MH2100 Calculus III

Midterm I (Solutions)

AY 2019/2020, Semester 1

Quantitative Research Society @NTU

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Overview

- **Coverage:** parametric curves and tangent lines; limits, continuity and partial derivatives; tangent planes, the Chain Rule, implicit differentiation, and critical points.
- **Format:** three questions, 100 marks in total, 90 minutes (official paper: September 2019).
- **Historical permitted material:** the September 2019 paper permitted one two-sided A4 piece of paper with handwritten notes. Consult current course materials for present assessment rules.
- **Source:** MH2100 Midterm I, AY2019/2020 Semester 1. Questions are transcribed from the official paper; solutions and marking schemes are unofficial QRS@NTU material.

Instructions (verbatim)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. ONE two-sided A4 piece of paper with handwritten notes is permitted.

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Question 1 (30 Marks)

QUESTION 1.

(i) Find parametric equations to represent the line segment from $(2, 3)$ to $(5, 5)$.

(ii) Write a vector equation for a parametrisation of the tangent line at the point when $t = \pi$ to the curve with parametric equations

$$x = \cos t, \qquad y = \sin t, \qquad z = t.$$

(iii) Write the parametrisation of a curve that has tangent vector $\langle e^t, 2t, \sin t\rangle$.

(iv) Find the point(s) on the following curve where the tangent is horizontal.

$$x = t + 4, \qquad y = t^3 - 3t + 7.$$

(v) Find the directional derivative of the function $f(x, y, z) = xy^2 - xz^2$ in the direction of the vector $\langle 1, 2, 3\rangle$.

Solution

Method 1: Vector form, derivative of the position vector, and the gradient formula

(i) Let $P = (2, 3)$ and $Q = (5, 5)$. The displacement vector from P to Q is

$$\overrightarrow{PQ} = \langle 5 - 2,\ 5 - 3\rangle = \langle 3, 2\rangle.$$

The line through P in the direction $\overrightarrow{PQ}$ is

$$\mathbf{r}(t) = \langle 2, 3\rangle + t\langle 3, 2\rangle = \langle 2 + 3t,\ 3 + 2t\rangle.$$

At the two ends,

$$\mathbf{r}(0) = \langle 2, 3\rangle, \qquad \mathbf{r}(1) = \langle 2 + 3,\ 3 + 2\rangle = \langle 5, 5\rangle,$$

so restricting the parameter to $0 \le t \le 1$ traces exactly the segment from $(2, 3)$ to $(5, 5)$ and not the whole line. Therefore

$$\boxed{x = 2 + 3t, \quad y = 3 + 2t, \quad 0 \le t \le 1.}$$

(ii) Let $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$. The point on the curve at $t = \pi$ is

$$\mathbf{r}(\pi) = \langle \cos \pi, \sin \pi, \pi\rangle = \langle -1, 0, \pi\rangle.$$

Differentiating componentwise,

$$\mathbf{r}'(t) = \langle -\sin t,\ \cos t,\ 1\rangle,$$

so

$$\mathbf{r}'(\pi) = \langle -\sin \pi,\ \cos \pi,\ 1\rangle = \langle 0, -1, 1\rangle.$$

Since $\mathbf{r}'(\pi) \neq \mathbf{0}$, it is a direction vector for the tangent line. The tangent line passes through $\mathbf{r}(\pi)$ with direction $\mathbf{r}'(\pi)$:

$$\boxed{\mathbf{l}(s) = \langle -1, 0, \pi\rangle + s\langle 0, -1, 1\rangle, \qquad s \in \mathbb{R}.}$$

(iii) A curve $\mathbf{r}(t)$ has tangent vector $\mathbf{r}'(t)$, so we need

$$\mathbf{r}'(t) = \langle \mathrm{e}^t,\ 2t,\ \sin t \rangle.$$

Integrate each component:

$$\int \mathrm{e}^t\,\mathrm{d}t = \mathrm{e}^t + c_1, \qquad \int 2t\,\mathrm{d}t = t^2 + c_2, \qquad \int \sin t\,\mathrm{d}t = -\cos t + c_3.$$

Hence every such curve has the form

$$\mathbf{r}(t) = \langle \mathrm{e}^t,\ t^2,\ -\cos t \rangle + \mathbf{c}, \qquad \mathbf{c} = \langle c_1, c_2, c_3 \rangle \text{ constant}.$$

Take $\mathbf{c} = \mathbf{0}$. Differentiating back,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathrm{e}^t,\ t^2,\ -\cos t \rangle = \langle \mathrm{e}^t,\ 2t,\ \sin t \rangle.$$

Therefore one such curve is

$$\boxed{\mathbf{r}(t) = \langle \mathrm{e}^t, t^2, -\cos t \rangle}$$

(any constant vector may be added).

(iv) Let $x(t) = t + 4$ and $y(t) = t^3 - 3t + 7$. Then

$$x'(t) = 1, \qquad y'(t) = 3t^2 - 3.$$

Since $x'(t) = 1 \neq 0$ for every t, the slope of the tangent is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y'(t)}{x'(t)} = 3t^2 - 3.$$

The tangent is horizontal exactly when this slope is 0:

$$3t^2 - 3 = 0 \iff t^2 = 1 \iff t = 1 \text{ or } t = -1.$$

At $t = 1$:

$$x = 1 + 4 = 5, \qquad y = 1 - 3 + 7 = 5.$$

At $t = -1$:

$$x = -1 + 4 = 3, \qquad y = -1 + 3 + 7 = 9.$$

Therefore the tangent is horizontal at

$$\boxed{(5, 5) \text{ and } (3, 9).}$$

(v) The function $f(x, y, z) = xy^2 - xz^2$ is a polynomial, so its partial derivatives are continuous and f is differentiable; hence $D_{\mathbf{u}} f = \operatorname{grad} f \cdot \mathbf{u}$ for every unit vector $\mathbf{u}$. The direction vector has length

$$|\langle 1, 2, 3 \rangle| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \qquad \text{so} \qquad \mathbf{u} = \frac{1}{\sqrt{14}}\langle 1, 2, 3 \rangle.$$

