\mathrm{d}z = 2\sin x\left[z - \frac{z^3}{3}\right]_0^1 = 2\sin x\cdot\frac{2}{3} = \frac{4}{3}\sin x.$$

The outer integral is

$$\int_0^\pi \frac{4}{3}\sin x\,\mathrm{d}x = \frac{4}{3}\left[-\cos x\right]_0^\pi = \frac{4}{3}\cdot 2 = \frac{8}{3}.$$

Hence

$$\boxed{\text{Answer (a): } \frac{8}{3}.}$$

(viii) The region of integration is

$$T = \{(x,y) : 1 \le y \le 3,\ y \le x \le 3\} = \{(x,y) : 1 \le y \le x \le 3\},$$

the triangle with vertices $(1,1)$, $(3,1)$, $(3,3)$. Described with x outside,

$$T = \{(x,y) : 1 \le x \le 3,\ 1 \le y \le x\}.$$

The integrand $12xy$ is continuous on T, so by Fubini's Theorem the two iterated integrals agree. With x held fixed,

$$\int_1^x 12xy\,\mathrm{d}y = 12x\left[\frac{y^2}{2}\right]_{y=1}^{y=x} = 6x(x^2-1) = 6x^3 - 6x.$$

Then

$$\begin{aligned}\int_1^3 (6x^3 - 6x)\,\mathrm{d}x &= \left[\frac{3}{2}x^4 - 3x^2\right]_1^3 = \left(\frac{243}{2} - 27\right) - \left(\frac{3}{2} - 3\right) \\ &= \frac{189}{2} + \frac{3}{2} = \frac{192}{2} = 96.\end{aligned}$$

Hence

Answer (a): 96.

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Method 3: Shell and slice decompositions (parts (ii), (iii), (v), (vi))

When the integrand depends only on one distance s (or the region is cut by one height z), integrate over all the other coordinates first. What remains is a single integral in s whose weight is the size of a thin shell or slice. This is the polar, cylindrical or spherical formula with the angular (and axial) integrations carried out first, or Fubini's Theorem with z as the outer variable.

Note: parts (ii), (iii), (v), (vi) name the coordinate system to use; this method should be used only if the course allows this interpretation.

(ii) The solid is

$$S = \{(x, y, z) : x^2 + y^2 \leq 9,\ 0 \leq z \leq x^2 + y^2\}.$$

At height z the horizontal slice consists of the points of the disk with $x^2+y^2 \geq z$, namely the annulus $\sqrt{z} \leq r \leq 3$. This slice is non-empty exactly for $0 \leq z \leq 9$, and its area is

$$A(z) = \pi \cdot 3^2 - \pi(\sqrt{z})^2 = \pi(9 - z).$$

By Fubini's Theorem with z outermost,

$$V = \int_0^9 A(z)\,\mathrm{d}z = \int_0^9 \pi(9-z)\,\mathrm{d}z = \pi\left[9z - \frac{z^2}{2}\right]_0^9 = \pi\left(81 - \frac{81}{2}\right) = \frac{81\pi}{2}.$$

Hence

$$\boxed{\text{Answer (e): None of the above.}}$$

(iii) The region is the right half-disk $D = \{x^2 + y^2 \leq 25,\ x \geq 0\}$ and the integrand $(x^2 + y^2)^{3/2} = r^3$ depends only on r. The part of D with distance at most r from the origin is a half-disk of area

$$A(r) = \frac{1}{2}\pi r^2, \qquad \text{so} \qquad A'(r) = \pi r,$$

which is the length of the half-circle of radius r (the area of a thin half-ring is $\pi r\,\mathrm{d}r$). Therefore

$$\iint_D (x^2 + y^2)^{3/2}\,\mathrm{d}A = \int_0^5 r^3 \cdot \pi r\,\mathrm{d}r = \pi\left[\frac{r^5}{5}\right]_0^5 = \pi \cdot \frac{3125}{5} = 625\pi.$$

Hence

$$\boxed{\text{Answer (a): } 625\pi.}$$

(v) The integrand $\sqrt{x^2 + y^2} = r$ depends only on the distance r from the z-axis. The part of T with distance at most r is a solid cylinder of radius r and height $8 - 4 = 4$, of volume

$$V(r) = 4\pi r^2, \qquad \text{so} \qquad V'(r) = 8\pi r,$$

the volume per unit thickness of the thin cylindrical shell at radius r. Therefore

$$\iiint_T \sqrt{x^2 + y^2}\,\mathrm{d}V = \int_0^1 r \cdot 8\pi r\,\mathrm{d}r = 8\pi\left[\frac{r^3}{3}\right]_0^1 = \frac{8\pi}{3}.$$

Hence

$$\boxed{\text{Answer (a): } \frac{8\pi}{3}.}$$

(vi) The integrand $(x^2+y^2+z^2)^2 = \rho^4$ depends only on the distance ρ from the origin. The part of B with distance at most ρ is a ball of volume

$$V(\rho) = \frac{4}{3}\pi\rho^3, \qquad \text{so} \qquad V'(\rho) = 4\pi\rho^2,$$

the surface area of the sphere of radius ρ (the thin spherical shell has volume $4\pi\rho^2\,\mathrm{d}\rho$). Therefore

$$\iiint_B (x^2+y^2+z^2)^2\,\mathrm{d}V = \int_0^4 \rho^4 \cdot 4\pi\rho^2\,\mathrm{d}\rho = 4\pi\left[\frac{\rho^7}{7}\right]_0^4 = \frac{4\pi\cdot 16384}{7} = \frac{65536\pi}{7}.$$

Hence

$$\boxed{\text{Answer (a): } \frac{65536\pi}{7}.}$$

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Method 4: Cartesian coordinates with a trigonometric substitution (part (ii))

Note: part (ii) asks for polar coordinates; this method should be used only if the course allows this interpretation.

(ii) Describe the disk as a type I region:

$$D = \{(x, y) : -3 \le x \le 3,\ -\sqrt{9 - x^2} \le y \le \sqrt{9 - x^2}\}.$$

Write $h(x) = \sqrt{9 - x^2}$. The integrand $x^2 + y^2$ is continuous on D, so by Fubini's Theorem

$$V = \iint_D (x^2 + y^2)\,\mathrm{d}A = \int_{-3}^{3} \int_{-h(x)}^{h(x)} (x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x.$$

With x held fixed, the inner integral is

$$\int_{-h}^{h} (x^2 + y^2)\,\mathrm{d}y = \left[x^2 y + \frac{y^3}{3}\right]_{-h}^{h} = 2x^2 h + \frac{2}{3}h^3.$$

Hence

$$V = \int_{-3}^{3} \left(2x^2\sqrt{9 - x^2} + \frac{2}{3}(9 - x^2)^{3/2}\right)\,\mathrm{d}x.$$

