MH2100 Calculus III

Midterm II (Solutions)

AY 2019/2020, Semester 1

Quantitative Research Society @NTU

November 2019

Overview

- **Historical paper:** the original paper was held in November 2019 with **90 minutes** allowed.
- **Historical permitted material:** the November 2019 paper permitted a two-sided A4 page of handwritten notes. Consult current course materials for present assessment rules.
- **2026 scope note:** the current midterm scope ends at double integrals, including polar coordinates. Triple integrals, cylindrical and spherical coordinates, general substitutions, and line integrals are beyond that scope. They are retained and solved because they appear in this historical paper.

Instructions to candidates (historical)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. One two-sided A4 piece of paper with handwritten notes is permitted.

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Question 1 (32 Marks)

(beyond 2026 midterm scope): Part (v) uses cylindrical coordinates.

(i) Use Lagrange Multipliers to find the maximum value of the function

$$f(x, y, z) = 14x + 8y + 12z$$

subject to the constraint $x^2 + y^2 + z^2 = 101$.

(ii) Evaluate the iterated integral.

$$\int_{-1}^{1} \int_{0}^{1} y\mathrm{e}^{xy} \, \mathrm{d}x \, \mathrm{d}y$$

(iii) Find the volume of the solid bounded by the surface $z = 5 + (x-4)^2 + 2y$ and the planes $x = 0$, $y = 0$, $z = 0$, $x = 3$, $y = 4$.

(iv) Evaluate the iterated integral by converting to polar coordinates.

$$\int_{0}^{\sqrt{\pi}} \int_{0}^{\sqrt{\pi - y^2}} \sin(x^2 + y^2) \, \mathrm{d}x \, \mathrm{d}y$$

(v) *(beyond 2026 midterm scope)* Use cylindrical coordinates to evaluate

$$\iiint_E y \, \mathrm{d}V,$$

where E is the solid that lies between the cylinders $x^2 + y^2 = 3$ and $x^2 + y^2 = 7$, above the xy-plane and below the plane $z = x + 4$.

Solution

Part (i), Method 1: Lagrange multipliers

Let

$$f(x, y, z) = 14x + 8y + 12z, \qquad g(x, y, z) = x^2 + y^2 + z^2,$$

and let $S = \{(x, y, z) : g(x, y, z) = 101\}$.

Existence of the maximum (EVT). S is a sphere of radius $\sqrt{101}$, so S is closed and bounded. The function f is a polynomial, hence continuous. By the Extreme Value Theorem, f attains a maximum value and a minimum value on S.

Lagrange condition. The partial derivatives are

$$f_x = 14, \qquad f_y = 8, \qquad f_z = 12,$$

$$g_x = 2x, \qquad g_y = 2y, \qquad g_z = 2z.$$

Hence

$$\nabla f = (14, 8, 12), \qquad \nabla g = (2x, 2y, 2z).$$

On S we have $x^2+y^2+z^2=101\neq 0$, so $(x,y,z)\neq(0,0,0)$ and therefore $\nabla g\neq \mathbf{0}$ at every point of S. Both f and g have continuous first partial derivatives. By the Lagrange Multiplier Theorem, every extreme point of f on S satisfies $\nabla f=\lambda\nabla g$ for some $\lambda\in\mathbb{R}$, that is

$$\begin{aligned} 14&=2\lambda x,\\ 8&=2\lambda y,\\ 12&=2\lambda z,\\ x^2+y^2+z^2&=101. \end{aligned}$$

Solving the system. If $\lambda=0$, the first equation reads $14=0$, which is impossible. Hence $\lambda\neq 0$, and the first three equations give

$$x=\frac{7}{\lambda},\qquad y=\frac{4}{\lambda},\qquad z=\frac{6}{\lambda}.$$

Substitute these into the constraint:

$$\frac{49}{\lambda^2}+\frac{16}{\lambda^2}+\frac{36}{\lambda^2}=101 \implies \frac{101}{\lambda^2}=101 \implies \lambda^2=1.$$

So $\lambda=1$ or $\lambda=-1$.

Branch $\lambda=1$. Then $(x,y,z)=(7,4,6)$, and

$$f(7,4,6)=14(7)+8(4)+12(6)=98+32+72=202.$$

Branch $\lambda=-1$. Then $(x,y,z)=(-7,-4,-6)$, and

$$f(-7,-4,-6)=14(-7)+8(-4)+12(-6)=-98-32-72=-202.$$

Comparison. The maximum exists by the EVT and occurs at one of the two candidates. Comparing $202>-202$,

$$\boxed{\max f=202,\quad \text{attained at } (7,4,6).}$$

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Part (i), Method 2: Cauchy–Schwarz inequality

This method should be used only if the course allows this interpretation (the question asks for Lagrange multipliers).

Write f as a dot product:

$$f(x,y,z) = (14,8,12)\cdot(x,y,z).$$

The lengths of the two vectors are

$$\|(14,8,12)\| = \sqrt{196+64+144} = \sqrt{404} = 2\sqrt{101}, \qquad \|(x,y,z)\| = \sqrt{x^2+y^2+z^2} = \sqrt{101}.$$

The Cauchy–Schwarz inequality $\mathbf{a}\cdot\mathbf{b} \le \|\mathbf{a}\|\,\|\mathbf{b}\|$ gives, for every point on the sphere,

$$f(x,y,z) \le 2\sqrt{101}\cdot\sqrt{101} = 202.$$

Equality in Cauchy–Schwarz holds exactly when $(x,y,z) = c(14,8,12)$ with $c \ge 0$. Substitute this into the constraint:

$$c^2(14^2+8^2+12^2) = 101 \implies 404c^2 = 101 \implies c^2 = \frac{1}{4} \implies c = \frac{1}{2} \quad (c \ge 0).$$

Hence the equality point is

$$(x,y,z) = \tfrac{1}{2}(14,8,12) = (7,4,6), \qquad f(7,4,6) = 98+32+72 = 202.$$

Therefore the bound 202 is attained, and

$$\boxed{\max f = 202, \quad \text{attained at } (7,4,6).}$$

Part (i), Method 3: Completing the square

This method should be used only if the course allows this interpretation (the question asks for Lagrange multipliers).

Expand the squared distance from (x,y,z) to $(7,4,6)$:

$$\begin{aligned}(x-7)^2+(y-4)^2+(z-6)^2 &= x^2-14x+49+y^2-8y+16+z^2-12z+36\\ &= (x^2+y^2+z^2)-(14x+8y+12z)+101.\end{aligned}$$

On the constraint $x^2+y^2+z^2 = 101$ this becomes

$$(x-7)^2+(y-4)^2+(z-6)^2 = 101 - f(x,y,z) + 101 = 202 - f(x,y,z).$$

Hence, on the sphere,

$$f(x,y,z) = 202 - \left[(x-7)^2+(y-4)^2+(z-6)^2\right] \le 202,$$

with equality if and only if $(x,y,z) = (7,4,6)$. This point lies on the sphere, since

$$7^2+4^2+6^2 = 49+16+36 = 101.$$

Therefore

$$\boxed{\max f = 202, \quad \text{attained at } (7,4,6).}$$

Part (ii), Method 1: Integrate in x first (given order)

The integrand ye^{xy} is continuous on the rectangle $[0,1]\times[-1,1]$, so the iterated integral is well defined and may be computed from the inside out.

Inner integral. With y fixed,

$$\frac{\partial}{\partial x}\,\mathrm{e}^{xy} = y\mathrm{e}^{xy},$$

so e^{xy} is an antiderivative of ye^{xy} with respect to x. Therefore

$$\int_0^1 y\mathrm{e}^{xy}\,\mathrm{d}x = \left[\mathrm{e}^{xy}\right]_{x=0}^{x=1} = \mathrm{e}^{y} - \mathrm{e}^{0} = \mathrm{e}^{y} - 1.$$

Outer integral.

$$\begin{aligned}\int_{-1}^{1} (\mathrm{e}^{y} - 1)\,\mathrm{d}y &= \left[\mathrm{e}^{y} - y\right]_{-1}^{1} \\ &= \left(\mathrm{e}^{1} - 1\right) - \left(\mathrm{e}^{-1} - (-1)\right) \\ &= \mathrm{e} - 1 - \mathrm{e}^{-1} - 1 \\ &= \mathrm{e} - \mathrm{e}^{-1} - 2.\end{aligned}$$

Hence

$$\boxed{\int_{-1}^{1}\int_{0}^{1} y\mathrm{e}^{xy}\,\mathrm{d}x\,\mathrm{d}y = \mathrm{e} - \mathrm{e}^{-1} - 2.}$$

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Part (ii), Method 2: Fubini in the other order

The integrand is continuous on the rectangle $R = [0,1] \times [-1,1]$, so by Fubini's Theorem

$$\int_{-1}^{1}\int_{0}^{1} y\mathrm{e}^{xy}\,\mathrm{d}x\,\mathrm{d}y = \int_{0}^{1}\int_{-1}^{1} y\mathrm{e}^{xy}\,\mathrm{d}y\,\mathrm{d}x.$$

Inner integral at $x = 0$.

$$\int_{-1}^{1} y\,\mathrm{d}y = \left[\frac{y^2}{2}\right]_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = 0.$$

Inner integral for $0 < x \le 1$. Integrate by parts with $u = y,\ \mathrm{d}v = \mathrm{e}^{xy}\,\mathrm{d}y$, so $\mathrm{d}u = \mathrm{d}y$ and $v = \mathrm{e}^{xy}/x$:

$$\begin{aligned}\int_{-1}^{1} y\mathrm{e}^{xy}\,\mathrm{d}y &= \left[\frac{y\mathrm{e}^{xy}}{x}\right]_{y=-1}^{y=1} - \int_{-1}^{1}\frac{\mathrm{e}^{xy}}{x}\,\mathrm{d}y \\ &= \frac{\mathrm{e}^{x}}{x} - \frac{-\mathrm{e}^{-x}}{x} - \left[\frac{\mathrm{e}^{xy}}{x^2}\right]_{y=-1}^{y=1} \\ &= \frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{x} - \frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{x^2}.\end{aligned}$$

Recognising a derivative. Let

$$F(x) = \frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{x} \qquad (x > 0).$$

By the quotient rule,

$$F'(x) = \frac{(\mathrm{e}^{x}+\mathrm{e}^{-x})\,x - (\mathrm{e}^{x}-\mathrm{e}^{-x})}{x^2} = \frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{x} - \frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{x^2},$$

which is exactly the inner integral for $0 < x \le 1$.