The partial derivatives are

$$f_x = y^2 - z^2, \qquad f_y = 2xy, \qquad f_z = -2xz,$$

so $\operatorname{grad} f = \langle y^2 - z^2,\ 2xy,\ -2xz \rangle$. Then

$$D_{\mathbf{u}} f = \operatorname{grad} f \cdot \mathbf{u} = \frac{1 \cdot (y^2 - z^2) + 2 \cdot (2xy) + 3 \cdot (-2xz)}{\sqrt{14}} = \frac{y^2 - z^2 + 4xy - 6xz}{\sqrt{14}}.$$

Therefore

$$\boxed{D_{\mathbf{u}} f(x, y, z) = \frac{y^2 - z^2 + 4xy - 6xz}{\sqrt{14}}.}$$

Method 2: Point-slope form, secant limit, base-point integral, parameter elimination, and the limit definition

(i) The slope of the segment is

$$m = \frac{5-3}{5-2} = \frac{2}{3},$$

so the segment lies on the line

$$y - 3 = \frac{2}{3}(x-2).$$

Along the segment x runs from 2 to 5. Put $x = 2+3t$; then $x = 2$ at $t = 0$ and $x = 5$ at $t = 1$, so $0 \le t \le 1$. Substituting,

$$y = 3 + \frac{2}{3}\big((2+3t)-2\big) = 3 + \frac{2}{3}\cdot 3t = 3 + 2t.$$

Therefore

$$\boxed{x = 2+3t, \quad y = 3+2t, \quad 0 \le t \le 1.}$$

(ii) Let $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, so $\mathbf{r}(\pi) = \langle -1, 0, \pi\rangle$. Compute the tangent direction as the limit of secant vectors,

$$\mathbf{r}'(\pi) = \lim_{h\to 0} \frac{\mathbf{r}(\pi+h) - \mathbf{r}(\pi)}{h},$$

using $\cos(\pi+h) = -\cos h$ and $\sin(\pi+h) = -\sin h$. First component:

$$\frac{-\cos h - (-1)}{h} = \frac{1-\cos h}{h} = \frac{\sin^2 h}{h(1+\cos h)} = \frac{\sin h}{h}\cdot\frac{\sin h}{1+\cos h} \longrightarrow 1\cdot\frac{0}{2} = 0.$$

Second component:

$$\frac{-\sin h - 0}{h} = -\frac{\sin h}{h} \longrightarrow -1.$$

Third component:

$$\frac{(\pi+h)-\pi}{h} = 1 \longrightarrow 1.$$

Hence the tangent direction is $\langle 0, -1, 1\rangle$, and the tangent line through $\mathbf{r}(\pi)$ is

$$\boxed{\mathbf{l}(s) = \langle -1, 0, \pi\rangle + s\langle 0, -1, 1\rangle, \qquad s \in \mathbb{R}.}$$

(iii) By the Fundamental Theorem of Calculus applied to each component, a curve with $\mathbf{r}'(t) = \langle \mathrm{e}^t, 2t, \sin t\rangle$ satisfies

$$\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \langle \mathrm{e}^s,\ 2s,\ \sin s\rangle \,\mathrm{d}s.$$

The three definite integrals are

$$\int_0^t \mathrm{e}^s \,\mathrm{d}s = \Big[\mathrm{e}^s\Big]_0^t = \mathrm{e}^t - 1,$$
$$\int_0^t 2s \,\mathrm{d}s = \Big[s^2\Big]_0^t = t^2,$$
$$\int_0^t \sin s \,\mathrm{d}s = \Big[-\cos s\Big]_0^t = -\cos t - (-\cos 0) = 1 - \cos t.$$

Choose the starting point $\mathbf{r}(0) = \langle 1, 0, -1\rangle$. Then

$$\mathbf{r}(t) = \langle 1, 0, -1\rangle + \langle \mathrm{e}^t - 1,\ t^2,\ 1 - \cos t\rangle = \langle \mathrm{e}^t,\ t^2,\ -\cos t\rangle.$$

Therefore one such curve is

$$\boxed{\mathbf{r}(t) = \langle \mathrm{e}^t, t^2, -\cos t\rangle}$$

(a different starting point adds a constant vector).

(iv) Eliminate the parameter: $x = t + 4$ gives $t = x - 4$, so the curve is the graph

$$y = (x-4)^3 - 3(x-4) + 7.$$

Differentiating with respect to x,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3(x-4)^2 - 3.$$

Horizontal tangent:

$$3(x-4)^2 - 3 = 0 \iff (x-4)^2 = 1 \iff x = 5 \text{ or } x = 3.$$

The corresponding heights are

$$y(5) = 1^3 - 3(1) + 7 = 5, \qquad y(3) = (-1)^3 - 3(-1) + 7 = -1 + 3 + 7 = 9.$$

Therefore the tangent is horizontal at

$$\boxed{(5,5) \text{ and } (3,9).}$$

(v) Use the definition of the directional derivative with the unit vector $\mathbf{u} = \langle a, b, c\rangle$, where

$$a = \frac{1}{\sqrt{14}}, \qquad b = \frac{2}{\sqrt{14}}, \qquad c = \frac{3}{\sqrt{14}},$$

namely

$$D_{\mathbf{u}} f(x,y,z) = \lim_{s\to 0} \frac{f(x + sa,\ y + sb,\ z + sc) - f(x,y,z)}{s}.$$

Expand the two terms of f along the line:

$$\begin{aligned}(x + sa)(y + sb)^2 &= xy^2 + s\,(ay^2 + 2bxy) + s^2\,(2aby + b^2x) + s^3\,ab^2,\\ (x + sa)(z + sc)^2 &= xz^2 + s\,(az^2 + 2cxz) + s^2\,(2acz + c^2x) + s^3\,ac^2.\end{aligned}$$

Subtracting and dividing by $s \neq 0$,

$$\frac{f(x + sa, y + sb, z + sc) - f(x,y,z)}{s} = \big(ay^2 + 2bxy - az^2 - 2cxz\big) + s\,(\cdots) + s^2\,(ab^2 - ac^2),$$

where $(\cdots) = 2aby + b^2x - 2acz - c^2x$. Letting $s \to 0$,

$$D_{\mathbf{u}} f = a(y^2 - z^2) + 2bxy - 2cxz = \frac{y^2 - z^2}{\sqrt{14}} + \frac{4xy}{\sqrt{14}} - \frac{6xz}{\sqrt{14}}.$$

Therefore

$$\boxed{D_{\mathbf{u}} f(x,y,z) = \frac{y^2 - z^2 + 4xy - 6xz}{\sqrt{14}}.}$$

Unofficial QRS marking scheme (30 marks)

(i) **(6 marks)** 3: correct displacement vector; 2: correct affine parametrisation; 1: restricts the parameter to $0 \leq t \leq 1$.