Let $x = 3\sin t$ with $-\pi/2 \le t \le \pi/2$. Then $\mathrm{d}x = 3\cos t\,\mathrm{d}t$, the limits become $x = -3 \mapsto t = -\pi/2$ and $x = 3 \mapsto t = \pi/2$, and, because $\cos t \ge 0$ on this interval,

$$\sqrt{9 - x^2} = \sqrt{9\cos^2 t} = 3\cos t.$$

Substituting term by term,

$$2x^2\sqrt{9 - x^2}\,\mathrm{d}x = 2 \cdot 9\sin^2 t \cdot 3\cos t \cdot 3\cos t\,\mathrm{d}t = 162\sin^2 t\cos^2 t\,\mathrm{d}t,$$
$$\frac{2}{3}(9 - x^2)^{3/2}\,\mathrm{d}x = \frac{2}{3} \cdot 27\cos^3 t \cdot 3\cos t\,\mathrm{d}t = 54\cos^4 t\,\mathrm{d}t.$$

Using the double-angle identities $\sin t\cos t = \frac{1}{2}\sin 2t$ and $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$,

$$\sin^2 t\cos^2 t = \frac{1}{4}\sin^2 2t = \frac{1}{8}(1 - \cos 4t),$$
$$\cos^4 t = \frac{1}{4}(1 + \cos 2t)^2 = \frac{1}{4}\left(1 + 2\cos 2t + \frac{1 + \cos 4t}{2}\right) = \frac{3}{8} + \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t.$$

Since $\sin 2t$ and $\sin 4t$ vanish at $t = \pm\pi/2$, the cosine terms integrate to 0 over $[-\pi/2, \pi/2]$, and

$$\int_{-\pi/2}^{\pi/2} \sin^2 t\cos^2 t\,\mathrm{d}t = \frac{1}{8} \cdot \pi = \frac{\pi}{8}, \qquad \int_{-\pi/2}^{\pi/2} \cos^4 t\,\mathrm{d}t = \frac{3}{8} \cdot \pi = \frac{3\pi}{8}.$$

Therefore

$$V = 162 \cdot \frac{\pi}{8} + 54 \cdot \frac{3\pi}{8} = \frac{162\pi}{8} + \frac{162\pi}{8} = \frac{324\pi}{8} = \frac{81\pi}{2}.$$

Hence

Answer (e): None of the above.

Method 5: Jacobian of a composition of maps (part (vii))

(vii) Write the transformation as the composition of

$$(u, v) \mapsto (p, q) = (u \sin v,\ u \cos v) \qquad \text{and} \qquad (p, q) \mapsto (x, y) = (5p,\ 4q).$$

For the first map,

$$\frac{\partial p}{\partial u} = \sin v, \quad \frac{\partial p}{\partial v} = u \cos v, \quad \frac{\partial q}{\partial u} = \cos v, \quad \frac{\partial q}{\partial v} = -u \sin v,$$

so

$$\frac{\partial(p, q)}{\partial(u, v)} = \begin{vmatrix} \sin v & u \cos v \\ \cos v & -u \sin v \end{vmatrix} = -u \sin^2 v - u \cos^2 v = -u.$$

For the second map,

$$\frac{\partial(x, y)}{\partial(p, q)} = \begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} = 20.$$

By the Chain Rule the Jacobian matrix of a composition is the product of the Jacobian matrices, and $\det(AB) = \det A \det B$. Therefore

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(x, y)}{\partial(p, q)} \cdot \frac{\partial(p, q)}{\partial(u, v)} = 20 \cdot (-u) = -20u.$$

Hence

$$\boxed{\text{Answer (c): } -20u.}$$

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Method 6: Reflection symmetry (parts (iii), (viii))

(iii) Let D be the full disk $x^2 + y^2 \le 25$, $D_+ = D \cap \{x \ge 0\}$ and $D_- = D \cap \{x \le 0\}$. The reflection $(x, y) \mapsto (-x, y)$ maps D_+ onto D_-, has Jacobian of absolute value 1, and leaves $g(x, y) = (x^2 + y^2)^{3/2}$ unchanged. Hence

$$\iint_{D_+} g \, \mathrm{d}A = \iint_{D_-} g \, \mathrm{d}A.$$

The two halves overlap only on the segment $x = 0$, which has zero area, so

$$\iint_{D_+} g \, \mathrm{d}A = \frac{1}{2} \iint_{D} g \, \mathrm{d}A.$$

In polar coordinates $D = \{0 \le r \le 5,\ 0 \le \theta \le 2\pi\}$ and $g = r^3$, so

$$\frac{1}{2} \iint_D g \, \mathrm{d}A = \frac{1}{2} \int_0^{2\pi} \int_0^5 r^3 \cdot r \, \mathrm{d}r \, \mathrm{d}\theta = \frac{1}{2} \cdot 2\pi \cdot \left[\frac{r^5}{5} \right]_0^5 = \pi \cdot 625 = 625\pi.$$

Hence

$$\boxed{\text{Answer (a): } 625\pi.}$$

(viii) The region $T = \{1 \le y \le x \le 3\}$ and its mirror image $T' = \{1 \le x \le y \le 3\}$ under $(x, y) \mapsto (y, x)$ together make up the square $Q = [1, 3]^2$, and they overlap only on the diagonal $x = y$, which has zero area. The reflection has Jacobian of absolute value 1 and the integrand satisfies $12xy = 12yx$, so

$$\iint_T 12xy \, \mathrm{d}A = \iint_{T'} 12xy \, \mathrm{d}A = \frac{1}{2} \iint_Q 12xy \, \mathrm{d}A.$$

On the square the integrand is a product of a function of x and a function of y, so the integral factors:

$$\frac{1}{2} \iint_Q 12xy \, \mathrm{d}A = 6 \int_1^3 x \, \mathrm{d}x \int_1^3 y \, \mathrm{d}y = 6 \left[\frac{x^2}{2} \right]_1^3 \left[\frac{y^2}{2} \right]_1^3 = 6 \cdot 4 \cdot 4 = 96.$$

Hence

$$\boxed{\text{Answer (a): } 96.}$$

Unofficial QRS marking scheme (40 marks)

(i) **(5 marks)** 2 marks for correct integration in y; 2 for integration in x; 1 for 120 and answer (e).

(ii) **(5 marks)** 2 marks for polar volume setup; 2 for correct evaluation; 1 for recognising that $81\pi/2$ is answer (e).

(iii) **(5 marks)** 2 marks for the half-disk bounds; 2 for the polar integrand including the Jacobian; 1 for answer (a).

(iv) **(5 marks)** 2 marks for integrating in z; 2 for the remaining integrations; 1 for answer (a).

(v) **(5 marks)** 2 marks for cylindrical bounds; 2 for the integrand and volume factor; 1 for answer (a).

(vi) **(5 marks)** 2 marks for spherical conversion; 2 for all bounds and integrations; 1 for answer (a).

(vii) **(5 marks)** 3 marks for the determinant; 1 for simplification; 1 for answer (c).