Outer integral. The inner integral is a continuous function of x on $[0,1]$ (integral of a continuous function over a fixed interval), so

$$\int_{0}^{1}\int_{-1}^{1} y\mathrm{e}^{xy}\,\mathrm{d}y\,\mathrm{d}x = \lim_{a\to 0^+}\int_{a}^{1} F'(x)\,\mathrm{d}x = F(1) - \lim_{a\to 0^+} F(a).$$

Here

$$F(1) = \mathrm{e} - \mathrm{e}^{-1},$$

and, by the definition of the derivative of $h(a) = \mathrm{e}^{a} - \mathrm{e}^{-a}$ at $a = 0$ (where $h(0) = 0$),

$$\lim_{a\to 0^+} F(a) = \lim_{a\to 0^+}\frac{h(a)-h(0)}{a} = h'(0) = \mathrm{e}^{0} + \mathrm{e}^{0} = 2.$$

Therefore

$$\boxed{\int_{-1}^{1}\int_{0}^{1} y\mathrm{e}^{xy}\,\mathrm{d}x\,\mathrm{d}y = \mathrm{e} - \mathrm{e}^{-1} - 2.}$$

Part (iii), Method 1: Integrate in y first

The solid lies over the rectangle

$$R = \{(x, y) : 0 \le x \le 3,\ 0 \le y \le 4\},$$

between the plane $z = 0$ and the surface $z = 5 + (x-4)^2 + 2y$. On R,

$$5 + (x-4)^2 + 2y \ge 5 > 0,$$

so the surface lies above the xy-plane and the volume is

$$V = \iint_R \left(5 + (x-4)^2 + 2y\right)\, \mathrm{d}A.$$

The integrand is continuous on R, so by Fubini's Theorem

$$V = \int_0^3 \int_0^4 \left(5 + (x-4)^2 + 2y\right)\, \mathrm{d}y\, \mathrm{d}x.$$

Inner integral. With x fixed, $5 + (x-4)^2$ is a constant, so

$$\begin{aligned}\int_0^4 \left(5 + (x-4)^2 + 2y\right)\, \mathrm{d}y &= \left[\left(5 + (x-4)^2\right) y + y^2\right]_{y=0}^{y=4} \\ &= 4\left(5 + (x-4)^2\right) + 16 \\ &= 36 + 4(x-4)^2.\end{aligned}$$

Outer integral.

$$\begin{aligned}\int_0^3 \left(36 + 4(x-4)^2\right)\, \mathrm{d}x &= \left[36x + \frac{4(x-4)^3}{3}\right]_0^3 \\ &= \left(108 + \frac{4(-1)^3}{3}\right) - \left(0 + \frac{4(-4)^3}{3}\right) \\ &= 108 - \frac{4}{3} + \frac{256}{3} \\ &= 108 + 84 = 192.\end{aligned}$$

Hence

$$\boxed{V = 192.}$$

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Part (iii), Method 2: Fubini in the other order

By Fubini's Theorem (continuous integrand on a rectangle) we may integrate in x first:

$$V = \int_0^4 \int_0^3 \left(5 + (x-4)^2 + 2y\right)\, \mathrm{d}x\, \mathrm{d}y.$$

Inner integral. With y fixed, $5 + 2y$ is a constant, so

$$\begin{aligned}\int_0^3 \left(5 + (x-4)^2 + 2y\right)\, \mathrm{d}x &= \left[(5+2y)x + \frac{(x-4)^3}{3}\right]_{x=0}^{x=3} \\ &= 3(5+2y) + \frac{(-1)^3}{3} - \frac{(-4)^3}{3} \\ &= 15 + 6y - \frac{1}{3} + \frac{64}{3} \\ &= 36 + 6y.\end{aligned}$$

Outer integral.

$$\int_0^4 (36 + 6y)\, \mathrm{d}y = \left[36y + 3y^2\right]_0^4 = 144 + 48 = 192.$$

Hence

$$\boxed{V = 192.}$$

Part (iii), Method 3: Splitting into separable terms

Split the integrand into three terms, each a product of a function of x and a function of y:

$$V = \iint_R 5\, \mathrm{d}A + \iint_R (x-4)^2\, \mathrm{d}A + \iint_R 2y\, \mathrm{d}A.$$

On a rectangle, $\iint_R p(x)q(y)\, \mathrm{d}A = \left(\int_0^3 p(x)\, \mathrm{d}x\right)\left(\int_0^4 q(y)\, \mathrm{d}y\right)$ by Fubini's Theorem. The one-variable integrals needed are

$$\int_0^3 \mathrm{d}x = 3, \qquad \int_0^4 \mathrm{d}y = 4,$$

$$\int_0^3 (x-4)^2\, \mathrm{d}x = \left[\frac{(x-4)^3}{3}\right]_0^3 = \frac{-1}{3} - \frac{-64}{3} = 21, \qquad \int_0^4 2y\, \mathrm{d}y = \left[y^2\right]_0^4 = 16.$$

Therefore

$$\begin{aligned}V &= 5 \cdot 3 \cdot 4 + 21 \cdot 4 + 3 \cdot 16 \\ &= 60 + 84 + 48 = 192,\end{aligned}$$

and

$$\boxed{V = 192.}$$

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Part (iv), Method 1: Polar coordinates

Region. The bounds are $0 \le y \le \sqrt{\pi}$ and $0 \le x \le \sqrt{\pi - y^2}$. The right boundary $x = \sqrt{\pi - y^2}$ is the part of the circle $x^2 + y^2 = \pi$ with $x \ge 0$. Hence the region of integration is the quarter-disc

$$D = \{(x, y) : x^2 + y^2 \le \pi,\ x \ge 0,\ y \ge 0\}.$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ this is

$$D = \left\{(r, \theta) : 0 \le r \le \sqrt{\pi},\ 0 \le \theta \le \frac{\pi}{2}\right\}.$$

Jacobian.

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r,$$

so $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. Also $x^2 + y^2 = r^2$, so the integrand becomes $\sin(r^2)$.

Integral.

$$\int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi - y^2}} \sin(x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\pi/2} \int_0^{\sqrt{\pi}} \sin(r^2)\, r\,\mathrm{d}r\,\mathrm{d}\theta.$$

For the inner integral substitute $u = r^2$, $\mathrm{d}u = 2r\,\mathrm{d}r$; the limits $r = 0$ and $r = \sqrt{\pi}$ become $u = 0$ and $u = \pi$:

$$\begin{aligned} \int_0^{\sqrt{\pi}} \sin(r^2)\, r\,\mathrm{d}r &= \frac{1}{2}\int_0^{\pi} \sin u\,\mathrm{d}u = \frac{1}{2}\left[-\cos u\right]_0^{\pi} \\ &= \frac{1}{2}\left(-\cos\pi + \cos 0\right) = \frac{1}{2}(1 + 1) = 1. \end{aligned}$$

The outer integral is

$$\int_0^{\pi/2} 1\,\mathrm{d}\theta = \frac{\pi}{2}.$$

Hence

$$\boxed{\frac{\pi}{2}.}$$

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Part (iv), Method 2: Squared-radius coordinates $s = r^2$

This method uses a general change of variables, which is beyond the 2026 midterm scope. *It should be used only if the course allows this interpretation* (the question asks for polar coordinates).

Because the integrand depends only on $s = x^2 + y^2$, use the coordinates (s, θ) with

$$x = \sqrt{s}\cos\theta, \qquad y = \sqrt{s}\sin\theta, \qquad 0 \le s \le \pi,\ 0 \le \theta \le \frac{\pi}{2}.$$

This maps the rectangle $[0, \pi] \times [0, \pi/2]$ onto the quarter-disc D, one-to-one on the interior.

Jacobian. The partial derivatives are

$$x_s = \frac{\cos\theta}{2\sqrt{s}}, \qquad x_\theta = -\sqrt{s}\sin\theta, \qquad y_s = \frac{\sin\theta}{2\sqrt{s}}, \qquad y_\theta = \sqrt{s}\cos\theta,$$

so

$$\frac{\partial(x,y)}{\partial(s,\theta)} = \begin{vmatrix} \dfrac{\cos\theta}{2\sqrt{s}} & -\sqrt{s}\sin\theta \\ \dfrac{\sin\theta}{2\sqrt{s}} & \sqrt{s}\cos\theta \end{vmatrix} = \frac{\cos^2\theta}{2} + \frac{\sin^2\theta}{2} = \frac{1}{2},$$

and $\mathrm{d}A = \frac{1}{2}\,\mathrm{d}s\,\mathrm{d}\theta$.