(ii) **(6 marks)** 2: finds the point at $t = \pi$; 2: finds the tangent vector; 2: gives a line through that point with that direction.

(iii) **(6 marks)** 4: integrates all three components correctly; 2: verifies the derivative or allows an arbitrary constant vector.

(iv) **(6 marks)** 2: differentiates x and y; 2: solves for both parameter values and checks $x' \neq 0$; 2: gives both points.

(v) **(6 marks)** 2: normalizes the direction vector; 2: computes the gradient; 2: takes the correct dot product.

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Question 2 (35 Marks)

QUESTION 2.

(i) Evaluate each of the following limits or show that the limit does not exist. Justify your answers.

(a)
$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y + 5}{\sin(xy) + 1}$$

(b)
$$\lim_{(x,y)\to(0,0)} \frac{7xy^3}{xy^3 - (x-y)^2}$$

(c)
$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$$

(ii) Define
$$f(x,y) = \begin{cases} y\cos\left(\dfrac{xy}{x^2+y^2}\right); & \text{if } x^2+y^2 \neq 0, \\ 0; & \text{if } x^2+y^2 = 0. \end{cases}$$

(a) Is f continuous at $(0,0)$? Justify your answer.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

(c) Is f differentiable at $(0,0)$? Justify your answer.

Solution

Method 1: Limit laws, two-path test, Squeeze Theorem, and the definition of differentiability

(i) (a) Let $N(x,y) = x^2 - y + 5$ and $D(x,y) = \sin(xy) + 1$. N is a polynomial, so it is continuous and
$$\lim_{(x,y)\to(0,0)} N(x,y) = N(0,0) = 0 - 0 + 5 = 5.$$

D is the continuous function sin composed with the polynomial xy, plus the constant 1, so it is continuous and
$$\lim_{(x,y)\to(0,0)} D(x,y) = D(0,0) = \sin 0 + 1 = 1.$$

The limit of the denominator is $1 \neq 0$, so the Quotient Law applies:
$$\lim_{(x,y)\to(0,0)} \frac{N(x,y)}{D(x,y)} = \frac{5}{1} = 5.$$

Therefore
$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2 - y + 5}{\sin(xy) + 1} = 5.}$$

(b) Let $q(x,y) = \dfrac{7xy^3}{xy^3-(x-y)^2}$. Two-Path Test: if q has different limits along two paths into $(0,0)$, then the limit does not exist.

Path 1, the y-axis $x=0$, $y\neq 0$: the numerator is 0 and the denominator is $0-(0-y)^2=-y^2\neq 0$, so

$$q(0,y)=\frac{0}{-y^2}=0 \longrightarrow 0 \qquad (y\to 0).$$

Path 2, the line $x=y$, $y\neq 0$: the numerator is $7y\cdot y^3=7y^4$ and the denominator is $y\cdot y^3-(y-y)^2=y^4$, so

$$q(y,y)=\frac{7y^4}{y^4}=7 \longrightarrow 7 \qquad (y\to 0).$$

Since $0\neq 7$,

$$\boxed{\text{the limit does not exist.}}$$

(c) Let $q(x,y)=\dfrac{xy^3}{x^2+y^6}$. Along every line $y=kx$, $x\neq 0$,

$$q(x,kx)=\frac{k^3x^4}{x^2+k^6x^6}=\frac{k^3x^2}{1+k^6x^4} \longrightarrow 0,$$

and along $x=0$, $q(0,y)=0/y^6=0$, so lines do not decide the question. Use the curve $x=y^3$, which passes through the origin since $(y^3,y)\to(0,0)$ as $y\to 0$. For $y\neq 0$,

$$q(y^3,y)=\frac{y^3\cdot y^3}{(y^3)^2+y^6}=\frac{y^6}{2y^6}=\frac{1}{2} \longrightarrow \frac{1}{2}.$$

The path $x=0$ gives 0 and the path $x=y^3$ gives $\frac{1}{2}$; since $0\neq\frac{1}{2}$,

$$\boxed{\text{the limit does not exist.}}$$

(ii) (a) For $(x,y)\neq(0,0)$, since $|\cos w|\leq 1$ for every real w,

$$|f(x,y)-f(0,0)|=|y|\left|\cos\left(\frac{xy}{x^2+y^2}\right)\right|\leq |y|\leq\sqrt{x^2+y^2}.$$

Thus

$$0\leq|f(x,y)-f(0,0)|\leq\sqrt{x^2+y^2} \longrightarrow 0 \qquad ((x,y)\to(0,0)),$$

and by the Squeeze Theorem $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0=f(0,0)$. Therefore

$$\boxed{f \text{ is continuous at } (0,0).}$$

(b) On the axes, for $h\neq 0$ and $k\neq 0$,

$$f(h,0)=0\cdot\cos\left(\frac{0}{h^2}\right)=0, \qquad f(0,k)=k\cos\left(\frac{0}{k^2}\right)=k\cos 0=k.$$

From the definitions of the partial derivatives,

$$f_x(0,0)=\lim_{h\to0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to0}\frac{0-0}{h}=0,$$
$$f_y(0,0)=\lim_{k\to0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to0}\frac{k-0}{k}=1.$$