(viii) **(5 marks)** 3 marks for the inner integral; 1 for the outer integral; 1 for answer (a).

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Question 2 (30 Marks)

(beyond 2026 midterm scope): Part (b) uses a general Jacobian substitution and part (c) is a triple integral.

(a) Evaluate the iterated integral

$$\int_0^4 \int_{\sqrt{y}}^2 \mathrm{e}^{x^3}\,\mathrm{d}x\,\mathrm{d}y.$$

(b) Use the transformation $x = \sqrt{2}u - \sqrt{\frac{2}{3}}v$; $y = \sqrt{2}u + \sqrt{\frac{2}{3}}v$ to evaluate the integral

$$\iint_R x^2 - xy + y^2\,\mathrm{d}A,$$

where R is the region bounded by the ellipse $x^2 - xy + y^2 = 2$.

(c) Evaluate the triple integral

$$\iiint_R z^2\,\mathrm{d}V,$$

where R is the solid bounded by the sphere $x^2 + y^2 + z^2 = 1$, the cylinder $x^2 + y^2 = 1$, and the plane $z = 2$.

Solution

Method 1: Reversed order, linear change of variables, cylindrical coordinates

(a) The function e^{x^3} has no elementary antiderivative in x, so the inner integral cannot be done in the given order. The limits describe the region

$$D = \{(x, y) : 0 \le y \le 4,\ \sqrt{y} \le x \le 2\}.$$

For $y \ge 0$ and $x \ge 0$, the inequality $x \ge \sqrt{y}$ is equivalent to $y \le x^2$. As y ranges over $[0, 4]$, x ranges over $[0, 2]$, so with x outside,

$$D = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le x^2\}.$$

The integrand e^{x^3} is continuous on the closed bounded region D, which is both of type I and of type II, so by Fubini's Theorem

$$\int_0^4 \int_{\sqrt{y}}^2 \mathrm{e}^{x^3}\,\mathrm{d}x\,\mathrm{d}y = \int_0^2 \int_0^{x^2} \mathrm{e}^{x^3}\,\mathrm{d}y\,\mathrm{d}x.$$

The inner integral is

$$\int_0^{x^2} \mathrm{e}^{x^3}\,\mathrm{d}y = \mathrm{e}^{x^3}\big[y\big]_0^{x^2} = x^2\mathrm{e}^{x^3}.$$

For the outer integral let $w = x^3$. Then $\mathrm{d}w = 3x^2\,\mathrm{d}x$, and the limits become $x = 0 \mapsto w = 0$, $x = 2 \mapsto w = 8$:

$$\int_0^2 x^2\mathrm{e}^{x^3}\,\mathrm{d}x = \frac{1}{3}\int_0^8 \mathrm{e}^w\,\mathrm{d}w = \frac{1}{3}\big[\mathrm{e}^w\big]_0^8 = \frac{\mathrm{e}^8 - 1}{3}.$$

Hence

$$\boxed{\frac{\mathrm{e}^8 - 1}{3}.}$$

(b) Write $a = \sqrt{2}u$ and $b = \sqrt{2/3}\,v$, so that $x = a - b$ and $y = a + b$. Then

$$x^2 = a^2 - 2ab + b^2, \qquad y^2 = a^2 + 2ab + b^2, \qquad xy = a^2 - b^2,$$

and

$$\begin{aligned} x^2 - xy + y^2 &= (a^2 - 2ab + b^2) - (a^2 - b^2) + (a^2 + 2ab + b^2) = a^2 + 3b^2 \\ &= 2u^2 + 3 \cdot \frac{2}{3} v^2 = 2u^2 + 2v^2. \end{aligned}$$

So the ellipse $x^2 - xy + y^2 = 2$ corresponds to $2u^2 + 2v^2 = 2$, that is $u^2 + v^2 = 1$, and its interior $x^2 - xy + y^2 < 2$ corresponds to $u^2 + v^2 < 1$. Thus R is the image of the unit disk

$$D = \{(u, v) : u^2 + v^2 \le 1\}.$$

The partial derivatives of the transformation are

$$\frac{\partial x}{\partial u} = \sqrt{2}, \qquad \frac{\partial x}{\partial v} = -\sqrt{\tfrac{2}{3}}, \qquad \frac{\partial y}{\partial u} = \sqrt{2}, \qquad \frac{\partial y}{\partial v} = \sqrt{\tfrac{2}{3}},$$

so, using $\sqrt{2}\sqrt{2/3} = \sqrt{4/3} = 2/\sqrt{3}$,

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \sqrt{2}\sqrt{\tfrac{2}{3}} - \left(-\sqrt{\tfrac{2}{3}}\right)\sqrt{2} = \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4}{\sqrt{3}}.$$

The map $(u, v) \mapsto (x, y)$ is linear with non-zero determinant, hence C^1 and one-to-one from D onto R, so the Change of Variables Theorem applies with $\mathrm{d}A = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v = \frac{4}{\sqrt{3}}\,\mathrm{d}u\,\mathrm{d}v$:

$$\iint_R (x^2 - xy + y^2)\,\mathrm{d}A = \iint_D (2u^2 + 2v^2) \cdot \frac{4}{\sqrt{3}}\,\mathrm{d}u\,\mathrm{d}v = \frac{8}{\sqrt{3}} \iint_D (u^2 + v^2)\,\mathrm{d}u\,\mathrm{d}v.$$

In polar coordinates $u = \rho\cos\theta$, $v = \rho\sin\theta$, the disk is $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $u^2 + v^2 = \rho^2$ and $\mathrm{d}u\,\mathrm{d}v = \rho\,\mathrm{d}\rho\,\mathrm{d}\theta$:

$$\frac{8}{\sqrt{3}} \int_0^{2\pi} \int_0^1 \rho^2 \cdot \rho\,\mathrm{d}\rho\,\mathrm{d}\theta = \frac{8}{\sqrt{3}} \int_0^{2\pi} \left[\frac{\rho^4}{4}\right]_0^1 \mathrm{d}\theta = \frac{8}{\sqrt{3}} \cdot \frac{1}{4} \cdot 2\pi = \frac{4\pi}{\sqrt{3}}.$$

Hence

$$\boxed{\frac{4\pi}{\sqrt{3}}}$$

(c) The wording is ambiguous. On the sphere $x^2 + y^2 + z^2 = 1$ one has $|z| \le 1$, so the plane $z = 2$ does not meet the sphere; the sphere lies inside the cylinder $x^2 + y^2 = 1$ and touches it along the circle $z = 0$. Inside the cylinder and outside the sphere there are two pieces: the piece above the upper hemisphere, which the plane $z = 2$ closes off, and the piece below the lower hemisphere, which is unbounded. Take the bounded piece:

$$R = \{(x, y, z) : x^2 + y^2 \le 1,\ \sqrt{1 - x^2 - y^2} \le z \le 2\}.$$

In cylindrical coordinates, with $\mathrm{d}V = r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$,

$$R = \{(r, \theta, z) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ \sqrt{1 - r^2} \le z \le 2\}.$$