Integral. The integrand is $\sin(x^2 + y^2) = \sin s$, so

$$\begin{aligned} \iint_D \sin(x^2+y^2)\,\mathrm{d}A &= \int_0^{\pi/2}\int_0^{\pi} \frac{1}{2}\sin s\,\mathrm{d}s\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} \frac{1}{2}\left[-\cos s\right]_0^{\pi}\,\mathrm{d}\theta = \int_0^{\pi/2}\frac{1}{2}(1+1)\,\mathrm{d}\theta \\ &= \int_0^{\pi/2} 1\,\mathrm{d}\theta = \frac{\pi}{2}. \end{aligned}$$

Hence

$$\boxed{\frac{\pi}{2}.}$$

Unofficial solutions

Part (v), Method 1: Cylindrical coordinates

Beyond 2026 midterm scope. Use $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

Jacobian.

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\cdot\left(r\cos^2\theta + r\sin^2\theta\right) = r,$$

so $\mathrm{d}V = r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$.

Region. Between the cylinders: $3 \le r^2 \le 7$, i.e. $\sqrt{3} \le r \le \sqrt{7}$, with $0 \le \theta \le 2\pi$. Above the xy-plane and below $z = x + 4$: $0 \le z \le r\cos\theta + 4$. This interval is non-empty because

$$r\cos\theta + 4 \ge 4 - r \ge 4 - \sqrt{7} > 0 \qquad (\sqrt{7} < 3).$$

So

$$E = \left\{(r,\theta,z) : \sqrt{3} \le r \le \sqrt{7},\ 0 \le \theta \le 2\pi,\ 0 \le z \le r\cos\theta + 4\right\}.$$

Integral. The integrand is $y = r\sin\theta$, and with the Jacobian r,

$$\iiint_E y\,\mathrm{d}V = \int_0^{2\pi}\int_{\sqrt{3}}^{\sqrt{7}}\int_0^{r\cos\theta+4} r^2\sin\theta\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

Innermost (z):

$$\int_0^{r\cos\theta+4} r^2\sin\theta\,\mathrm{d}z = r^2\sin\theta\,(r\cos\theta + 4) = r^3\sin\theta\cos\theta + 4r^2\sin\theta.$$

Middle (r): the two radial integrals are

$$\int_{\sqrt{3}}^{\sqrt{7}} r^3\,\mathrm{d}r = \left[\frac{r^4}{4}\right]_{\sqrt{3}}^{\sqrt{7}} = \frac{49-9}{4} = 10, \qquad \int_{\sqrt{3}}^{\sqrt{7}} r^2\,\mathrm{d}r = \left[\frac{r^3}{3}\right]_{\sqrt{3}}^{\sqrt{7}} = \frac{7\sqrt{7}-3\sqrt{3}}{3},$$

so

$$\int_{\sqrt{3}}^{\sqrt{7}} \left(r^3\sin\theta\cos\theta + 4r^2\sin\theta\right)\,\mathrm{d}r = 10\sin\theta\cos\theta + \frac{4\left(7\sqrt{7}-3\sqrt{3}\right)}{3}\sin\theta.$$

Outer (θ): over a full period,

$$\int_0^{2\pi}\sin\theta\cos\theta\,\mathrm{d}\theta = \left[\frac{\sin^2\theta}{2}\right]_0^{2\pi} = 0, \qquad \int_0^{2\pi}\sin\theta\,\mathrm{d}\theta = \left[-\cos\theta\right]_0^{2\pi} = -1+1 = 0.$$

Therefore

$$\iiint_E y\,\mathrm{d}V = 10\cdot 0 + \frac{4\left(7\sqrt{7}-3\sqrt{3}\right)}{3}\cdot 0 = 0,$$

that is,

$$\boxed{\iiint_E y\,\mathrm{d}V = 0.}$$

Part (v), Method 2: Cartesian iterated integral

This method should be used only if the course allows this interpretation (the question asks for cylindrical coordinates).

Let $D = \{(x, y) : 3 \le x^2 + y^2 \le 7\}$ be the annulus under E. Over each point of D the solid is $0 \le z \le x + 4$, and $x + 4 \ge 4 - \sqrt{7} > 0$ there because $|x| \le \sqrt{x^2 + y^2} \le \sqrt{7} < 3$. The integrand y is continuous on E, so Fubini's Theorem gives

$$\iiint_E y \,\mathrm{d}V = \iint_D \int_0^{x+4} y \,\mathrm{d}z \,\mathrm{d}A = \iint_D y\,(x+4)\,\mathrm{d}A.$$

Integrate in y first, with x fixed in $[-\sqrt{7}, \sqrt{7}]$. The factor $x + 4$ does not depend on y, so

$$\int y\,(x+4)\,\mathrm{d}y = (x+4)\int y\,\mathrm{d}y.$$

The vertical slice of D at x has two cases.

Case $|x| < \sqrt{3}$. The slice is $\sqrt{3 - x^2} \le |y| \le \sqrt{7 - x^2}$, two intervals:

$$\int_{-\sqrt{7-x^2}}^{-\sqrt{3-x^2}} y\,\mathrm{d}y = \left[\frac{y^2}{2}\right]_{-\sqrt{7-x^2}}^{-\sqrt{3-x^2}} = \frac{(3-x^2)-(7-x^2)}{2} = -2,$$
$$\int_{\sqrt{3-x^2}}^{\sqrt{7-x^2}} y\,\mathrm{d}y = \left[\frac{y^2}{2}\right]_{\sqrt{3-x^2}}^{\sqrt{7-x^2}} = \frac{(7-x^2)-(3-x^2)}{2} = 2.$$

The slice integral is $-2 + 2 = 0$.

Case $\sqrt{3} \le |x| \le \sqrt{7}$. The slice is $-\sqrt{7 - x^2} \le y \le \sqrt{7 - x^2}$:

$$\int_{-\sqrt{7-x^2}}^{\sqrt{7-x^2}} y\,\mathrm{d}y = \left[\frac{y^2}{2}\right]_{-\sqrt{7-x^2}}^{\sqrt{7-x^2}} = \frac{7-x^2}{2} - \frac{7-x^2}{2} = 0.$$

In both cases the inner integral is 0, hence

$$\iiint_E y \,\mathrm{d}V = \int_{-\sqrt{7}}^{\sqrt{7}} (x+4)\cdot 0 \,\mathrm{d}x = 0,$$

and

$$\boxed{\iiint_E y \,\mathrm{d}V = 0.}$$

Part (v), Method 3: Reflection symmetry

This method should be used only if the course allows this interpretation (the question asks for cylindrical coordinates).

Consider the reflection $(x, y, z) = T(X, Y, Z) = (X, -Y, Z)$. Its Jacobian is

$$\frac{\partial(x,y,z)}{\partial(X,Y,Z)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1, \qquad |-1| = 1.$$

T maps E onto itself, because the conditions defining E involve y only through y^2:

$$(X, -Y, Z) \in E \iff 3 \le X^2 + Y^2 \le 7,\ 0 \le Z \le X + 4 \iff (X, Y, Z) \in E.$$

By the change-of-variables formula with $y = -Y$,

$$I = \iiint_E y \, \mathrm{d}V = \iiint_E (-Y) \cdot 1 \, \mathrm{d}V = -I.$$

Hence $2I = 0$, so

$$\boxed{\iiint_E y \, \mathrm{d}V = 0.}$$

Unofficial QRS marking scheme (32 marks)

(i) **(8 marks)** 3 marks for the Lagrange system; 2 for the feasible point $(7, 4, 6)$; 1 for a compactness or Cauchy–Schwarz global-maximum justification; 1 for evaluating f; 1 for identifying the maximum.

(ii) **(6 marks)** 3 marks for the inner integral; 2 for the outer integral; 1 for the correct final value.

(iii) **(6 marks)** 2 marks for the rectangular double-integral setup; 3 for correct evaluation; 1 for 192.

(iv) **(6 marks)** 2 marks for the quarter-disk polar bounds; 2 for the Jacobian; 1 for the radial integral; 1 for $\pi/2$.

(v) **(6 marks)** 2 marks for the cylindrical bounds; 2 for including $y = r \sin\theta$ and $\mathrm{d}V = r \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}z$; 1 for the full-period cancellation; 1 for 0.

Unofficial QRS@NTU solution

Question 2 (34 Marks)

(beyond 2026 midterm scope): Part (i) uses spherical integration; part (iv) uses a general Jacobian substitution.

(i) *(beyond 2026 midterm scope)* Use spherical coordinates to evaluate

$$\iiint_B \sqrt{x^2+y^2+z^2}\,\mathrm{d}V,$$

where B is the ball $x^2+y^2+z^2 \le 6$.

(ii) Find the area of the region bounded by the curves $y=\sqrt{x+1}$ and $x=3y-1$.

(iii) Evaluate the iterated integral

$$\int_0^{\sqrt{\pi/2}} \int_y^{\sqrt{\pi/2}} \cos(x^2)\,\mathrm{d}x\,\mathrm{d}y.$$

(iv) *(beyond 2026 midterm scope)* Use the transformation $x=u/v$; $y=v$ to evaluate the integral

$$\iint_R xy^3\,\mathrm{d}A,$$

where R is the region in the first quadrant ($x\ge 0$ and $y\ge 0$) bounded by the lines $y=x$, $y=3x$, and the hyperbolas $xy=1$ and $xy=4$.