Therefore

$$\boxed{f_x(0,0)=0, \qquad f_y(0,0)=1.}$$

(c) By definition, f is differentiable at $(0,0)$ if and only if

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\sqrt{h^2+k^2}} = 0.$$

By (b) the linear part is $L(h,k) = 0 \cdot h + 1 \cdot k = k$, so the quotient to examine is

$$E(h,k) = \frac{f(h,k) - k}{\sqrt{h^2+k^2}}.$$

Along the line $h = k = t$ with $t > 0$,

$$f(t,t) = t\cos\left(\frac{t\cdot t}{t^2+t^2}\right) = t\cos\frac{1}{2}, \qquad \sqrt{t^2+t^2} = \sqrt{2}\,t,$$

so

$$E(t,t) = \frac{t\cos\frac{1}{2} - t}{\sqrt{2}\,t} = \frac{\cos\frac{1}{2} - 1}{\sqrt{2}} \approx -0.0866.$$

Since $0 < \frac{1}{2} < 2\pi$, $\cos\frac{1}{2} \neq 1$, so $E(t,t)$ is a nonzero constant and $E(t,t) \not\to 0$ as $t \to 0^+$. Hence $E(h,k) \not\to 0$ and

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

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Method 2: Explicit error bounds, families of paths, axis restriction, and the directional-derivative test

(i) (a) Write the distance from the claimed limit as one fraction:

$$\frac{x^2-y+5}{\sin(xy)+1}-5=\frac{x^2-y+5-5\sin(xy)-5}{1+\sin(xy)}=\frac{x^2-y-5\sin(xy)}{1+\sin(xy)}.$$

Restrict to $x^2+y^2<1$. Then $|xy|\le\frac{1}{2}(x^2+y^2)<\frac{1}{2}$, and since $|\sin w|\le|w|$,

$$|\sin(xy)|\le|xy|<\frac{1}{2},\qquad\text{so}\qquad 1+\sin(xy)>1-\frac{1}{2}=\frac{1}{2}.$$

For the numerator, by the triangle inequality,

$$|x^2-y-5\sin(xy)|\le x^2+|y|+5|xy|.$$

Hence

$$0\le\left|\frac{x^2-y+5}{\sin(xy)+1}-5\right|\le 2\big(x^2+|y|+5|xy|\big)\ \longrightarrow\ 0,$$

because $x^2\to0$, $|y|\to0$ and $|xy|\to0$. By the Squeeze Theorem,

$$\boxed{\lim_{(x,y)\to(0,0)}\frac{x^2-y+5}{\sin(xy)+1}=5.}$$

(b) Let $q(x,y)=\dfrac{7xy^3}{xy^3-(x-y)^2}$ and use the family of parabolas $x=y+my^2$ (m constant), each passing through the origin. On such a curve,

$$x-y=my^2,\qquad (x-y)^2=m^2y^4,\qquad xy^3=(y+my^2)y^3=y^4(1+my).$$

Hence

$$xy^3-(x-y)^2=y^4(1+my)-m^2y^4=y^4(1-m^2+my),$$

and for $m\neq\pm1$ and $0<|y|$ small (so that $1-m^2+my\neq0$),

$$q(y+my^2,y)=\frac{7y^4(1+my)}{y^4(1-m^2+my)}=\frac{7(1+my)}{1-m^2+my}\ \longrightarrow\ \frac{7}{1-m^2}\qquad(y\to0).$$

For $m=0$ (the line $x=y$) the limit is 7; for $m=2$ it is $\dfrac{7}{1-4}=-\dfrac{7}{3}$. Two paths give different limits, so

$$\boxed{\text{the limit does not exist.}}$$

(c) Let $q(x,y)=\dfrac{xy^3}{x^2+y^6}$ and use the family of curves $x=my^3$ (m constant), each passing through the origin. For $y\neq0$,

$$q(my^3,y)=\frac{my^3\cdot y^3}{m^2y^6+y^6}=\frac{m\,y^6}{(m^2+1)y^6}=\frac{m}{m^2+1}.$$

The value is constant along each curve but depends on m: $m=0$ gives 0 and $m=1$ gives $\frac{1}{2}$. Hence

$$\boxed{\text{the limit does not exist.}}$$

(ii) (a) Use the ε–δ definition of continuity. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. If $0 < \sqrt{x^2+y^2} < \delta$, then, since $|\cos w| \le 1$,

$$|f(x,y) - f(0,0)| = |y| \left|\cos\left(\frac{xy}{x^2+y^2}\right)\right| \le |y| \le \sqrt{x^2+y^2} < \delta = \varepsilon.$$

At $(x,y) = (0,0)$ itself, $|f(0,0) - f(0,0)| = 0 < \varepsilon$. Hence

$$\boxed{f \text{ is continuous at } (0,0).}$$

(b) A partial derivative at $(0,0)$ is the ordinary derivative of the restriction of f to the corresponding axis. Let

$$g(x) = f(x,0), \qquad \varphi(y) = f(0,y).$$

For $x \neq 0$, $g(x) = 0 \cdot \cos 0 = 0$, and $g(0) = f(0,0) = 0$; so $g(x) = 0$ for every x and $g'(0) = 0$. For $y \neq 0$, $\varphi(y) = y\cos 0 = y$, and $\varphi(0) = 0$; so $\varphi(y) = y$ for every y and $\varphi'(0) = 1$. Hence

$$f_x(0,0) = g'(0) = 0, \qquad f_y(0,0) = \varphi'(0) = 1,$$

and

$$\boxed{f_x(0,0) = 0, \qquad f_y(0,0) = 1.}$$

(c) Theorem: if f is differentiable at $(0,0)$, then for every unit vector $\mathbf{u} = \langle a, b\rangle$ the directional derivative exists and

$$D_{\mathbf{u}} f(0,0) = \operatorname{grad} f(0,0) \cdot \mathbf{u} = 0 \cdot a + 1 \cdot b = b.$$

Compute $D_{\mathbf{u}} f(0,0)$ directly. For $t \neq 0$, using $a^2 + b^2 = 1$,

$$f(ta, tb) = tb \cos\left(\frac{t^2 ab}{t^2(a^2+b^2)}\right) = tb\cos(ab),$$

so

$$D_{\mathbf{u}} f(0,0) = \lim_{t\to 0} \frac{f(ta,tb) - f(0,0)}{t} = \lim_{t\to 0} \frac{tb\cos(ab)}{t} = b\cos(ab).$$

Take $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so $ab = \frac{1}{2}$:

$$D_{\mathbf{u}} f(0,0) = \frac{\cos\frac{1}{2}}{\sqrt{2}}, \qquad \operatorname{grad} f(0,0) \cdot \mathbf{u} = \frac{1}{\sqrt{2}}.$$

These differ because $\cos\frac{1}{2} \neq 1$. By the contrapositive of the theorem,

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

Method 3: Polar coordinates

Put $x = r\cos\theta$, $y = r\sin\theta$ with $r > 0$; then $(x, y) \to (0, 0)$ if and only if $r \to 0^+$. A limit L exists when $|F(r\cos\theta, r\sin\theta) - L|$ is bounded by a function of r alone that tends to 0; the limit does not exist when two fixed rays $\theta = \theta_1$, $\theta = \theta_2$ give different limits.