The inner integral is

$$\int_{\sqrt{1-r^2}}^{2} z^2 \,\mathrm{d}z = \left[\frac{z^3}{3}\right]_{\sqrt{1-r^2}}^{2} = \frac{1}{3}\big(8 - (1-r^2)^{3/2}\big).$$

The θ-integral contributes a factor 2π, so

$$\iiint_R z^2 \,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_{\sqrt{1-r^2}}^{2} z^2 r \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta = \frac{2\pi}{3}\int_0^1 \big(8r - r(1-r^2)^{3/2}\big)\,\mathrm{d}r.$$

The first term gives $\int_0^1 8r\,\mathrm{d}r = \big[4r^2\big]_0^1 = 4$. For the second let $w = 1-r^2$, so $\mathrm{d}w = -2r\,\mathrm{d}r$ and $r = 0 \mapsto w = 1$, $r = 1 \mapsto w = 0$:

$$\int_0^1 r(1-r^2)^{3/2}\,\mathrm{d}r = \int_1^0 w^{3/2}\left(-\frac{1}{2}\right)\mathrm{d}w = \frac{1}{2}\int_0^1 w^{3/2}\,\mathrm{d}w = \frac{1}{2}\left[\frac{2}{5}w^{5/2}\right]_0^1 = \frac{1}{5}.$$

Therefore

$$\iiint_R z^2 \,\mathrm{d}V = \frac{2\pi}{3}\left(4 - \frac{1}{5}\right) = \frac{2\pi}{3}\cdot\frac{19}{5} = \frac{38\pi}{15}.$$

Hence

$$\boxed{\frac{38\pi}{15}}$$

Unofficial QRS@NTU Question

Method 2: Integration by parts (a), completing the square (b), slicing in z (c)

(a) Keep the given order and let

$$F(y) = \int_{\sqrt{y}}^{2} \mathrm{e}^{x^3}\,\mathrm{d}x, \qquad 0 \le y \le 4,$$

so that the required integral is $\int_0^4 F(y)\,\mathrm{d}y$. Since e^{x^3} is continuous, the Fundamental Theorem of Calculus with the Chain Rule gives, for $0 < y \le 4$,

$$F'(y) = -\mathrm{e}^{(\sqrt{y})^3} \cdot \frac{\mathrm{d}}{\mathrm{d}y}\sqrt{y} = -\frac{\mathrm{e}^{y^{3/2}}}{2\sqrt{y}}.$$

Also $F(4) = \int_2^2 \mathrm{e}^{x^3}\,\mathrm{d}x = 0$, and $0 \le F(y) \le 2\mathrm{e}^8$ because the interval $[\sqrt{y}, 2]$ has length at most 2 and $\mathrm{e}^{x^3} \le \mathrm{e}^8$ on it. Fix $0 < \varepsilon < 4$ and integrate by parts on $[\varepsilon, 4]$ with $U = F(y)$ and $\mathrm{d}V = \mathrm{d}y$, $V = y$:

$$\begin{aligned}\int_\varepsilon^4 F(y)\,\mathrm{d}y &= \big[yF(y)\big]_\varepsilon^4 - \int_\varepsilon^4 yF'(y)\,\mathrm{d}y \\ &= 4F(4) - \varepsilon F(\varepsilon) + \int_\varepsilon^4 \frac{y\,\mathrm{e}^{y^{3/2}}}{2\sqrt{y}}\,\mathrm{d}y = -\varepsilon F(\varepsilon) + \int_\varepsilon^4 \frac{\sqrt{y}}{2}\mathrm{e}^{y^{3/2}}\,\mathrm{d}y.\end{aligned}$$

Let $\varepsilon \to 0^+$. Then $0 \le \varepsilon F(\varepsilon) \le 2\mathrm{e}^8\varepsilon \to 0$, and both integrands are continuous on $[0,4]$, so

$$\int_0^4 F(y)\,\mathrm{d}y = \int_0^4 \frac{\sqrt{y}}{2}\mathrm{e}^{y^{3/2}}\,\mathrm{d}y.$$

Let $w = y^{3/2}$. Then $\mathrm{d}w = \frac{3}{2}\sqrt{y}\,\mathrm{d}y$, so $\frac{\sqrt{y}}{2}\,\mathrm{d}y = \frac{1}{3}\,\mathrm{d}w$, and $y = 0 \mapsto w = 0$, $y = 4 \mapsto w = 8$:

$$\int_0^4 \frac{\sqrt{y}}{2}\mathrm{e}^{y^{3/2}}\,\mathrm{d}y = \frac{1}{3}\int_0^8 \mathrm{e}^w\,\mathrm{d}w = \frac{1}{3}\big[\mathrm{e}^w\big]_0^8 = \frac{\mathrm{e}^8 - 1}{3}.$$

Hence

$$\boxed{\frac{\mathrm{e}^8 - 1}{3}}.$$

(b) Note: part (b) asks for the given transformation; this method should be used only if the course allows this interpretation.

Complete the square in y:

$$x^2 - xy + y^2 = \left(y^2 - xy + \frac{x^2}{4}\right) + \frac{3}{4}x^2 = \left(y - \frac{x}{2}\right)^2 + \frac{3}{4}x^2.$$

A point lies in R when $\left(y - \frac{x}{2}\right)^2 + \frac{3}{4}x^2 \le 2$. This needs $\frac{3}{4}x^2 \le 2$, that is $|x| \le c$ with $c = \sqrt{8/3} = \frac{2\sqrt{2}}{\sqrt{3}}$, and then $\left|y - \frac{x}{2}\right| \le h(x)$ with $h(x) = \sqrt{2 - \frac{3}{4}x^2}$. So

$$R = \left\{(x,y) : -c \le x \le c,\ \frac{x}{2} - h(x) \le y \le \frac{x}{2} + h(x)\right\}.$$

The integrand is continuous on R, so by Fubini's Theorem integrate first in y. With x fixed let $s = y - \frac{x}{2}$; then $\mathrm{d}s = \mathrm{d}y$ and $y = \frac{x}{2} \pm h \mapsto s = \pm h$:

$$\int_{x/2-h}^{x/2+h} \left[\left(y - \frac{x}{2}\right)^2 + \frac{3}{4}x^2\right]\,\mathrm{d}y = \int_{-h}^{h} \left(s^2 + \frac{3}{4}x^2\right)\,\mathrm{d}s = \left[\frac{s^3}{3} + \frac{3}{4}x^2 s\right]_{-h}^{h}$$

$$= \frac{2}{3}h^3 + \frac{3}{2}x^2 h.$$

So

$$\iint_R (x^2 - xy + y^2)\,\mathrm{d}A = \int_{-c}^{c} \left(\frac{2}{3}h(x)^3 + \frac{3}{2}x^2 h(x)\right)\,\mathrm{d}x.$$

