# MH2100 Calculus III

## Midterm I (Solutions)

AY 2021/2022, Semester 1

*Quantitative Research Society @NTU*

September 2021

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**Overview**

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- **Coverage:** parametric curves and tangent lines; limits, continuity and partial derivatives; directional derivatives, tangent planes, the Chain Rule, implicit differentiation, and critical points.
- **Format:** three questions, 50 marks in total, 90 minutes (official paper: September 2021).
- **Historical permitted material:** the September 2021 paper permitted one two-sided A4 piece of paper with handwritten notes. Consult current course materials for present assessment rules.
- **Source:** MH2100 Midterm I, AY2021/2022 Semester 1. Questions are transcribed from the official paper; solutions and marking schemes are unofficial QRS@NTU material.
- **Question themes:** Q1 tangent lines to parametric curves and directional derivatives; Q2 two-variable limits, continuity, partial derivatives and differentiability; Q3 a directional derivative from the definition, a tangent plane, the Chain Rule, implicit differentiation and the classification of critical points.

# Instructions (verbatim)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. ONE two-sided A4 piece of paper with handwritten notes is permitted.

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# Contents

**Question 1 (17 Marks) 3**
Solution . . . 3
Method 1: Tangent vectors, the slope formula and the gradient dot product . . . 3
Method 2: First-order expansion about the point . . . 6
Method 3: Differentiating the dot product and completing the square . . . 8
Method 4: Angle parametrisation of the unit direction . . . 8
Method 5: Quarter-turn rotation of the gradient . . . 9
Unofficial QRS marking scheme (17 marks) . . . 9

**Question 2 (16 Marks) 10**
Solution . . . 10
Method 1: Continuity of quotients, path test and squeeze bounds . . . 10
Method 2: Polar coordinates centred at the limit point . . . 13
Method 3: Splitting off a squeezable term, a one-parameter curve family, and axis restrictions . . . 14
Method 4: Linearisation error from the definition of differentiability . . . 14
Unofficial QRS marking scheme (16 marks) . . . 15

**Question 3 (17 Marks) 16**
Solution . . . 16
Method 1: Definitions and standard formulas . . . 16
Method 2: Substitute first, then differentiate . . . 19
Method 3: Increment expansion about the point . . . 20
Method 4: Gradient of the level-surface function as a normal vector . . . 21
Unofficial QRS marking scheme (17 marks) . . . 21

Unofficial QRS marking scheme

# Question 1 (17 Marks)

**QUESTION 1.**

(i) [3 marks] Find parametric equations for the line that is tangent to the curve parametrised by

$$\mathbf{r}(t) = \begin{pmatrix} 5\sin t \\ t^3 \\ 4\mathrm{e}^{3t} \end{pmatrix},$$

at the point $(0, 0, 4)$.

(ii) [3 marks] The cartesian equation $y = ax + b$ describes the line tangent to the curve

$$\begin{cases} x(t) = \sin(2t), \\ y(t) = t\cos t \end{cases}$$

at the point defined by $t = \pi/2$. Find the values of $a$ and $b$.

(iii) [3 marks] At the point defined by $t = a$ the tangent vector of the curve parametrised by

$$\mathbf{r}(t) = \begin{pmatrix} \mathrm{e}^{\sin t}\tan t \\ t^2 - 10t \\ 23 \end{pmatrix},$$

is othogonal to the vector $\langle 0, 1, 2\rangle$. Find the value of $a$.

(iv) [3 marks] Find the directional derivative of the function

$$f(x, y, z) = x^3 + y^2 - 4z^2 + z\ln x$$

in the direction of the vector $\langle 1, 4, 1\rangle$.

(v) [5 marks] In which directions is the directional derivative of

$$f(x, y) = x^2y - y^8 + 2x$$

at $(6, 1)$ equal to zero? Write the unit vector for each direction.

## Solution

**Method 1: Tangent vectors, the slope formula and the gradient dot product**

(i) First find the parameter value of the point $(0, 0, 4)$. The second component gives

$$t^3 = 0 \implies t = 0.$$

The other two components at $t = 0$ are

$$5\sin 0 = 0, \qquad 4\mathrm{e}^{3\cdot 0} = 4,$$

so $\mathbf{r}(0) = \langle 0, 0, 4\rangle$ and the point corresponds to $t = 0$.