(v) Find the volume of the solid inside the cylinder $x^2+y^2=16$ and the ellipsoid $2x^2+2y^2+z^2=32$.

Solution

Part (i), Method 1: Spherical coordinates

Beyond 2026 midterm scope. Use $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$.

Jacobian.

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}.$$

Expand along the third row. The two cofactors needed are

$$C_{31} = +\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} = \rho^2\sin\phi\cos\phi\left(\cos^2\theta+\sin^2\theta\right) = \rho^2\sin\phi\cos\phi,$$

$$C_{32} = -\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} = -\rho\sin^2\phi\left(\cos^2\theta+\sin^2\theta\right) = -\rho\sin^2\phi.$$

Hence

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \cos\phi\cdot\rho^2\sin\phi\cos\phi + (-\rho\sin\phi)(-\rho\sin^2\phi) = \rho^2\sin\phi\left(\cos^2\phi+\sin^2\phi\right) = \rho^2\sin\phi.$$

Since $\sin\phi\ge 0$ for $0\le\phi\le\pi$, $\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$.

Region and integrand. The ball $x^2+y^2+z^2 \le 6$ is $0 \le \rho \le \sqrt{6}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$, and $\sqrt{x^2+y^2+z^2} = \rho$.

Integral. The integrand $\rho \cdot \rho^2 \sin\phi = \rho^3 \sin\phi$ is a product of one-variable factors over a box, so

$$\begin{aligned}\iiint_B \sqrt{x^2+y^2+z^2}\,\mathrm{d}V &= \int_0^{2\pi}\int_0^{\pi}\int_0^{\sqrt{6}} \rho^3 \sin\phi \,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\ &= \left(\int_0^{2\pi} \mathrm{d}\theta\right)\left(\int_0^{\pi} \sin\phi\,\mathrm{d}\phi\right)\left(\int_0^{\sqrt{6}} \rho^3\,\mathrm{d}\rho\right).\end{aligned}$$

The three factors are

$$\int_0^{2\pi} \mathrm{d}\theta = 2\pi, \qquad \int_0^{\pi} \sin\phi\,\mathrm{d}\phi = [-\cos\phi]_0^{\pi} = 1+1 = 2, \qquad \int_0^{\sqrt{6}} \rho^3\,\mathrm{d}\rho = \left[\frac{\rho^4}{4}\right]_0^{\sqrt{6}} = \frac{36}{4} = 9.$$

Therefore the integral is $2\pi \cdot 2 \cdot 9$, that is,

$$\boxed{36\pi.}$$

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Part (i), Method 2: Cylindrical coordinates

This method should be used only if the course allows this interpretation (the question asks for spherical coordinates).

Use $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, with $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta$ (Jacobian r, as for polar coordinates). The ball is

$$-\sqrt{6} \le z \le \sqrt{6}, \qquad 0 \le r \le \sqrt{6 - z^2}, \qquad 0 \le \theta \le 2\pi,$$

and the integrand is $\sqrt{r^2 + z^2}$. So

$$\iiint_B \sqrt{x^2 + y^2 + z^2}\,\mathrm{d}V = \int_0^{2\pi} \int_{-\sqrt{6}}^{\sqrt{6}} \int_0^{\sqrt{6-z^2}} \sqrt{r^2 + z^2}\, r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta.$$

Inner integral (r). With z fixed, substitute $u = r^2 + z^2$, $\mathrm{d}u = 2r\,\mathrm{d}r$; the limits $r = 0$ and $r = \sqrt{6 - z^2}$ become $u = z^2$ and $u = 6$:

$$\int_0^{\sqrt{6-z^2}} \sqrt{r^2 + z^2}\, r\,\mathrm{d}r = \frac{1}{2}\int_{z^2}^{6} u^{1/2}\,\mathrm{d}u = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_{z^2}^{6} = \frac{1}{3}\left(6\sqrt{6} - |z|^3\right),$$

using $(z^2)^{3/2} = |z|^3$.

Middle integral (z). The integrand is even in z, so

$$\begin{aligned}\int_{-\sqrt{6}}^{\sqrt{6}} \frac{1}{3}\left(6\sqrt{6} - |z|^3\right)\,\mathrm{d}z &= \frac{2}{3}\int_0^{\sqrt{6}} \left(6\sqrt{6} - z^3\right)\,\mathrm{d}z = \frac{2}{3}\left[6\sqrt{6}\,z - \frac{z^4}{4}\right]_0^{\sqrt{6}} \\ &= \frac{2}{3}\,(36 - 9) = 18.\end{aligned}$$

Outer integral (θ).

$$\int_0^{2\pi} 18\,\mathrm{d}\theta = 36\pi.$$

Hence

$$\boxed{36\pi.}$$

Part (i), Method 3: Spherical shells

This method should be used only if the course allows this interpretation (the question asks for the spherical-coordinate triple integral).

The integrand $\sqrt{x^2 + y^2 + z^2} = \rho$ is constant on each sphere $\rho = \text{const}$. The volume of the ball of radius ρ is

$$V(\rho) = \frac{4}{3}\pi\rho^3, \qquad V'(\rho) = 4\pi\rho^2,$$

so the shell between radii ρ and $\rho + \mathrm{d}\rho$ has volume $4\pi\rho^2\,\mathrm{d}\rho$. (This is the spherical-coordinate integral of Method 1 with the ϕ and θ integrals done first: $2\pi \cdot 2 \cdot \rho^2 = 4\pi\rho^2$.) Hence

$$\iiint_B \rho\,\mathrm{d}V = \int_0^{\sqrt{6}} \rho \cdot 4\pi\rho^2\,\mathrm{d}\rho = 4\pi\left[\frac{\rho^4}{4}\right]_0^{\sqrt{6}} = 4\pi \cdot 9 = 36\pi,$$

that is,

$$\boxed{36\pi.}$$

Part (ii), Method 1: Horizontal strips

The curves. $y = \sqrt{x+1}$ means $y \geq 0$ and $x = y^2 - 1$: the upper half of a parabola opening to the right. The line is $x = 3y - 1$.

Intersections. Set the x-values equal (with $y \geq 0$):

$$y^2 - 1 = 3y - 1 \iff y^2 - 3y = 0 \iff y(y-3) = 0 \iff y = 0 \text{ or } y = 3.$$

The intersection points are $(-1, 0)$ and $(8, 3)$.

Which curve is on the right. For $0 \leq y \leq 3$,

$$(3y - 1) - (y^2 - 1) = 3y - y^2 = y(3 - y) \geq 0,$$

so the line is to the right of the parabola. The region is horizontally simple:

$$D = \{(x, y) : 0 \leq y \leq 3,\ y^2 - 1 \leq x \leq 3y - 1\}.$$

Area.

$$\begin{aligned} A = \iint_D 1 \, \mathrm{d}A &= \int_0^3 \int_{y^2-1}^{3y-1} \mathrm{d}x \, \mathrm{d}y = \int_0^3 \left[(3y-1) - (y^2-1)\right] \, \mathrm{d}y \\ &= \int_0^3 \left(3y - y^2\right) \, \mathrm{d}y = \left[\frac{3y^2}{2} - \frac{y^3}{3}\right]_0^3 \\ &= \frac{27}{2} - 9 = \frac{9}{2}. \end{aligned}$$

Hence

$$\boxed{A = \frac{9}{2}.}$$

Unofficial Question Solutions

Part (ii), Method 2: Vertical strips

Solve both curves for y: the parabola is $y = \sqrt{x+1}$ (defined for $x \geq -1$) and the line is $y = \dfrac{x+1}{3}$.

Intersections. Let $t = x + 1 \geq 0$. The curves meet where

$$\sqrt{t} = \frac{t}{3} \iff \sqrt{t}\left(1 - \frac{\sqrt{t}}{3}\right) = 0 \iff \sqrt{t} = 0 \text{ or } \sqrt{t} = 3 \iff t = 0 \text{ or } t = 9,$$

that is, at $x = -1$ and $x = 8$.

Which curve is on top. For $t = x + 1 \in [0, 9]$,

$$\sqrt{t} - \frac{t}{3} = \sqrt{t}\left(1 - \frac{\sqrt{t}}{3}\right) \geq 0 \qquad (0 \leq \sqrt{t} \leq 3),$$

so the parabola is above the line. The region is vertically simple:

$$D = \left\{(x, y) : -1 \leq x \leq 8,\ \frac{x+1}{3} \leq y \leq \sqrt{x+1}\right\}.$$

Area.

$$A = \int_{-1}^{8} \int_{(x+1)/3}^{\sqrt{x+1}} \mathrm{d}y\,\mathrm{d}x = \int_{-1}^{8} \left(\sqrt{x+1} - \frac{x+1}{3}\right) \mathrm{d}x.$$

Substitute $t = x + 1,\ \mathrm{d}t = \mathrm{d}x$; the limits $x = -1$ and $x = 8$ become $t = 0$ and $t = 9$:

$$\begin{aligned} A &= \int_0^9 \left(t^{1/2} - \frac{t}{3}\right) \mathrm{d}t = \left[\frac{2}{3}t^{3/2} - \frac{t^2}{6}\right]_0^9 \\ &= \frac{2}{3} \cdot 27 - \frac{81}{6} = 18 - \frac{27}{2} = \frac{9}{2}. \end{aligned}$$