Let $x = c\sin t$ with $-\pi/2 \le t \le \pi/2$. Then $\mathrm{d}x = c\cos t\,\mathrm{d}t$, $x^2 = \frac{8}{3}\sin^2 t$, $\frac{3}{4}x^2 = 2\sin^2 t$, and, since $\cos t \ge 0$,

$$h = \sqrt{2 - 2\sin^2 t} = \sqrt{2}\cos t.$$

Substituting term by term,

$$\frac{2}{3}h^3 = \frac{2}{3}\cdot 2\sqrt{2}\cos^3 t = \frac{4\sqrt{2}}{3}\cos^3 t, \qquad \frac{3}{2}x^2 h = \frac{3}{2}\cdot\frac{8}{3}\sin^2 t\cdot\sqrt{2}\cos t = 4\sqrt{2}\sin^2 t\cos t.$$

Multiplying by $\mathrm{d}x = c\cos t\,\mathrm{d}t$ with $c = \frac{2\sqrt{2}}{\sqrt{3}}$, and using $\frac{2\sqrt{2}}{\sqrt{3}}\cdot 4\sqrt{2} = \frac{16}{\sqrt{3}}$,

$$\iint_R (x^2 - xy + y^2)\,\mathrm{d}A = \frac{16}{\sqrt{3}}\int_{-\pi/2}^{\pi/2}\left(\frac{1}{3}\cos^4 t + \sin^2 t\cos^2 t\right)\,\mathrm{d}t.$$

From $\cos^4 t = \frac{3}{8} + \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t$ and $\sin^2 t\cos^2 t = \frac{1}{8} - \frac{1}{8}\cos 4t$ (double-angle identities), and since $\sin 2t$, $\sin 4t$ vanish at $t = \pm\pi/2$,

$$\int_{-\pi/2}^{\pi/2}\cos^4 t\,\mathrm{d}t = \frac{3\pi}{8}, \qquad \int_{-\pi/2}^{\pi/2}\sin^2 t\cos^2 t\,\mathrm{d}t = \frac{\pi}{8}.$$

Therefore

$$\iint_R (x^2 - xy + y^2)\,\mathrm{d}A = \frac{16}{\sqrt{3}}\left(\frac{1}{3}\cdot\frac{3\pi}{8} + \frac{\pi}{8}\right) = \frac{16}{\sqrt{3}}\cdot\frac{\pi}{4} = \frac{4\pi}{\sqrt{3}}.$$

Hence

$$\boxed{\frac{4\pi}{\sqrt{3}}}$$

(c) Use the same region R as in Method 1 and slice it horizontally (Fubini's Theorem with z outermost). At height z the slice is

$$R_z = \{(x,y) : x^2 + y^2 \le 1,\ \sqrt{1 - x^2 - y^2} \le z\}.$$

- If $z < 0$, then $\sqrt{1 - x^2 - y^2} \ge 0 > z$, so R_z is empty.
- If $0 \le z \le 1$, then $\sqrt{1 - x^2 - y^2} \le z \iff x^2 + y^2 \ge 1 - z^2$, so R_z is the annulus $1 - z^2 \le x^2 + y^2 \le 1$, of area $\pi\cdot 1 - \pi(1 - z^2) = \pi z^2$.
- If $1 \le z \le 2$, then $\sqrt{1 - x^2 - y^2} \le 1 \le z$ for every point of the unit disk, so R_z is the whole unit disk, of area π.

The integrand z^2 is constant on each slice, so

$$\iiint_R z^2\,\mathrm{d}V = \int_0^1 z^2\cdot\pi z^2\,\mathrm{d}z + \int_1^2 z^2\cdot\pi\,\mathrm{d}z = \pi\left[\frac{z^5}{5}\right]_0^1 + \pi\left[\frac{z^3}{3}\right]_1^2$$
$$= \frac{\pi}{5} + \frac{7\pi}{3} = \frac{3\pi + 35\pi}{15} = \frac{38\pi}{15}.$$

Hence

$$\boxed{\frac{38\pi}{15}}$$

Method 3: Sublevel ellipses (b), subtraction of a half-ball in spherical coordinates (c)

(b) Note: part (b) asks for the given transformation; this method should be used only if the course allows this interpretation.

Write $Q(x,y) = x^2 - xy + y^2 \geq 0$ and $Q(x,y) = \int_0^{Q(x,y)} \mathrm{d}t$. Then $\iint_R Q \,\mathrm{d}A$ is the volume of

$$W = \{(x,y,t) : (x,y) \in R,\ 0 \leq t \leq Q(x,y)\}.$$

By Fubini's Theorem with t outermost ($0 \leq t \leq 2$ on R), the slice of W at level t is $\{(x,y) : t \leq Q(x,y) \leq 2\}$, of area $A(2) - A(t)$, where $A(t)$ is the area of $\{Q \leq t\}$. By the completed square of Method 2,

$$\{Q \leq t\} = \left\{ \left(y - \frac{x}{2}\right)^2 + \frac{3}{4}x^2 \leq t \right\}.$$

The shear $(x,y) \mapsto (x,s) = (x,\ y - \frac{x}{2})$ has Jacobian $\begin{vmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{vmatrix} = 1$, so it preserves area, and it maps $\{Q \leq t\}$ onto the ellipse

$$\frac{s^2}{t} + \frac{x^2}{4t/3} \leq 1,$$

with semi-axes $\sqrt{t}$ and $\frac{2\sqrt{t}}{\sqrt{3}}$. The area of an ellipse with semi-axes a, b is πab, so

$$A(t) = \pi\sqrt{t} \cdot \frac{2\sqrt{t}}{\sqrt{3}} = \frac{2\pi t}{\sqrt{3}}.$$

Therefore

$$\iint_R Q \,\mathrm{d}A = \int_0^2 \big(A(2) - A(t)\big)\,\mathrm{d}t = \frac{2\pi}{\sqrt{3}} \int_0^2 (2-t)\,\mathrm{d}t = \frac{2\pi}{\sqrt{3}} \left[2t - \frac{t^2}{2}\right]_0^2 = \frac{2\pi}{\sqrt{3}} \cdot 2 = \frac{4\pi}{\sqrt{3}}.$$

Hence

$$\boxed{\frac{4\pi}{\sqrt{3}}}$$

(c) With R as in Method 1, let $C = \{x^2 + y^2 \leq 1,\ 0 \leq z \leq 2\}$ be the solid cylinder and $H = \{x^2 + y^2 + z^2 \leq 1,\ z \geq 0\}$ the upper half-ball. Then $C = R \cup H$, and $R \cap H$ is the upper hemisphere, which has zero volume. Hence

$$\iiint_R z^2 \,\mathrm{d}V = \iiint_C z^2 \,\mathrm{d}V - \iiint_H z^2 \,\mathrm{d}V.$$