(i)(a) Here $xy = r^2\cos\theta\sin\theta$ and $|r^2\cos\theta\sin\theta| \le r^2$. As in Method 2,

$$\frac{x^2 - y + 5}{\sin(xy) + 1} - 5 = \frac{r^2\cos^2\theta - r\sin\theta - 5\sin(r^2\cos\theta\sin\theta)}{1 + \sin(r^2\cos\theta\sin\theta)}.$$

For $r < \frac{1}{\sqrt{2}}$, $|\sin(r^2\cos\theta\sin\theta)| \le r^2 < \frac{1}{2}$, so the denominator exceeds $\frac{1}{2}$, and the numerator is at most $r^2 + r + 5r^2$ in absolute value. Hence, for every θ,

$$\left|\frac{x^2 - y + 5}{\sin(xy) + 1} - 5\right| \le 2\big(r + 6r^2\big) \longrightarrow 0 \qquad (r \to 0^+),$$

a bound free of θ. Therefore

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{x^2 - y + 5}{\sin(xy) + 1} = 5.}$$

(i)(b) Here $xy^3 = r^4\cos\theta\sin^3\theta$ and $(x - y)^2 = r^2(\cos\theta - \sin\theta)^2$, so

$$\frac{7xy^3}{xy^3 - (x-y)^2} = \frac{7r^4\cos\theta\sin^3\theta}{r^4\cos\theta\sin^3\theta - r^2(\cos\theta - \sin\theta)^2} = \frac{7r^2\cos\theta\sin^3\theta}{r^2\cos\theta\sin^3\theta - (\cos\theta - \sin\theta)^2}.$$

On the ray $\theta = \frac{\pi}{4}$: $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$, so $\cos\theta - \sin\theta = 0$ and $\cos\theta\sin^3\theta = \frac{1}{4} \neq 0$; the quotient equals $\dfrac{7r^2/4}{r^2/4} = 7$ for every $r > 0$, with limit 7. On the ray $\theta = \frac{\pi}{2}$: $\cos\theta = 0$, so the quotient equals $\dfrac{0}{0 - 1} = 0$, with limit 0. Since $7 \neq 0$,

$$\boxed{\text{the limit does not exist.}}$$

(i)(c) Directly in polar coordinates,

$$\frac{xy^3}{x^2 + y^6} = \frac{r^4\cos\theta\sin^3\theta}{r^2\cos^2\theta + r^6\sin^6\theta} = \frac{r^2\cos\theta\sin^3\theta}{\cos^2\theta + r^4\sin^6\theta},$$

which tends to 0 on every fixed ray (for $\cos\theta \neq 0$ the denominator tends to $\cos^2\theta \neq 0$; for $\cos\theta = 0$ the quotient is 0). Rays are inconclusive here, so first substitute $w = y^3$. The map $y \mapsto y^3$ is a continuous bijection of $\mathbb{R}$ with continuous inverse $w \mapsto w^{1/3}$, so $(x, y) \to (0, 0)$ if and only if $(x, w) \to (0, 0)$, and

$$\frac{xy^3}{x^2 + y^6} = \frac{xw}{x^2 + w^2}.$$

Now use polar coordinates in the (x, w)-plane, $x = \rho\cos\phi$, $w = \rho\sin\phi$:

$$\frac{xw}{x^2 + w^2} = \frac{\rho^2\cos\phi\sin\phi}{\rho^2} = \cos\phi\sin\phi = \frac{1}{2}\sin 2\phi.$$

The ray $\phi = 0$ (the curve $y = 0$) gives 0; the ray $\phi = \frac{\pi}{4}$ (the curve $x = y^3$, $y > 0$) gives $\frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}$. Since $0 \neq \frac{1}{2}$,

$$\boxed{\text{the limit does not exist.}}$$

(ii)(a) For $r > 0$,

$$\frac{xy}{x^2+y^2} = \frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta, \qquad f(r\cos\theta, r\sin\theta) = r\sin\theta\,\cos(\cos\theta\sin\theta).$$

Hence, for every θ,

$$|f(r\cos\theta, r\sin\theta) - f(0,0)| = r\,|\sin\theta|\,|\cos(\cos\theta\sin\theta)| \le r \;\longrightarrow\; 0 \qquad (r \to 0^+),$$

a bound free of θ. Therefore

$$\boxed{f \text{ is continuous at } (0,0).}$$

(ii)(c) With $f_x(0,0) = 0$ and $f_y(0,0) = 1$ from Method 1 (b), write the differentiability quotient with $h = r\cos\theta$, $k = r\sin\theta$:

$$E = \frac{f(h,k) - k}{\sqrt{h^2+k^2}} = \frac{r\sin\theta\cos(\cos\theta\sin\theta) - r\sin\theta}{r} = \sin\theta\big(\cos(\cos\theta\sin\theta) - 1\big).$$

E does not depend on r, so on the ray θ its limit is $\sin\theta\big(\cos(\cos\theta\sin\theta) - 1\big)$. On the ray $\theta = \frac{\pi}{4}$,

$$E = \frac{1}{\sqrt{2}}\Big(\cos\frac{1}{2} - 1\Big) \neq 0,$$

so $E \not\to 0$ as $(h,k) \to (0,0)$. Therefore

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

Unofficial QRS marking scheme (35 marks)

(i) **(15 marks)**

(a) **(5 marks)** 2: observes a nonzero limiting denominator; 2: evaluates the numerator and denominator; 1: gives 5.