Hence

$$\boxed{A = \frac{9}{2}.}$$

Part (ii), Method 3: Affine substitution for a quadratic width

Write the curves as $x = y^2 - 1$ $(y \geq 0)$ and $x = 3y - 1$. The horizontal width of the region at height y is

$$w(y) = (3y - 1) - (y^2 - 1) = 3y - y^2 = (y - \alpha)(\beta - y), \qquad \alpha = 0,\ \beta = 3,$$

which vanishes at the intersection heights $y = 0$ and $y = 3$ and is non-negative for $0 \leq y \leq 3$, so $A = \int_0^3 w(y)\,\mathrm{d}y$. For any $\alpha < \beta$, substitute $y = \alpha + (\beta - \alpha)s,\ \mathrm{d}y = (\beta - \alpha)\,\mathrm{d}s$; the limits $y = \alpha$ and $y = \beta$ become $s = 0$ and $s = 1$, and $(y - \alpha)(\beta - y) = (\beta - \alpha)^2 s(1 - s)$. Hence

$$\int_\alpha^\beta (y - \alpha)(\beta - y)\,\mathrm{d}y = (\beta - \alpha)^3 \int_0^1 \left(s - s^2\right) \mathrm{d}s = (\beta - \alpha)^3 \left[\frac{s^2}{2} - \frac{s^3}{3}\right]_0^1 = \frac{(\beta - \alpha)^3}{6}.$$

With $\alpha = 0$, $\beta = 3$:

$$A = \frac{3^3}{6} = \frac{27}{6} = \frac{9}{2},$$

so

$$\boxed{A = \frac{9}{2}.}$$

Part (iii), Method 1: Reverse the order of integration

Region. Let $a = \sqrt{\pi/2}$. The given bounds are $0 \le y \le a$ and $y \le x \le a$, so the region is the triangle

$$T = \{(x, y) : 0 \le y \le a,\ y \le x \le a\},$$

with vertices $(0, 0)$, $(a, 0)$, (a, a). Described with x outside,

$$T = \{(x, y) : 0 \le x \le a,\ 0 \le y \le x\}.$$

The inner integral in the given order has no elementary antiderivative ($\int \cos(x^2)\,\mathrm{d}x$), so we reverse the order. The integrand $\cos(x^2)$ is continuous on the closed triangle T, so Fubini's Theorem gives

$$\int_0^a \int_y^a \cos(x^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \int_0^x \cos(x^2)\,\mathrm{d}y\,\mathrm{d}x.$$

Inner integral. $\cos(x^2)$ does not depend on y, so

$$\int_0^x \cos(x^2)\,\mathrm{d}y = \left[y\cos(x^2)\right]_{y=0}^{y=x} = x\cos(x^2).$$

Outer integral. Substitute $u = x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$; the limits $x = 0$ and $x = \sqrt{\pi/2}$ become $u = 0$ and $u = \pi/2$:

$$\int_0^{\sqrt{\pi/2}} x\cos(x^2)\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi/2} \cos u\,\mathrm{d}u = \frac{1}{2}\left[\sin u\right]_0^{\pi/2} = \frac{1}{2}(1 - 0) = \frac{1}{2}.$$

Hence

$$\boxed{\frac{1}{2}.}$$

Unofficial QRS@NTU Solutions

Part (iii), Method 2: Integration by parts in the given order

Keep the given order. Let $a = \sqrt{\pi/2}$ and

$$G(y) = \int_y^a \cos(x^2)\,\mathrm{d}x, \qquad 0 \le y \le a,$$

so the required integral is $I = \int_0^a G(y)\,\mathrm{d}y$. By the Fundamental Theorem of Calculus (the integrand $\cos(x^2)$ is continuous),

$$G'(y) = -\cos(y^2), \qquad G(a) = \int_a^a \cos(x^2)\,\mathrm{d}x = 0.$$

Integrate by parts with $U = G(y),\ \mathrm{d}W = \mathrm{d}y$, so $\mathrm{d}U = -\cos(y^2)\,\mathrm{d}y$ and $W = y$:

$$\begin{aligned} I &= \Big[y\,G(y)\Big]_0^a - \int_0^a y\,\big(-\cos(y^2)\big)\,\mathrm{d}y \\ &= (a\cdot G(a) - 0\cdot G(0)) + \int_0^a y\cos(y^2)\,\mathrm{d}y \\ &= 0 + \int_0^a y\cos(y^2)\,\mathrm{d}y. \end{aligned}$$

Substitute $u = y^2,\ \mathrm{d}u = 2y\,\mathrm{d}y$; the limits $y = 0$ and $y = a$ become $u = 0$ and $u = \pi/2$:

$$I = \frac{1}{2}\int_0^{\pi/2} \cos u\,\mathrm{d}u = \frac{1}{2}\,[\sin u]_0^{\pi/2} = \frac{1}{2}.$$

Hence

$$\boxed{\frac{1}{2}.}$$

Unofficial Q&A Document

Part (iv), Method 1: The given transformation $x = u/v$, $y = v$

Beyond 2026 midterm scope.

Inverse map. From $y = v$ and $x = u/v$ we get $v = y$ and $u = xv = xy$. On R we have $xy \geq 1$, so $x > 0$ and $y > 0$; in particular $v > 0$.

Image of the region.

$$xy = 1 \iff u = 1, \qquad xy = 4 \iff u = 4,$$
$$y = x \iff v = \frac{u}{v} \iff v^2 = u, \qquad y = 3x \iff v = \frac{3u}{v} \iff v^2 = 3u.$$

The region between the two lines is $x \leq y \leq 3x$, that is $1 \leq y/x \leq 3$, and

$$\frac{y}{x} = \frac{v}{u/v} = \frac{v^2}{u}.$$

So $1 \leq v^2/u \leq 3$, i.e. $u \leq v^2 \leq 3u$, and with $v > 0$,

$$S = \{(u, v) : 1 \leq u \leq 4,\ \sqrt{u} \leq v \leq \sqrt{3u}\}.$$

Jacobian. The partial derivatives are

$$x_u = \frac{1}{v}, \qquad x_v = -\frac{u}{v^2}, \qquad y_u = 0, \qquad y_v = 1,$$

so

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1/v & -u/v^2 \\ 0 & 1 \end{vmatrix} = \frac{1}{v} \cdot 1 - \left(-\frac{u}{v^2}\right) \cdot 0 = \frac{1}{v}, \qquad \left|\frac{1}{v}\right| = \frac{1}{v} \quad (v > 0).$$

Integrand.

$$xy^3 = \frac{u}{v} \cdot v^3 = uv^2.$$

Integral.

$$\begin{aligned} \iint_R xy^3 \, \mathrm{d}A &= \int_1^4 \int_{\sqrt{u}}^{\sqrt{3u}} uv^2 \cdot \frac{1}{v} \, \mathrm{d}v \, \mathrm{d}u = \int_1^4 \int_{\sqrt{u}}^{\sqrt{3u}} uv \, \mathrm{d}v \, \mathrm{d}u \\ &= \int_1^4 u \left[\frac{v^2}{2}\right]_{v=\sqrt{u}}^{v=\sqrt{3u}} \mathrm{d}u = \int_1^4 u \cdot \frac{3u - u}{2} \, \mathrm{d}u \\ &= \int_1^4 u^2 \, \mathrm{d}u = \left[\frac{u^3}{3}\right]_1^4 = \frac{64 - 1}{3} = 21. \end{aligned}$$

Hence

$$\boxed{21.}$$

Unofficial QRS@NTU solutions

Part (iv), Method 2: The substitution $u = xy$, $w = y/x$

This method should be used only if the course allows this interpretation (the question asks for the transformation $x = u/v$, $y = v$).

The four boundary curves are level sets of $u = xy$ and $w = y/x$, so these coordinates turn R into the rectangle

$$S' = \{(u, w) : 1 \le u \le 4,\ 1 \le w \le 3\}.$$

Solving for $x, y > 0$: $uw = y^2$ and $u/w = x^2$, so

$$x = \sqrt{\frac{u}{w}}, \qquad y = \sqrt{uw}.$$

Jacobian. The partial derivatives are

$$x_u = \frac{1}{2\sqrt{uw}}, \qquad x_w = -\frac{\sqrt{u}}{2w^{3/2}}, \qquad y_u = \frac{\sqrt{w}}{2\sqrt{u}}, \qquad y_w = \frac{\sqrt{u}}{2\sqrt{w}},$$

so

$$\frac{\partial(x, y)}{\partial(u, w)} = x_u y_w - x_w y_u = \frac{1}{2\sqrt{uw}} \cdot \frac{\sqrt{u}}{2\sqrt{w}} + \frac{\sqrt{u}}{2w^{3/2}} \cdot \frac{\sqrt{w}}{2\sqrt{u}} = \frac{1}{4w} + \frac{1}{4w} = \frac{1}{2w} > 0.$$

Integrand.

$$xy^3 = (xy)\, y^2 = u \cdot uw = u^2 w.$$

Integral.

$$\begin{aligned} \iint_R xy^3 \, \mathrm{d}A &= \int_1^4 \int_1^3 u^2 w \cdot \frac{1}{2w} \, \mathrm{d}w \, \mathrm{d}u = \int_1^4 \int_1^3 \frac{u^2}{2} \, \mathrm{d}w \, \mathrm{d}u \\ &= \int_1^4 \frac{u^2}{2} \cdot (3 - 1) \, \mathrm{d}u = \int_1^4 u^2 \, \mathrm{d}u = \left[\frac{u^3}{3}\right]_1^4 = 21. \end{aligned}$$

Hence

$$\boxed{21.}$$

Unofficial QRS@NTU study notes

Part (iv), Method 3: Cartesian coordinates with three vertical strips

This method should be used only if the course allows this interpretation (it integrates directly in x and y without the transformation).