In cylindrical coordinates $C = \{0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 2\}$, so

$$\iiint_C z^2 \,\mathrm{d}V = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 r\,\mathrm{d}r \int_0^2 z^2\,\mathrm{d}z = 2\pi \cdot \frac{1}{2} \cdot \frac{8}{3} = \frac{8\pi}{3}.$$

In spherical coordinates $z = \rho\cos\phi$, $\mathrm{d}V = \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$ and $H = \{0 \leq \rho \leq 1,\ 0 \leq \phi \leq \pi/2,\ 0 \leq \theta \leq 2\pi\}$, so

$$\iiint_H z^2 \,\mathrm{d}V = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \rho^4\,\mathrm{d}\rho \int_0^{\pi/2} \cos^2\phi \sin\phi\,\mathrm{d}\phi = 2\pi \cdot \frac{1}{5} \cdot \int_0^{\pi/2} \cos^2\phi \sin\phi\,\mathrm{d}\phi.$$

For the last integral let $w = \cos\phi$, so $\mathrm{d}w = -\sin\phi\,\mathrm{d}\phi$ and $\phi = 0 \mapsto w = 1$, $\phi = \pi/2 \mapsto w = 0$:

$$\int_0^{\pi/2} \cos^2\phi \sin\phi\,\mathrm{d}\phi = \int_1^0 w^2(-\mathrm{d}w) = \int_0^1 w^2\,\mathrm{d}w = \frac{1}{3}.$$

So $\iiint_H z^2\,\mathrm{d}V = 2\pi \cdot \frac{1}{5} \cdot \frac{1}{3} = \frac{2\pi}{15}$, and

$$\iiint_R z^2\,\mathrm{d}V = \frac{8\pi}{3} - \frac{2\pi}{15} = \frac{40\pi}{15} - \frac{2\pi}{15} = \frac{38\pi}{15}.$$

Hence

$$\boxed{\frac{38\pi}{15}}$$

Unofficial QRS marking scheme (30 marks)

(a) **(10 marks)** 3 marks for the reversed region; 3 for the reversed integral; 3 for the substitution $u = x^3$; 1 for the final value.

(b) **(10 marks)** 3 marks for diagonalising the ellipse; 3 for the Jacobian; 3 for the unit-disk polar integral; 1 for $4\pi/\sqrt{3}$.

(c) **(10 marks)** 2 marks for identifying the literal ambiguity; 3 for an explicit bounded-component interpretation and bounds; 3 for the cylindrical integral; 2 for $38\pi/15$ under that interpretation.

Unofficial QRS@NTU solutions

Question 3 (30 Marks)

(beyond 2026 midterm scope): All parts involve vector fields or line integrals.

(a) Find the work done by the force field $\mathbf{F}(x,y) = x\sin(y)\mathrm{i} + y\mathrm{j}$ on a particle that moves along the curve $y = x^2$ from $(0,0)$ to $(\sqrt{\pi},\pi)$.

(b) Evaluate the line integral $6\int_C yz\cos x\,\mathrm{d}s$ where C has parametrisation $x = t$, $y = \cos t$ and $z = \sin t$, with $0 \le t \le \pi$.

(c) Let $\mathbf{F}(x,y) = x^3y^2\mathrm{i} + y\left(\dfrac{x^4}{2} - 2\right)\mathrm{j}$ and let C be a curve with parametrisation $\mathbf{r}(t) = \sqrt{t}\mathrm{i} + (1+t^3)\mathrm{j}$ with $0 \le t \le 1$ (starting at $\mathbf{r}(0)$ and terminating at $\mathbf{r}(1)$).

(i) Find a function f such that $\mathbf{F} = \nabla f$.

(ii) Hence, or otherwise, evaluate the line integral $\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r}$.

Solution

Method 1: Parametrisation by x, potential from the x-partial, Fundamental Theorem for Line Integrals

(a) Parametrise the curve $y = x^2$ by $x = t$:

$$\mathbf{r}(t) = (t, t^2), \qquad 0 \le t \le \sqrt{\pi},$$

which starts at $\mathbf{r}(0) = (0,0)$ and ends at $\mathbf{r}(\sqrt{\pi}) = (\sqrt{\pi},\pi)$. Then

$$\mathbf{r}'(t) = (1, 2t), \qquad \mathbf{F}(\mathbf{r}(t)) = \big(t\sin(t^2),\ t^2\big),$$

so

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = t\sin(t^2)\cdot 1 + t^2\cdot 2t = t\sin(t^2) + 2t^3.$$

The work is

$$W = \int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \int_0^{\sqrt{\pi}} \big(t\sin(t^2) + 2t^3\big)\,\mathrm{d}t.$$

For the first term let $w = t^2$. Then $\mathrm{d}w = 2t\,\mathrm{d}t$, and $t = 0 \mapsto w = 0$, $t = \sqrt{\pi} \mapsto w = \pi$:

$$\int_0^{\sqrt{\pi}} t\sin(t^2)\,\mathrm{d}t = \frac{1}{2}\int_0^{\pi}\sin w\,\mathrm{d}w = \frac{1}{2}\big[-\cos w\big]_0^{\pi} = \frac{1}{2}\big(1+1\big) = 1.$$

The second term is

$$\int_0^{\sqrt{\pi}} 2t^3\,\mathrm{d}t = \left[\frac{t^4}{2}\right]_0^{\sqrt{\pi}} = \frac{\pi^2}{2}.$$

Therefore $W = 1 + \frac{\pi^2}{2}$, that is,

$$\boxed{1 + \frac{\pi^2}{2}}$$

(b) With $\mathbf{r}(t) = (t, \cos t, \sin t)$,

$$\mathbf{r}'(t) = (1, -\sin t, \cos t), \qquad \|\mathbf{r}'(t)\| = \sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2},$$

so $\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t = \sqrt{2}\,\mathrm{d}t$. Along C the integrand is

$$yz\cos x = \cos t\cdot\sin t\cdot\cos t = \cos^2 t\sin t.$$

Hence

$$6\int_C yz\cos x\,\mathrm{d}s = 6\int_0^{\pi} \cos^2 t\sin t\cdot\sqrt{2}\,\mathrm{d}t = 6\sqrt{2}\int_0^{\pi}\cos^2 t\sin t\,\mathrm{d}t.$$

Let $w=\cos t$. Then $\mathrm{d}w=-\sin t\,\mathrm{d}t$, and $t=0\mapsto w=1$, $t=\pi\mapsto w=-1$:

$$\int_0^{\pi}\cos^2 t\sin t\,\mathrm{d}t = \int_1^{-1} w^2(-\mathrm{d}w) = \int_{-1}^{1} w^2\,\mathrm{d}w = \left[\frac{w^3}{3}\right]_{-1}^{1} = \frac{2}{3}.$$