(b) **(5 marks)** 2: computes the $x = 0$ restricted limit; 2: computes the $x = y$ restricted limit; 1: concludes nonexistence.

(c) **(5 marks)** 2: computes the $x = 0$ restricted limit; 2: computes the $x = y^3$ restricted limit; 1: concludes nonexistence.

(ii) **(20 marks)**

(a) **(5 marks)** 3: establishes a valid bound tending to zero; 2: concludes continuity.

(b) **(5 marks)** 2: correct difference quotient for f_x; 2: correct difference quotient for f_y; 1: states both values.

(c) **(10 marks)** 2: identifies the only possible linear map $L(h,k) = k$; 4: evaluates a valid path; 2: normalizes the remainder; 2: concludes non-differentiability.

Question 3 (35 Marks)

QUESTION 3.

(i) Find an equation for the tangent plane to the surface $z = 5x^2y - 7xy^2$ at the point $(3, 2, 6)$.

(ii) Find an equation for the tangent plane to the surface given by the equation $x^2y - yz + xz^2 = 17$ at the point $(1, 2, 5)$.

(iii) Use the Chain Rule to find $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ for

$$f(x, y, z) = 2xy^2z^4, \qquad x = 2uv, \qquad y = u^2, \qquad z = \sin v.$$

(iv) Find $\dfrac{\partial z}{\partial y}$ where z is defined implicitly by

$$xe^{4y} + 4y^2z^3 + ze^{8x} = 0.$$

(v) Let

$$f(x, y) = x^3 - 3x^2y + 3xy^2 - 3x - 9y.$$

Find and classify all critical points of f. (For each critical point, determine whether or not it is a local max/min or a saddle point.)

Solution

Method 1: Graph tangent-plane formula, gradient normal, Chain Rule, Implicit Function Theorem, and the Second Derivative Test

(i) Let $g(x, y) = 5x^2y - 7xy^2$. The point lies on the surface:

$$g(3, 2) = 5 \cdot 9 \cdot 2 - 7 \cdot 3 \cdot 4 = 90 - 84 = 6.$$

The partial derivatives are

$$g_x = 10xy - 7y^2, \qquad g_y = 5x^2 - 14xy.$$

Both are polynomials, hence continuous, so g is differentiable and the tangent plane at $(3, 2, 6)$ exists. At $(3, 2)$,

$$g_x(3, 2) = 10 \cdot 3 \cdot 2 - 7 \cdot 4 = 60 - 28 = 32, \qquad g_y(3, 2) = 5 \cdot 9 - 14 \cdot 3 \cdot 2 = 45 - 84 = -39.$$

The tangent plane to $z = g(x, y)$ is $z - z_0 = g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0)$:

$$\begin{aligned} z - 6 &= 32(x - 3) - 39(y - 2) \\ z - 6 &= 32x - 96 - 39y + 78 \\ z &= 32x - 39y - 12. \end{aligned}$$

Therefore

$$\boxed{32x - 39y - z = 12.}$$

(ii) Let $F(x,y,z) = x^2y - yz + xz^2 - 17$, so the surface is the level set $F = 0$. The point lies on it:

$$F(1,2,5) = 1\cdot 2 - 2\cdot 5 + 1\cdot 25 - 17 = 2 - 10 + 25 - 17 = 0.$$

The partial derivatives are

$$F_x = 2xy + z^2, \qquad F_y = x^2 - z, \qquad F_z = -y + 2xz.$$

At $(1,2,5)$,

$$F_x = 4 + 25 = 29, \qquad F_y = 1 - 5 = -4, \qquad F_z = -2 + 10 = 8.$$

So $\operatorname{grad} F(1,2,5) = \langle 29, -4, 8\rangle \neq \mathbf{0}$, which is a normal vector to the tangent plane:

$$\begin{aligned} 29(x-1) - 4(y-2) + 8(z-5) &= 0 \\ 29x - 29 - 4y + 8 + 8z - 40 &= 0 \\ 29x - 4y + 8z &= 61. \end{aligned}$$

Therefore

$$\boxed{29x - 4y + 8z = 61.}$$

(iii) $f(x,y,z) = 2xy^2z^4$ is a polynomial, hence differentiable, and $x = 2uv$, $y = u^2$, $z = \sin v$ have continuous partial derivatives, so the Chain Rule applies:

$$\frac{\partial f}{\partial u} = f_x\, x_u + f_y\, y_u + f_z\, z_u, \qquad \frac{\partial f}{\partial v} = f_x\, x_v + f_y\, y_v + f_z\, z_v.$$

The partial derivatives of f are

$$f_x = 2y^2z^4, \qquad f_y = 4xyz^4, \qquad f_z = 8xy^2z^3,$$

and those of the substitutions are

$$x_u = 2v, \quad y_u = 2u, \quad z_u = 0; \qquad x_v = 2u, \quad y_v = 0, \quad z_v = \cos v.$$

For u:

$$\begin{aligned} \frac{\partial f}{\partial u} &= 2y^2z^4\cdot 2v + 4xyz^4\cdot 2u + 8xy^2z^3\cdot 0 = 4vy^2z^4 + 8uxyz^4 \\ &= 4v\,u^4\sin^4 v + 8u\,(2uv)\,u^2\sin^4 v = 4u^4v\sin^4 v + 16u^4v\sin^4 v = 20u^4v\sin^4 v. \end{aligned}$$

For v:

$$\begin{aligned} \frac{\partial f}{\partial v} &= 2y^2z^4\cdot 2u + 4xyz^4\cdot 0 + 8xy^2z^3\cos v = 4uy^2z^4 + 8xy^2z^3\cos v \\ &= 4u\,u^4\sin^4 v + 8(2uv)\,u^4\sin^3 v\cos v = 4u^5\sin^4 v + 16u^5v\sin^3 v\cos v. \end{aligned}$$