Vertices.

$$y=x,\ xy=1 \implies (1,1), \qquad y=x,\ xy=4 \implies (2,2),$$
$$y=3x,\ xy=1 \implies \left(\tfrac{1}{\sqrt{3}},\sqrt{3}\right), \qquad y=3x,\ xy=4 \implies \left(\tfrac{2}{\sqrt{3}},2\sqrt{3}\right).$$

So x runs from $1/\sqrt{3}$ to 2. For each x the lower boundary is $\max\{x,1/x\}$ and the upper boundary is $\min\{3x,4/x\}$. The switches are at $x=1/x \iff x=1$ and at $3x=4/x \iff x=2/\sqrt{3}$, with $1<2/\sqrt{3}$. This gives three vertical strips:

$$\begin{aligned} \tfrac{1}{\sqrt{3}} \le x \le 1: &\quad \tfrac{1}{x} \le y \le 3x,\\ 1 \le x \le \tfrac{2}{\sqrt{3}}: &\quad x \le y \le 3x,\\ \tfrac{2}{\sqrt{3}} \le x \le 2: &\quad x \le y \le \tfrac{4}{x}. \end{aligned}$$

In every strip the inner integral is $\int xy^3\,\mathrm{d}y = \dfrac{xy^4}{4}$.

Strip 1.

$$\begin{aligned} \int_{1/\sqrt{3}}^{1} \frac{x}{4}\left(81x^4-\frac{1}{x^4}\right)\mathrm{d}x &= \frac{1}{4}\int_{1/\sqrt{3}}^{1}\left(81x^5-x^{-3}\right)\mathrm{d}x = \frac{1}{4}\left[\frac{27x^6}{2}+\frac{1}{2x^2}\right]_{1/\sqrt{3}}^{1}\\ &= \frac{1}{4}\left[\left(\frac{27}{2}+\frac{1}{2}\right)-\left(\frac{27}{2}\cdot\frac{1}{27}+\frac{3}{2}\right)\right] = \frac{1}{4}(14-2)=3. \end{aligned}$$

Strip 2.

$$\begin{aligned} \int_{1}^{2/\sqrt{3}} \frac{x}{4}\left(81x^4-x^4\right)\mathrm{d}x &= \int_{1}^{2/\sqrt{3}} 20x^5\,\mathrm{d}x = \left[\frac{10x^6}{3}\right]_{1}^{2/\sqrt{3}}\\ &= \frac{10}{3}\left(\frac{64}{27}-1\right) = \frac{10}{3}\cdot\frac{37}{27} = \frac{370}{81}. \end{aligned}$$

Strip 3.

$$\begin{aligned} \int_{2/\sqrt{3}}^{2} \frac{x}{4}\left(\frac{256}{x^4}-x^4\right)\mathrm{d}x &= \int_{2/\sqrt{3}}^{2}\left(64x^{-3}-\frac{x^5}{4}\right)\mathrm{d}x = \left[-\frac{32}{x^2}-\frac{x^6}{24}\right]_{2/\sqrt{3}}^{2}\\ &= \left(-8-\frac{64}{24}\right)-\left(-24-\frac{64/27}{24}\right) = 16-\frac{8}{3}+\frac{8}{81} = \frac{1088}{81}. \end{aligned}$$

Total.

$$3+\frac{370}{81}+\frac{1088}{81} = 3+\frac{1458}{81} = 3+18 = 21.$$

Hence

$$\boxed{21.}$$

Part (v), Method 1: Polar coordinates for the height function

The solid. Write $r^2 = x^2 + y^2$. The ellipsoid condition $2r^2 + z^2 \le 32$ gives

$$-\sqrt{32 - 2r^2} \le z \le \sqrt{32 - 2r^2},$$

which needs $32 - 2r^2 \ge 0$, i.e. $r \le 4$. The cylinder condition is also $r \le 4$. So the ellipsoid lies entirely inside the cylinder (they touch along the circle $r = 4$, $z = 0$), and the solid is

$$\{(x, y, z) : x^2 + y^2 \le 16,\ |z| \le \sqrt{32 - 2(x^2 + y^2)}\}.$$

Its height over (x, y) is $2\sqrt{32 - 2(x^2 + y^2)}$, so with D the disc $x^2 + y^2 \le 16$,

$$V = \iint_D 2\sqrt{32 - 2(x^2 + y^2)}\,\mathrm{d}A.$$

Polar coordinates. D is $0 \le r \le 4$, $0 \le \theta \le 2\pi$, and $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$ (Jacobian r):

$$V = \int_0^{2\pi} \int_0^4 2\sqrt{32 - 2r^2}\, r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Radial integral. Substitute $u = 32 - 2r^2$, $\mathrm{d}u = -4r\,\mathrm{d}r$, so $r\,\mathrm{d}r = -\frac{1}{4}\,\mathrm{d}u$; the limits $r = 0$ and $r = 4$ become $u = 32$ and $u = 0$:

$$\int_0^4 2\sqrt{32 - 2r^2}\, r\,\mathrm{d}r = 2\int_{32}^0 u^{1/2}\left(-\frac{1}{4}\right)\mathrm{d}u = \frac{1}{2}\int_0^{32} u^{1/2}\,\mathrm{d}u = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_0^{32}$$
$$= \frac{32^{3/2}}{3} = \frac{32 \cdot 4\sqrt{2}}{3} = \frac{128\sqrt{2}}{3}.$$

Angular integral.

$$V = \int_0^{2\pi} \frac{128\sqrt{2}}{3}\,\mathrm{d}\theta = 2\pi \cdot \frac{128\sqrt{2}}{3} = \frac{256\sqrt{2}\,\pi}{3}.$$

Hence

$$\boxed{V = \frac{256\sqrt{2}\pi}{3}.}$$

Part (v), Method 2: Horizontal slices

As in Method 1, the solid is the whole ellipsoid $2x^2 + 2y^2 + z^2 \le 32$, because that ellipsoid already satisfies $x^2 + y^2 \le 16$. Its z-range is $z^2 \le 32$, i.e. $-4\sqrt{2} \le z \le 4\sqrt{2}$.

Cross-section at height z.

$$2(x^2 + y^2) \le 32 - z^2 \iff x^2 + y^2 \le 16 - \frac{z^2}{2}.$$

This is a disc whose squared radius is $16 - z^2/2 \le 16$ (inside the cylinder), so its area is

$$A(z) = \pi\left(16 - \frac{z^2}{2}\right).$$

Volume. By Fubini's Theorem (integrating the area of the cross-sections), and since A is even in z,

$$\begin{aligned} V &= \int_{-4\sqrt{2}}^{4\sqrt{2}} \pi\left(16 - \frac{z^2}{2}\right) \mathrm{d}z = 2\pi\left[16z - \frac{z^3}{6}\right]_0^{4\sqrt{2}} \\ &= 2\pi\left(64\sqrt{2} - \frac{128\sqrt{2}}{6}\right) = 2\pi\left(64\sqrt{2} - \frac{64\sqrt{2}}{3}\right) = 2\pi \cdot \frac{128\sqrt{2}}{3} = \frac{256\sqrt{2}\,\pi}{3}, \end{aligned}$$

using $(4\sqrt{2})^3 = 64 \cdot 2\sqrt{2} = 128\sqrt{2}$. Hence

$$\boxed{V = \frac{256\sqrt{2}\pi}{3}.}$$

Part (v), Method 3: Linear scaling to the unit ball

This method uses a general change of variables, which is beyond the 2026 midterm scope.

Divide the ellipsoid equation by 32:

$$\frac{x^2}{16} + \frac{y^2}{16} + \frac{z^2}{32} \le 1,$$

so the semi-axes are 4, 4, $4\sqrt{2}$. Substitute $x = 4X$, $y = 4Y$, $z = 4\sqrt{2}\,Z$. The ellipsoid becomes the unit ball $X^2 + Y^2 + Z^2 \le 1$, and

$$\frac{\partial(x, y, z)}{\partial(X, Y, Z)} = \begin{vmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4\sqrt{2} \end{vmatrix} = 64\sqrt{2}.$$

With the unit-ball volume $\frac{4}{3}\pi$,

$$V = 64\sqrt{2} \cdot \frac{4}{3}\pi = \frac{256\sqrt{2}\,\pi}{3},$$

that is,

$$\boxed{V = \frac{256\sqrt{2}\pi}{3}.}$$

Unofficial QRS marking scheme (34 marks)

(i) **(7 marks)** 2 marks for spherical bounds; 2 for the integrand and Jacobian; 2 for integration; 1 for 36π.

(ii) **(6 marks)** 2 marks for intersections; 2 for correct horizontal bounds; 1 for integration; 1 for $9/2$.

(iii) **(6 marks)** 3 marks for reversed bounds; 2 for the substitution $u = x^2$; 1 for $1/2$.

(iv) **(7 marks)** 2 marks for the transformed region; 2 for the Jacobian; 2 for the transformed integral; 1 for 21.

(v) **(8 marks)** 2 marks for the vertical height and polar double-integral setup; 2 for identifying the common radial endpoint $r = 4$; 3 for integration; 1 for $256\sqrt{2}\pi/3$.

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Question 3 (34 Marks)

(beyond 2026 midterm scope): All parts use scalar or vector line integrals.