Differentiate component by component:

$$\frac{\mathrm{d}}{\mathrm{d}t}(5\sin t) = 5\cos t, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(t^3\right) = 3t^2, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(4\mathrm{e}^{3t}\right) = 4\cdot 3\mathrm{e}^{3t} = 12\mathrm{e}^{3t}.$$

Hence

$$\mathbf{r}'(t) = \langle 5\cos t,\ 3t^2,\ 12\mathrm{e}^{3t}\rangle,$$

and at $t = 0$

$$\mathbf{r}'(0) = \langle 5\cos 0,\ 3\cdot 0^2,\ 12\mathrm{e}^{0}\rangle = \langle 5, 0, 12\rangle \neq \mathbf{0}.$$

Since $\mathbf{r}'(0) \neq \mathbf{0}$, it is a direction vector of the tangent line. The tangent line passes through $\mathbf{r}(0) = \langle 0, 0, 4\rangle$ with direction $\mathbf{r}'(0)$:

$$\boxed{\mathbf{l}(s) = \langle 0, 0, 4\rangle + s\langle 5, 0, 12\rangle, \quad s \in \mathbb{R},} \qquad \boxed{x = 5s,\ y = 0,\ z = 4 + 12s.}$$

(ii) The point defined by $t = \pi/2$ is

$$x\left(\tfrac{\pi}{2}\right) = \sin\pi = 0, \qquad y\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2}\cos\tfrac{\pi}{2} = \tfrac{\pi}{2}\cdot 0 = 0,$$

that is, the origin $(0, 0)$. Differentiate each coordinate (Chain Rule for $x$, Product Rule for $y$):

$$x'(t) = 2\cos(2t), \qquad y'(t) = \cos t - t\sin t.$$

At $t = \pi/2$,

$$x'\left(\tfrac{\pi}{2}\right) = 2\cos\pi = -2, \qquad y'\left(\tfrac{\pi}{2}\right) = \cos\tfrac{\pi}{2} - \tfrac{\pi}{2}\sin\tfrac{\pi}{2} = 0 - \tfrac{\pi}{2} = -\tfrac{\pi}{2}.$$

Since $x'(\pi/2) = -2 \neq 0$, the slope of the tangent line is

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\pi/2} = \frac{y'(\pi/2)}{x'(\pi/2)} = \frac{-\pi/2}{-2} = \frac{\pi}{4}.$$

The tangent line through $(0, 0)$ with slope $\pi/4$ is

$$y - 0 = \frac{\pi}{4}(x - 0) \iff y = \frac{\pi}{4}x + 0.$$

Comparing with $y = ax + b$:

$$\boxed{a = \frac{\pi}{4}, \qquad b = 0.}$$

(iii) Differentiate the first component with the Product Rule and the Chain Rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\sin t}\tan t\right) = \mathrm{e}^{\sin t}\cos t\cdot\tan t + \mathrm{e}^{\sin t}\sec^2 t = \mathrm{e}^{\sin t}\left(\sin t + \sec^2 t\right).$$

The other two components give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^2 - 10t\right) = 2t - 10, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(23) = 0.$$

Hence

$$\mathbf{r}'(t) = \left\langle \mathrm{e}^{\sin t}\left(\sin t + \sec^2 t\right),\ 2t - 10,\ 0\right\rangle.$$

Orthogonality to $\langle 0, 1, 2\rangle$ means a zero dot product:

$$\mathbf{r}'(t)\cdot\langle 0, 1, 2\rangle = 0\cdot\mathrm{e}^{\sin t}\left(\sin t + \sec^2 t\right) + 1\cdot(2t - 10) + 2\cdot 0 = 2t - 10.$$

Setting this equal to 0 at $t = a$:

$$2a - 10 = 0 \implies a = 5.$$

The tangent vector exists at $t = 5$, because $\cos 5 \neq 0$ ($\frac{3\pi}{2} < 5 < \frac{5\pi}{2}$). It is non-zero, because $\sec^2 5 > 1$ and $\sin 5 \geq -1$ give $\sin 5 + \sec^2 5 > 0$. Hence

$$\boxed{a = 5.}$$

(iv) Compute the three partial derivatives (the term $z\ln x$ contributes $z/x$ to $f_x$ and $\ln x$ to $f_z$):

$$f_x = 3x^2 + \frac{z}{x}, \qquad f_y = 2y, \qquad f_z = -8z + \ln x.$$

These are continuous on the domain $x > 0$, so $f$ is differentiable there and $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$ for a unit vector $\mathbf{u}$. Normalise the direction:

$$|\langle 1, 4, 1\rangle| = \sqrt{1^2 + 4^2 + 1^2} = \sqrt{18}, \qquad \mathbf{u} = \frac{\langle 1, 4, 1\rangle}{\sqrt{18}}.$$