Therefore

$$\boxed{\frac{\partial f}{\partial u} = 20u^4v\sin^4 v,}$$

$$\boxed{\frac{\partial f}{\partial v} = 4u^5\sin^4 v + 16u^5v\sin^3 v\cos v.}$$

(iv) Let $F(x,y,z) = xe^{4y} + 4y^2z^3 + ze^{8x}$, so the surface is $F = 0$. F has continuous partial derivatives, and

$$F_z = 12y^2z^2 + e^{8x} \geq e^{8x} > 0$$

at every point. By the Implicit Function Theorem, near every point of the surface z is a differentiable function of (x,y) and

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

Differentiating F in y with x, z held fixed,

$$F_y = \frac{\partial}{\partial y}\big(xe^{4y}\big) + \frac{\partial}{\partial y}\big(4y^2z^3\big) + \frac{\partial}{\partial y}\big(ze^{8x}\big) = 4xe^{4y} + 8yz^3 + 0.$$

Therefore

$$\boxed{\frac{\partial z}{\partial y} = -\frac{4xe^{4y} + 8yz^3}{12y^2z^2 + e^{8x}}.}$$

(v) Let $f(x,y) = x^3 - 3x^2y + 3xy^2 - 3x - 9y$. The first partial derivatives are

$$f_x = 3x^2 - 6xy + 3y^2 - 3 = 3\big[(x-y)^2 - 1\big],$$
$$f_y = -3x^2 + 6xy - 9 = -3\big[x^2 - 2xy + 3\big] = -3\big[(x-y)^2 - y^2 + 3\big].$$

Critical points solve $f_x = 0$ and $f_y = 0$. From $f_x = 0$,

$$(x-y)^2 = 1 \iff x - y = 1 \text{ or } x - y = -1.$$

Substituting $(x-y)^2 = 1$ into $f_y = 0$,

$$1 - y^2 + 3 = 0 \iff y^2 = 4 \iff y = 2 \text{ or } y = -2.$$

The four cases, with $x = (x-y) + y$:

$$x - y = 1,\ y = 2: \quad x = 3, \qquad x - y = -1,\ y = 2: \quad x = 1,$$
$$x - y = 1,\ y = -2: \quad x = -1, \qquad x - y = -1,\ y = -2: \quad x = -3.$$

So the critical points are $(3,2)$, $(1,2)$, $(-1,-2)$, $(-3,-2)$.

The second partial derivatives are

$$f_{xx} = 6x - 6y, \qquad f_{xy} = -6x + 6y, \qquad f_{yy} = 6x.$$

They are polynomials, hence continuous, so the Second Derivative Test applies with

$$D = f_{xx}f_{yy} - f_{xy}^2 = 6(x-y)\cdot 6x - 36(x-y)^2 = 36(x-y)\big[x - (x-y)\big] = 36y(x-y).$$

At each critical point:

$$(-3,-2):\ D = 36(-2)(-1) = 72 > 0,\ \ f_{xx} = 6(-3+2) = -6 < 0 \quad \text{local maximum;}$$
$$(3,2):\ D = 36(2)(1) = 72 > 0,\ \ f_{xx} = 6(3-2) = 6 > 0 \quad \text{local minimum;}$$
$$(-1,-2):\ D = 36(-2)(1) = -72 < 0 \quad \text{saddle point;}$$
$$(1,2):\ D = 36(2)(-1) = -72 < 0 \quad \text{saddle point.}$$

Summary:

(x,y)	$(-3,-2)$	$(3,2)$	$(-1,-2)$	$(1,2)$
D	72	72	-72	-72
f_{xx}	-6	6	6	-6

Therefore

$$\boxed{\begin{array}{c}(-3,-2):\ \text{strict local maximum}, \quad (3,2):\ \text{strict local minimum},\\ (-1,-2) \text{ and } (1,2):\ \text{saddle points.}\end{array}}$$

Method 2: Level-surface normal, implicit differentiation, direct substitution, and change of variables

(i) View the graph as the level surface $G(x,y,z) = 5x^2y - 7xy^2 - z = 0$; the point satisfies $G(3,2,6) = 90 - 84 - 6 = 0$. The partial derivatives are

$$G_x = 10xy - 7y^2, \qquad G_y = 5x^2 - 14xy, \qquad G_z = -1,$$

so at $(3,2,6)$,

$$\operatorname{grad} G(3,2,6) = \langle 60-28,\ 45-84,\ -1\rangle = \langle 32,-39,-1\rangle \neq \mathbf{0}.$$

The tangent plane has this normal:

$$\begin{aligned} 32(x-3) - 39(y-2) - (z-6) &= 0 \\ 32x - 96 - 39y + 78 - z + 6 &= 0 \\ 32x - 39y - z &= 12. \end{aligned}$$

Therefore

$$\boxed{32x - 39y - z = 12.}$$

(ii) Since $F_z(1,2,5) = -2 + 2\cdot 1\cdot 5 = 8 \neq 0$ for $F = x^2y - yz + xz^2 - 17$, the Implicit Function Theorem lets us write $z = z(x,y)$ near $(1,2)$ with $z(1,2) = 5$. Differentiate $x^2y - yz + xz^2 = 17$ implicitly. In x (with y fixed):

$$2xy - y\,z_x + z^2 + 2xz\,z_x = 0 \iff (2xz - y)\,z_x = -(2xy + z^2).$$

At $(1,2,5)$: $8z_x = -(4+25) = -29$, so $z_x = -\dfrac{29}{8}$. In y (with x fixed):

$$x^2 - z - y\,z_y + 2xz\,z_y = 0 \iff (2xz - y)\,z_y = z - x^2.$$

At $(1,2,5)$: $8z_y = 5 - 1 = 4$, so $z_y = \dfrac{1}{2}$. The tangent plane to the graph $z = z(x,y)$ is

$$\begin{aligned} z - 5 &= -\frac{29}{8}(x-1) + \frac{1}{2}(y-2) \\ 8z - 40 &= -29x + 29 + 4y - 8 \\ 29x - 4y + 8z &= 61. \end{aligned}$$