Therefore

$$6\int_C yz\cos x\,\mathrm{d}s = 6\sqrt{2}\cdot\frac{2}{3} = 4\sqrt{2},$$

that is,

$$\boxed{4\sqrt{2}}$$

(c) (i) Write $\mathbf{F}=P\mathrm{i}+Q\mathrm{j}$ with

$$P(x,y)=x^3y^2, \qquad Q(x,y)=y\left(\frac{x^4}{2}-2\right)=\frac{x^4y}{2}-2y.$$

First check that a potential exists. The partial derivatives are

$$\frac{\partial P}{\partial y}=2x^3y, \qquad \frac{\partial Q}{\partial x}=\frac{4x^3y}{2}=2x^3y.$$

They are equal and continuous on all of $\mathbb{R}^2$, which is open and simply connected, so $\mathbf{F}$ is conservative. Now integrate $f_x=P$ with respect to x, holding y fixed:

$$f(x,y)=\int x^3y^2\,\mathrm{d}x=\frac{x^4y^2}{4}+h(y),$$

where h depends on y only. Differentiate with respect to y and match with Q:

$$f_y=\frac{x^4y}{2}+h'(y)=\frac{x^4y}{2}-2y \implies h'(y)=-2y \implies h(y)=-y^2+K.$$

Taking $K=0$ gives $f(x,y)=\frac{x^4y^2}{4}-y^2$. Check: $f_x=x^3y^2=P$ and $f_y=\frac{x^4y}{2}-2y=Q$. Hence

$$\boxed{f(x,y)=\frac{x^4y^2}{4}-y^2.}$$

(ii) The endpoints of C are

$$\mathbf{r}(0)=\left(\sqrt{0},\ 1+0^3\right)=(0,1), \qquad \mathbf{r}(1)=\left(\sqrt{1},\ 1+1^3\right)=(1,2).$$

The Fundamental Theorem for Line Integrals requires a C^1 function f and a C^1 (or piecewise C^1) parametrisation of C. Here $\mathbf{r}'(t)=\left(\frac{1}{2\sqrt{t}},\ 3t^2\right)$ is undefined at $t=0$, so reparametrise with $t=s^2$, $0\le s\le 1$:

$$\tilde{\mathbf{r}}(s)=\mathbf{r}(s^2)=(s,\ 1+s^6), \qquad \tilde{\mathbf{r}}'(s)=(1,\ 6s^5).$$

This is C^1 on $[0,1]$, traces the same curve in the same direction (since $s\mapsto s^2$ is increasing), and runs from $(0,1)$ to $(1,2)$. The potential f from (i) is a polynomial, hence C^1, and $\mathbf{F}=\nabla f$. Therefore

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}=f(1,2)-f(0,1)=\left(\frac{1\cdot 4}{4}-4\right)-(0-1)=-3+1=-2.$$

Hence

$$\boxed{-2}$$

Method 2: Parametrisation by y (a), triple-angle identity (b), potential from the y-partial and direct line integral (c)

(a) On C one has $x = \sqrt{y} \geq 0$, so parametrise by y:

$$\mathbf{r}(y) = (\sqrt{y},\ y), \qquad 0 \leq y \leq \pi,$$

from $(0,0)$ to $(\sqrt{\pi}, \pi)$. For $y > 0$,

$$\mathbf{r}'(y) = \left(\frac{1}{2\sqrt{y}},\ 1\right), \qquad \mathbf{F}(\mathbf{r}(y)) = \left(\sqrt{y}\sin y,\ y\right),$$

so

$$\mathbf{F}(\mathbf{r}(y)) \cdot \mathbf{r}'(y) = \sqrt{y}\sin y \cdot \frac{1}{2\sqrt{y}} + y \cdot 1 = \frac{1}{2}\sin y + y.$$

The derivative $\mathbf{r}'(y)$ is unbounded as $y \to 0^+$, but for each $0 < \varepsilon < \pi$ the line integral over the arc from $(\sqrt{\varepsilon}, \varepsilon)$ to $(\sqrt{\pi}, \pi)$ is the same in either parametrisation (both are C^1 and increasing there). As $\varepsilon \to 0^+$ both sides tend to the full integral, because the dot product above extends continuously to $y = 0$. Hence

$$W = \int_0^\pi \left(\frac{1}{2}\sin y + y\right) \mathrm{d}y = \frac{1}{2}\big[-\cos y\big]_0^\pi + \left[\frac{y^2}{2}\right]_0^\pi = \frac{1}{2}\cdot 2 + \frac{\pi^2}{2} = 1 + \frac{\pi^2}{2}.$$

Hence

$$\boxed{1 + \frac{\pi^2}{2}}$$

(b) As in Method 1, $\mathrm{d}s = \sqrt{2}\,\mathrm{d}t$ and the integrand along C is $\cos^2 t \sin t$. Rewrite it without powers. From $\cos^2 t = 1 - \sin^2 t$ and the triple-angle identity $\sin 3t = 3\sin t - 4\sin^3 t$, that is $\sin^3 t = \frac{1}{4}(3\sin t - \sin 3t)$,

$$\cos^2 t \sin t = \sin t - \sin^3 t = \sin t - \frac{3}{4}\sin t + \frac{1}{4}\sin 3t = \frac{1}{4}(\sin t + \sin 3t).$$

Then

$$\int_0^\pi \cos^2 t \sin t\,\mathrm{d}t = \frac{1}{4}\left(\big[-\cos t\big]_0^\pi + \left[-\frac{\cos 3t}{3}\right]_0^\pi\right) = \frac{1}{4}\left(2 + \frac{1+1}{3}\right) = \frac{1}{4}\cdot\frac{8}{3} = \frac{2}{3}.$$

Therefore

$$6\int_C yz\cos x\,\mathrm{d}s = 6\sqrt{2}\cdot\frac{2}{3} = 4\sqrt{2},$$

that is,

$$\boxed{4\sqrt{2}}$$

(c) (i) The equality $\partial P/\partial y = 2x^3 y = \partial Q/\partial x$ on $\mathbb{R}^2$ was checked in Method 1, so a potential exists. This time integrate $f_y = Q$ with respect to y, holding x fixed:

$$f(x,y) = \int \left(\frac{x^4 y}{2} - 2y\right) \mathrm{d}y = \frac{x^4 y^2}{4} - y^2 + g(x),$$

where g depends on x only. Differentiate with respect to x and match with P:

$$f_x = x^3 y^2 + g'(x) = x^3 y^2 \implies g'(x) = 0 \implies g(x) = K.$$

Taking $K = 0$,

$$\boxed{f(x,y) = \frac{x^4 y^2}{4} - y^2.}$$

(ii) Compute the line integral directly with the C^1 parametrisation $\tilde{\mathbf{r}}(s) = (s,\ 1+s^6)$, $0 \le s \le 1$, from Method 1, for which $\tilde{\mathbf{r}}'(s) = (1,\ 6s^5)$. Along the curve,

$$P = s^3(1+s^6)^2, \qquad Q = (1+s^6)\left(\frac{s^4}{2} - 2\right).$$

Expand each term of $\mathbf{F}\cdot\tilde{\mathbf{r}}' = P\cdot 1 + Q\cdot 6s^5$:

$$s^3(1+s^6)^2 = s^3(1+2s^6+s^{12}) = s^3 + 2s^9 + s^{15},$$
$$6s^5(1+s^6)\left(\frac{s^4}{2}-2\right) = 6s^5\left(\frac{s^4}{2} - 2 + \frac{s^{10}}{2} - 2s^6\right) = 3s^9 - 12s^5 + 3s^{15} - 12s^{11}.$$