(i) *(beyond 2026 midterm scope)* Evaluate the line integral

$$\int_C xy \, \mathrm{d}s,$$

where C is the line segment from $(-2,-1)$ to $(4,5)$.

(ii) *(beyond 2026 midterm scope)* Find the work done by the force field $\mathbf{F}(x,y) = y\mathbf{i} - x\mathbf{j}$ on a particle that moves along the curve $x = 2y^3 - 6y^2 + 1$ from $(-3,1)$ to $(1,3)$.

(iii) *(beyond 2026 midterm scope)* Evaluate the line integral

$$\int_C x^2 y \cos z \, \mathrm{d}s,$$

where C has parametrisation $x = \cos t$, $y = \sin t$ and $z = t$, with $0 \le t \le \pi$.

(iv) *(beyond 2026 midterm scope)* Let $\mathbf{F}(x,y) = y\mathbf{i} + (x+10y)\mathbf{j}$ and let C be the upper semicircle that starts at $(1,2)$ and ends at $(5,2)$.

(1) Find a function f such that $\mathbf{F} = \nabla f$.

(2) Hence, or otherwise, evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

Solution

Part (i), Method 1: Parametrise by $t \in [0,1]$

Beyond 2026 midterm scope.

Parametrisation. The segment from $(-2,-1)$ to $(4,5)$ has direction $(4,5) - (-2,-1) = (6,6)$, so

$$\mathbf{r}(t) = (-2,-1) + t(6,6) = (-2+6t,\, -1+6t), \qquad 0 \le t \le 1.$$

Then

$$\mathbf{r}'(t) = (6,6), \qquad \|\mathbf{r}'(t)\| = \sqrt{36+36} = 6\sqrt{2}, \qquad \mathrm{d}s = 6\sqrt{2}\,\mathrm{d}t.$$

Integrand.

$$xy = (-2+6t)(-1+6t) = 2 - 12t - 6t + 36t^2 = 2 - 18t + 36t^2.$$

Integral.

$$\begin{aligned}\int_C xy \, \mathrm{d}s &= \int_0^1 \left(2 - 18t + 36t^2\right) 6\sqrt{2}\, \mathrm{d}t = 6\sqrt{2}\left[2t - 9t^2 + 12t^3\right]_0^1 \\ &= 6\sqrt{2}\,(2 - 9 + 12) = 6\sqrt{2} \cdot 5 = 30\sqrt{2}.\end{aligned}$$

Hence

$$\boxed{\int_C xy \, \mathrm{d}s = 30\sqrt{2}.}$$

Part (i), Method 2: Parametrise by x

The line through $(-2,-1)$ and $(4,5)$ has slope $\frac{5-(-1)}{4-(-2)} = 1$, so it is $y = x+1$, and C is the graph of $y = x+1$ for $-2 \le x \le 4$. For a graph $y = g(x)$,

$$\mathrm{d}s = \sqrt{1+g'(x)^2}\,\mathrm{d}x = \sqrt{1+1^2}\,\mathrm{d}x = \sqrt{2}\,\mathrm{d}x.$$

Hence

$$\begin{aligned}\int_C xy\,\mathrm{d}s &= \int_{-2}^{4} x(x+1)\sqrt{2}\,\mathrm{d}x = \sqrt{2}\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_{-2}^{4} \\ &= \sqrt{2}\left[\left(\frac{64}{3}+8\right)-\left(-\frac{8}{3}+2\right)\right] = \sqrt{2}\left(\frac{72}{3}+6\right) = \sqrt{2}\,(24+6) = 30\sqrt{2}.\end{aligned}$$

Therefore

$$\boxed{\int_C xy\,\mathrm{d}s = 30\sqrt{2}.}$$

Part (i), Method 3: Midpoint parametrisation with odd-term cancellation

The midpoint of the segment is $(1,2)$. Parametrise symmetrically about it:

$$x = 1+u, \qquad y = 2+u, \qquad -3 \le u \le 3,$$

so that $u=-3$ gives $(-2,-1)$ and $u=3$ gives $(4,5)$. Then $\mathbf{r}'(u) = (1,1)$ and $\mathrm{d}s = \sqrt{2}\,\mathrm{d}u$. The integrand is

$$xy = (1+u)(2+u) = 2+3u+u^2.$$

On the symmetric interval $[-3,3]$ the odd term $3u$ integrates to 0, so

$$\int_C xy\,\mathrm{d}s = \sqrt{2}\int_{-3}^{3}\left(2+u^2\right)\,\mathrm{d}u = \sqrt{2}\cdot 2\left[2u+\frac{u^3}{3}\right]_0^3 = 2\sqrt{2}\,(6+9) = 30\sqrt{2}.$$

Hence

$$\boxed{\int_C xy\,\mathrm{d}s = 30\sqrt{2}.}$$

Unofficial Solutions

Part (ii), Method 1: Parametrise by y

Beyond 2026 midterm scope.

Parametrisation. Use y as the parameter:

$$\mathbf{r}(y) = \left(2y^3 - 6y^2 + 1,\ y\right), \qquad 1 \le y \le 3.$$

The endpoints match:

$$x(1) = 2 - 6 + 1 = -3, \qquad x(3) = 54 - 54 + 1 = 1,$$

so $\mathbf{r}(1) = (-3, 1)$ and $\mathbf{r}(3) = (1, 3)$, in the required direction. The derivative is

$$\mathbf{r}'(y) = \left(6y^2 - 12y,\ 1\right).$$

Dot product. On C, $\mathbf{F} = (y,\ -x) = (y,\ -(2y^3 - 6y^2 + 1))$, so

$$\begin{aligned}\mathbf{F} \cdot \mathbf{r}'(y) &= y\left(6y^2 - 12y\right) - \left(2y^3 - 6y^2 + 1\right) \cdot 1 \\ &= 6y^3 - 12y^2 - 2y^3 + 6y^2 - 1 \\ &= 4y^3 - 6y^2 - 1.\end{aligned}$$

Work.

$$\begin{aligned}W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} &= \int_1^3 \left(4y^3 - 6y^2 - 1\right) \mathrm{d}y = \left[y^4 - 2y^3 - y\right]_1^3 \\ &= (81 - 54 - 3) - (1 - 2 - 1) = 24 - (-2) = 26.\end{aligned}$$

Hence

$$\boxed{W = 26.}$$

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Part (ii), Method 2: Close the curve and apply Green's Theorem

This method uses Green's Theorem (beyond the midterm scope).

F is not conservative. With $P = y$, $Q = -x$,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -1 - 1 = -2 \neq 0,$$

so the work depends on the path and a potential cannot be used.

Closing segment. Let L be the segment from $(1, 3)$ back to $(-3, 1)$. The line through these points is $x = 2y - 5$. Compare it with C, $x = 2y^3 - 6y^2 + 1$:

$$(2y - 5) - (2y^3 - 6y^2 + 1) = -2y^3 + 6y^2 + 2y - 6 = -2(y - 3)(y - 1)(y + 1) \geq 0 \quad (1 \leq y \leq 3),$$

with equality only at $y = 1$ and $y = 3$. So C lies to the left of L, and C followed by L is a simple closed curve bounding

$$D = \{(x, y) : 1 \leq y \leq 3,\ 2y^3 - 6y^2 + 1 \leq x \leq 2y - 5\}.$$

C goes up the left side of D and L comes down the right side, so $C \cup L$ is traversed clockwise, i.e. it is $-\partial D$.

Area of D.

$$\begin{aligned}\text{Area}(D) &= \int_1^3 \left(-2y^3 + 6y^2 + 2y - 6\right)\, \mathrm{d}y = \left[-\frac{y^4}{2} + 2y^3 + y^2 - 6y\right]_1^3 \\ &= \left(-\frac{81}{2} + 54 + 9 - 18\right) - \left(-\frac{1}{2} + 2 + 1 - 6\right) = \frac{9}{2} - \left(-\frac{7}{2}\right) = 8.\end{aligned}$$

Green's Theorem. P, Q have continuous partial derivatives on $\mathbb{R}^2$, so for the clockwise boundary

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_L \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = -\iint_D (-2)\, \mathrm{d}A = 2 \cdot 8 = 16.$$

Integral along L. Here $\mathbf{F} \cdot \mathrm{d}\mathbf{r} = P\,\mathrm{d}x + Q\,\mathrm{d}y = y\,\mathrm{d}x - x\,\mathrm{d}y$. Parametrise L by $x = 1 - 4t$, $y = 3 - 2t$, $0 \leq t \leq 1$, so $\mathrm{d}x = -4\,\mathrm{d}t$, $\mathrm{d}y = -2\,\mathrm{d}t$:

$$\begin{aligned}\int_L y\,\mathrm{d}x - x\,\mathrm{d}y &= \int_0^1 \left[(3 - 2t)(-4) - (1 - 4t)(-2)\right]\, \mathrm{d}t \\ &= \int_0^1 (-12 + 8t + 2 - 8t)\, \mathrm{d}t = \int_0^1 (-10)\, \mathrm{d}t = -10.\end{aligned}$$

Work along C.

$$W = 16 - (-10) = 26,$$

so

$$\boxed{W = 26.}$$

Part (iii), Method 1: Direct parametrisation and substitution

Beyond 2026 midterm scope.