Therefore

$$\boxed{29x - 4y + 8z = 61.}$$

(iii) Note: the question asks for the Chain Rule, so this route should be used only as a check of Method 1. Substitute $x = 2uv$, $y = u^2$, $z = \sin v$ into f first:

$$f = 2(2uv)(u^2)^2(\sin v)^4 = 4u^5 v \sin^4 v.$$

In u (with v fixed):

$$\frac{\partial f}{\partial u} = 4v\sin^4 v \cdot 5u^4 = 20u^4 v \sin^4 v.$$

In v (with u fixed), by the Product Rule on $v\cdot\sin^4 v$:

$$\frac{\partial f}{\partial v} = 4u^5\big(1\cdot\sin^4 v + v\cdot 4\sin^3 v\cos v\big) = 4u^5\sin^4 v + 16u^5 v\sin^3 v\cos v.$$

Therefore

$$\boxed{\frac{\partial f}{\partial u} = 20u^4 v \sin^4 v,}$$

$$\boxed{\frac{\partial f}{\partial v} = 4u^5 \sin^4 v + 16u^5 v \sin^3 v \cos v.}$$

(iv) By Method 1, $F_z > 0$ everywhere, so $z = z(x, y)$ is differentiable near every point of the surface. Differentiate $xe^{4y} + 4y^2z^3 + ze^{8x} = 0$ with respect to y, holding x fixed and treating z as a function of y:

$$\frac{\partial}{\partial y}\big(xe^{4y}\big) = 4xe^{4y}, \qquad \frac{\partial}{\partial y}\big(4y^2z^3\big) = 8yz^3 + 12y^2z^2\, z_y, \qquad \frac{\partial}{\partial y}\big(ze^{8x}\big) = e^{8x}\, z_y.$$

Adding,

$$4xe^{4y} + 8yz^3 + \big(12y^2z^2 + e^{8x}\big)z_y = 0.$$

Since $12y^2z^2 + e^{8x} > 0$, divide by it:

$$\boxed{\frac{\partial z}{\partial y} = -\frac{4xe^{4y} + 8yz^3}{12y^2z^2 + e^{8x}}.}$$

(v) Let $d = x - y$. Since

$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3,$$

we have $x^3 - 3x^2y + 3xy^2 = d^3 + y^3$, and with $x = d + y$,

$$f = d^3 + y^3 - 3(d + y) - 9y = (d^3 - 3d) + (y^3 - 12y) = g(d) + h(y),$$

where $g(d) = d^3 - 3d$ and $h(y) = y^3 - 12y$. The map $(d, y) \mapsto (x, y) = (d + y, y)$ is linear and invertible with continuous inverse $(x, y) \mapsto (x - y, y)$, so it carries neighbourhoods to neighbourhoods; hence local maxima, local minima and saddle points of $F(d, y) = g(d) + h(y)$ correspond exactly to those of f.

One-variable analysis:

$$\begin{aligned} &g'(d) = 3d^2 - 3 = 0 \iff d = \pm 1, \quad g''(d) = 6d: \quad g''(-1) = -6 < 0,\ g''(1) = 6 > 0; \\ &h'(y) = 3y^2 - 12 = 0 \iff y = \pm 2, \quad h''(y) = 6y: \quad h''(-2) = -12 < 0,\ h''(2) = 12 > 0. \end{aligned}$$

So g has a strict local maximum at $d = -1$ and a strict local minimum at $d = 1$; h has a strict local maximum at $y = -2$ and a strict local minimum at $y = 2$. Since $\operatorname{grad} F = \langle g'(d), h'(y)\rangle$, the critical points of F are exactly the four pairs (d, y) with $d = \pm 1$, $y = \pm 2$.

Equal types give extrema. At $(d, y) = (-1, -2)$: for (d, y) near $(-1, -2)$, $g(d) \le g(-1)$ and $h(y) \le h(-2)$, so $F(d, y) \le F(-1, -2)$, with equality only at $(-1, -2)$; a strict local maximum. At $(d, y) = (1, 2)$: $g(d) \ge g(1)$ and $h(y) \ge h(2)$ nearby, so a strict local minimum.

Opposite types give saddles. At $(d, y) = (1, -2)$: along $y = -2$, $F(d, -2) = g(d) + h(-2)$ has a strict local minimum at $d = 1$, so arbitrarily close points have $F > F(1, -2)$; along $d = 1$, $F(1, y) = g(1) + h(y)$ has a strict local maximum at $y = -2$, so arbitrarily close points have $F < F(1, -2)$. Hence a saddle point. The same argument with the roles exchanged shows $(d, y) = (-1, 2)$ is a saddle point.

Map back with $x = d + y$:

$$\begin{aligned}(d, y) = (-1, -2) &\mapsto (x, y) = (-3, -2) \quad \text{local maximum},\\ (d, y) = (1, 2) &\mapsto (x, y) = (3, 2) \quad \text{local minimum},\\ (d, y) = (1, -2) &\mapsto (x, y) = (-1, -2) \quad \text{saddle point},\\ (d, y) = (-1, 2) &\mapsto (x, y) = (1, 2) \quad \text{saddle point}.\end{aligned}$$

Therefore

$$\boxed{\begin{array}{c}(-3, -2) : \text{ strict local maximum}, \quad (3, 2) : \text{ strict local minimum},\\ (-1, -2) \text{ and } (1, 2) : \text{ saddle points}.\end{array}}$$

Unofficial QRS marking scheme (35 marks)

(i) **(7 marks)** 3: correct partial derivatives; 2: evaluates both slopes at $(3, 2)$; 2: gives the correct plane through $(3, 2, 6)$.

(ii) **(7 marks)** 3: defines F and computes its gradient; 2: evaluates the normal at $(1, 2, 5)$; 2: gives the correct plane.

(iii) **(7 marks)** 3: writes the correct Chain Rule; 2: differentiates the substitutions; 1: correct f_u; 1: correct f_v.

(iv) **(6 marks)** 3: differentiates the implicit equation correctly; 2: collects the z_y terms; 1: solves for z_y with its nonzero-denominator condition.

(v) **(8 marks)** Either method earns full credit:

- Method 1: 2 for f_x, f_y; 2 for all four critical points; 2 for the Hessian discriminants; 2 for the maximum, minimum, and two saddles.
- Method 2: 2 for the invertible change $d = x - y$ and separation; 2 for the one-variable maxima and minima; 2 for mapping all four points; 2 for the resulting classifications.