Adding,

$$\mathbf{F}(\tilde{\mathbf{r}}(s))\cdot\tilde{\mathbf{r}}'(s) = s^3 - 12s^5 + 5s^9 - 12s^{11} + 4s^{15}.$$

Therefore

$$\begin{aligned}\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} &= \int_0^1 \left(s^3 - 12s^5 + 5s^9 - 12s^{11} + 4s^{15}\right)\mathrm{d}s \\ &= \left[\frac{s^4}{4} - 2s^6 + \frac{s^{10}}{2} - s^{12} + \frac{s^{16}}{4}\right]_0^1 \\ &= \frac{1}{4} - 2 + \frac{1}{2} - 1 + \frac{1}{4} = -2.\end{aligned}$$

Hence

$$\boxed{-2}$$

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Method 3: Splitting off a gradient (a), path independence along a broken line (c)

(a) Split $\mathbf{F} = \mathbf{G} + \mathbf{H}$ with

$$\mathbf{G}(x,y) = x\sin(y)\mathrm{i}, \qquad \mathbf{H}(x,y) = y\mathrm{j} = \nabla\left(\frac{y^2}{2}\right).$$

The field $\mathbf{F}$ itself is not conservative: $\partial(x\sin y)/\partial y = x\cos y$ while $\partial(y)/\partial x = 0$. Only the part $\mathbf{H}$ has a potential, and by the Fundamental Theorem for Line Integrals (with the C^1 parametrisation $\mathbf{r}(t) = (t, t^2)$),

$$\int_C \mathbf{H}\cdot\,\mathrm{d}\mathbf{r} = \frac{y^2}{2}\Big|_{(0,0)}^{(\sqrt{\pi},\pi)} = \frac{\pi^2}{2} - 0 = \frac{\pi^2}{2}.$$

For $\mathbf{G}$, with $\mathbf{r}(t) = (t, t^2)$ and $\mathbf{r}'(t) = (1, 2t)$, one has $\mathbf{G}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = t\sin(t^2)$. With $w = t^2$, $\mathrm{d}w = 2t\,\mathrm{d}t$, $t = 0 \mapsto w = 0$, $t = \sqrt{\pi} \mapsto w = \pi$,

$$\int_C \mathbf{G}\cdot\,\mathrm{d}\mathbf{r} = \int_0^{\sqrt{\pi}} t\sin(t^2)\,\mathrm{d}t = \frac{1}{2}\int_0^{\pi}\sin w\,\mathrm{d}w = \frac{1}{2}\big[-\cos w\big]_0^{\pi} = 1.$$

Therefore

$$W = \int_C \mathbf{G}\cdot\,\mathrm{d}\mathbf{r} + \int_C \mathbf{H}\cdot\,\mathrm{d}\mathbf{r} = 1 + \frac{\pi^2}{2},$$

that is,

$$\boxed{1 + \frac{\pi^2}{2}}$$

(c) (i) Integrate $\mathbf{F}$ along the segment from the origin to (x, y), $\mathbf{c}(t) = (tx, ty)$, $0 \le t \le 1$, with $\mathbf{c}'(t) = (x, y)$. Since $\mathbf{F}$ is conservative on $\mathbb{R}^2$ (Method 1), this defines a potential with $f(0,0) = 0$:

$$f(x,y) = \int_0^1 \mathbf{F}(tx, ty)\cdot(x,y)\,\mathrm{d}t.$$

Here $\mathbf{F}(tx,ty) = \left(t^5x^3y^2,\ \frac{t^5x^4y}{2} - 2ty\right)$, so

$$\mathbf{F}(tx,ty)\cdot(x,y) = t^5x^4y^2 + \frac{t^5x^4y^2}{2} - 2ty^2 = \frac{3}{2}t^5x^4y^2 - 2ty^2.$$

Therefore

$$f(x,y) = \frac{3}{2}x^4y^2\left[\frac{t^6}{6}\right]_0^1 - 2y^2\left[\frac{t^2}{2}\right]_0^1 = \frac{x^4y^2}{4} - y^2.$$

Hence

$$\boxed{f(x,y) = \frac{x^4y^2}{4} - y^2.}$$

(ii) Since $\mathbf{F} = \nabla f$ on $\mathbb{R}^2$, the line integral depends only on the endpoints $(0,1)$ and $(1,2)$, so replace C by the broken line C_1 from $(0,1)$ to $(1,1)$ followed by C_2 from $(1,1)$ to $(1,2)$.
On C_1: $\mathbf{r}_1(t) = (t,1)$, $0 \le t \le 1$, $\mathbf{r}_1'(t) = (1,0)$, so $\mathbf{F}\cdot\mathbf{r}_1' = P(t,1) = t^3$ and

$$\int_{C_1}\mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \int_0^1 t^3\,\mathrm{d}t = \frac{1}{4}.$$

On C_2: $\mathbf{r}_2(t) = (1, t)$, $1 \le t \le 2$, $\mathbf{r}_2'(t) = (0, 1)$, so $\mathbf{F} \cdot \mathbf{r}_2' = Q(1, t) = t\left(\frac{1}{2} - 2\right) = -\frac{3}{2}t$ and

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_1^2 \left(-\frac{3}{2}t\right) \mathrm{d}t = -\frac{3}{2}\left[\frac{t^2}{2}\right]_1^2 = -\frac{3}{2} \cdot \frac{3}{2} = -\frac{9}{4}.$$

Therefore

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \frac{1}{4} - \frac{9}{4} = -2.$$

Hence

$$\boxed{-2}$$

Unofficial QRS marking scheme (30 marks)

(a) **(10 marks)** 3 marks for a valid parametrisation; 3 for $\mathbf{F} \cdot \mathbf{r}'$; 3 for integration; 1 for $1 + \pi^2/2$.

(b) **(10 marks)** 3 marks for $\mathrm{d}s = \sqrt{2}\,\mathrm{d}t$; 3 for substitution into the scalar line integral; 3 for integration; 1 for $4\sqrt{2}$.

(c) **(10 marks)**

(i) 5 marks: 3 for integrating f_x and 2 for determining $h(y)$.

(ii) 5 marks: 2 for endpoints, 2 for the potential-difference theorem, and 1 for -2.

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