Arc-length element. With $\mathbf{r}(t) = (\cos t, \sin t, t)$, $0 \le t \le \pi$,

$$\mathbf{r}'(t) = (-\sin t, \cos t, 1), \qquad \|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}, \qquad \mathrm{d}s = \sqrt{2}\,\mathrm{d}t.$$

Integrand.

$$x^2 y \cos z = \cos^2 t \cdot \sin t \cdot \cos t = \cos^3 t \sin t.$$

Integral.

$$\int_C x^2 y \cos z \,\mathrm{d}s = \sqrt{2} \int_0^{\pi} \cos^3 t \sin t \,\mathrm{d}t.$$

Substitute $u = \cos t$, $\mathrm{d}u = -\sin t \,\mathrm{d}t$; the limits $t = 0$ and $t = \pi$ become $u = 1$ and $u = -1$:

$$\sqrt{2} \int_0^{\pi} \cos^3 t \sin t \,\mathrm{d}t = \sqrt{2} \int_1^{-1} u^3 (-\mathrm{d}u) = \sqrt{2} \int_{-1}^{1} u^3 \,\mathrm{d}u = \sqrt{2} \left[\frac{u^4}{4}\right]_{-1}^{1} = \sqrt{2} \left(\frac{1}{4} - \frac{1}{4}\right) = 0.$$

Hence

$$\boxed{\int_C x^2 y \cos z \,\mathrm{d}s = 0.}$$

Part (iii), Method 2: Symmetry $t \mapsto \pi - t$

With $\mathbf{r}(t) = (\cos t, \sin t, t)$ we have $\mathbf{r}'(t) = (-\sin t, \cos t, 1)$, so $\mathrm{d}s = \sqrt{\sin^2 t + \cos^2 t + 1}\,\mathrm{d}t = \sqrt{2}\,\mathrm{d}t$, and $x^2 y \cos z = \cos^2 t \cdot \sin t \cdot \cos t$. Hence the integral is $\sqrt{2}\,I$ with

$$I = \int_0^{\pi} g(t)\,\mathrm{d}t, \qquad g(t) = \cos^3 t \sin t.$$

Since $\cos(\pi - t) = -\cos t$ and $\sin(\pi - t) = \sin t$,

$$g(\pi - t) = (-\cos t)^3 \sin t = -g(t).$$

Substitute $w = \pi - t$, $\mathrm{d}w = -\mathrm{d}t$; the limits $t = 0$ and $t = \pi$ become $w = \pi$ and $w = 0$:

$$I = \int_{\pi}^{0} g(\pi - w)(-\mathrm{d}w) = \int_0^{\pi} g(\pi - w)\,\mathrm{d}w = -\int_0^{\pi} g(w)\,\mathrm{d}w = -I.$$

Hence $2I = 0$, so $I = 0$ and

$$\boxed{\int_C x^2 y \cos z \,\mathrm{d}s = 0.}$$

Part (iv)(1), Method 1: Integrate f_x with respect to x

Beyond 2026 midterm scope.

Test for a potential. Write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ with $P = y$, $Q = x + 10y$. Both have continuous partial derivatives on $\mathbb{R}^2$, which is open and simply connected, and

$$\frac{\partial P}{\partial y} = 1, \qquad \frac{\partial Q}{\partial x} = 1.$$

Since $P_y = Q_x$ on $\mathbb{R}^2$, $\mathbf{F}$ is conservative, so a potential f exists.

Finding f. We need $f_x = y$ and $f_y = x + 10y$. Integrate the first with respect to x (y held fixed):

$$f(x, y) = \int y \, \mathrm{d}x = xy + h(y),$$

where h depends on y only. Differentiate with respect to y and compare with Q:

$$f_y = x + h'(y) = x + 10y \implies h'(y) = 10y \implies h(y) = 5y^2 + K.$$

Taking $K = 0$,

$$\boxed{f(x, y) = xy + 5y^2.}$$

Check: $f_x = y = P$ and $f_y = x + 10y = Q$.

Part (iv)(1), Method 2: Integrate f_y with respect to y

Integrate $f_y = x + 10y$ with respect to y (x held fixed):

$$f(x, y) = \int (x + 10y) \, \mathrm{d}y = xy + 5y^2 + k(x).$$

Differentiate with respect to x and compare with P:

$$f_x = y + k'(x) = y \implies k'(x) = 0 \implies k(x) = K.$$

Taking $K = 0$,

$$\boxed{f(x, y) = xy + 5y^2.}$$

Part (iv)(1), Method 3: Line integral from the origin

Since $\mathbf{F}$ is conservative, $f(x, y) = \int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ along any path from $(0, 0)$ to (x, y) is a potential. Use the path along the x-axis from $(0, 0)$ to $(x, 0)$, then vertically to (x, y):

$$\begin{aligned} f(x, y) &= \int_0^x P(t, 0) \, \mathrm{d}t + \int_0^y Q(x, s) \, \mathrm{d}s = \int_0^x 0 \, \mathrm{d}t + \int_0^y (x + 10s) \, \mathrm{d}s \\ &= 0 + \left[xs + 5s^2\right]_{s=0}^{s=y} = xy + 5y^2. \end{aligned}$$

Hence

$$\boxed{f(x, y) = xy + 5y^2.}$$

Part (iv)(2), Method 1: Fundamental Theorem for Line Integrals

By part (1), $\mathbf{F} = \nabla f$ with $f(x, y) = xy + 5y^2$, and f has continuous partial derivatives. C is a piecewise smooth curve from $(1, 2)$ to $(5, 2)$. The Fundamental Theorem for Line Integrals gives

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(5, 2) - f(1, 2).$$

Evaluate:

$$f(5, 2) = 5 \cdot 2 + 5 \cdot 4 = 10 + 20 = 30, \qquad f(1, 2) = 1 \cdot 2 + 5 \cdot 4 = 2 + 20 = 22.$$

Hence $30 - 22 = 8$, and

$$\boxed{\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 8.}$$

Part (iv)(2), Method 2: Direct parametrisation of the semicircle

The endpoints $(1, 2)$ and $(5, 2)$ are the ends of a diameter, so the circle has centre $(3, 2)$ and radius 2. The upper semicircle from $(1, 2)$ to $(5, 2)$ is

$$\mathbf{r}(t) = (3 - 2\cos t,\ 2 + 2\sin t), \qquad 0 \le t \le \pi,$$

since $\mathbf{r}(0) = (1, 2)$, $\mathbf{r}(\pi/2) = (3, 4)$ (upper half) and $\mathbf{r}(\pi) = (5, 2)$. Then

$$\mathbf{r}'(t) = (2\sin t,\ 2\cos t).$$

On C,

$$P = y = 2 + 2\sin t, \qquad Q = x + 10y = 3 - 2\cos t + 20 + 20\sin t = 23 - 2\cos t + 20\sin t.$$

So

$$\begin{aligned}\mathbf{F} \cdot \mathbf{r}'(t) &= (2 + 2\sin t)(2\sin t) + (23 - 2\cos t + 20\sin t)(2\cos t) \\ &= 4\sin t + 4\sin^2 t + 46\cos t - 4\cos^2 t + 40\sin t\cos t.\end{aligned}$$

Integrate each term over $[0, \pi]$:

$$\int_0^\pi 4\sin t\,\mathrm{d}t = 4\left[-\cos t\right]_0^\pi = 8, \qquad \int_0^\pi 46\cos t\,\mathrm{d}t = 46\left[\sin t\right]_0^\pi = 0,$$

$$\int_0^\pi 4\sin^2 t\,\mathrm{d}t = 2\left[t - \frac{\sin 2t}{2}\right]_0^\pi = 2\pi, \qquad \int_0^\pi 4\cos^2 t\,\mathrm{d}t = 2\left[t + \frac{\sin 2t}{2}\right]_0^\pi = 2\pi,$$

$$\int_0^\pi 40\sin t\cos t\,\mathrm{d}t = 20\left[\sin^2 t\right]_0^\pi = 0.$$

Therefore

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 8 + 2\pi + 0 - 2\pi + 0 = 8,$$

and

$$\boxed{\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 8.}$$

Part (iv)(2), Method 3: Path independence with a straight segment

Since $\mathbf{F}$ is conservative on $\mathbb{R}^2$ (part (1)), the line integral depends only on the endpoints. Replace C by the segment

$$\mathbf{r}(x) = (x, 2), \qquad 1 \le x \le 5, \qquad \mathbf{r}'(x) = (1, 0).$$

On it $\mathbf{F} = (2,\, x + 20)$, so $\mathbf{F} \cdot \mathbf{r}'(x) = 2$ and

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_1^5 2\,\mathrm{d}x = [2x]_1^5 = 10 - 2 = 8.$$

Hence

$$\boxed{\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 8.}$$

Unofficial QRS marking scheme (34 marks)

(i) **(8 marks)** 3 marks for a valid parametrisation; 2 for $\mathrm{d}s = 6\sqrt{2}\,\mathrm{d}t$; 2 for integration; 1 for $30\sqrt{2}$.

(ii) **(8 marks)** 2 marks for parametrising with y; 3 for $\mathbf{F} \cdot \mathrm{d}\mathbf{r}$; 2 for integration; 1 for 26.

(iii) **(8 marks)** 2 marks for the parametrisation; 2 for $\mathrm{d}s = \sqrt{2}\,\mathrm{d}t$; 3 for the scalar line integral; 1 for 0.

(iv) **(10 marks)**

(1) 5 marks: 3 marks for integrating f_x; 2 marks for determining $h(y)$.

(2) 5 marks: 2 marks for the endpoints; 2 marks for applying the potential-difference theorem; 1 mark for 8.

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