Then

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{1 \cdot \left(3x^2 + \frac{z}{x}\right) + 4 \cdot 2y + 1 \cdot (-8z + \ln x)}{\sqrt{18}},$$

that is,

$$\boxed{D_{\mathbf{u}}f(x, y, z) = \frac{3x^2 + z/x + 8y - 8z + \ln x}{\sqrt{18}} \quad (x > 0).}$$

(v) Compute the partial derivatives:

$$f_x = 2xy + 2, \qquad f_y = x^2 - 8y^7.$$

At $(6, 1)$:

$$f_x(6, 1) = 2 \cdot 6 \cdot 1 + 2 = 14, \qquad f_y(6, 1) = 6^2 - 8 \cdot 1^7 = 36 - 8 = 28,$$

so

$$\nabla f(6, 1) = \langle 14, 28\rangle = 14\langle 1, 2\rangle.$$

Since $f$ is a polynomial, it is differentiable, and for a unit vector $\mathbf{u} = \langle u_1, u_2\rangle$

$$D_{\mathbf{u}}f(6, 1) = \nabla f(6, 1) \cdot \mathbf{u} = 14u_1 + 28u_2 = 14(u_1 + 2u_2).$$

Hence

$$D_{\mathbf{u}}f(6, 1) = 0 \iff u_1 + 2u_2 = 0 \iff u_1 = -2u_2.$$

Substitute into the unit condition $u_1^2 + u_2^2 = 1$:

$$(-2u_2)^2 + u_2^2 = 5u_2^2 = 1 \implies u_2 = \pm\frac{1}{\sqrt{5}}.$$

For $u_2 = -\frac{1}{\sqrt{5}}$, $u_1 = \frac{2}{\sqrt{5}}$; for $u_2 = \frac{1}{\sqrt{5}}$, $u_1 = -\frac{2}{\sqrt{5}}$. Therefore

$$\boxed{\mathbf{u} = \frac{1}{\sqrt{5}}\langle 2, -1\rangle \quad \text{or} \quad \mathbf{u} = \frac{1}{\sqrt{5}}\langle -2, 1\rangle.}$$

**Method 2: First-order expansion about the point**

This method applies to parts (i), (ii) and (iv). The tangent line, or the directional derivative, is the part of the expansion that is of first order in the increment of the parameter.

(i) The point $(0,0,4)$ corresponds to $t=0$, since $t^3=0$ forces $t=0$ and

$$\mathbf{r}(0) = \langle 5\sin 0,\ 0,\ 4\mathrm{e}^0\rangle = \langle 0,0,4\rangle.$$

Recall the Maclaurin series

$$\sin t = t - \frac{t^3}{6} + \cdots, \qquad \mathrm{e}^w = 1 + w + \frac{w^2}{2} + \cdots.$$

With $w = 3t$,

$$\begin{aligned} 5\sin t &= 5t - \frac{5}{6}t^3 + \cdots = 0 + 5t + O(t^3), \\ t^3 &= 0 + 0\cdot t + O(t^3), \\ 4\mathrm{e}^{3t} &= 4\left(1 + 3t + \frac{9}{2}t^2 + \cdots\right) = 4 + 12t + O(t^2). \end{aligned}$$

Hence

$$\mathbf{r}(t) = \langle 0,0,4\rangle + t\langle 5,0,12\rangle + O(t^2).$$

The tangent line is the first-order part. Writing $s$ for the parameter of the line,

$$\boxed{\mathbf{l}(s) = \langle 0,0,4\rangle + s\langle 5,0,12\rangle, \quad s\in\mathbb{R},} \qquad \boxed{x = 5s,\ y = 0,\ z = 4 + 12s.}$$

(ii) Let $t = \frac{\pi}{2} + s$. Then

$$x = \sin(2t) = \sin(\pi + 2s) = -\sin(2s) = -2s + O(s^3),$$

and, using $\cos\left(\frac{\pi}{2} + s\right) = -\sin s$,

$$y = t\cos t = -\left(\tfrac{\pi}{2} + s\right)\sin s = -\left(\tfrac{\pi}{2} + s\right)\left(s + O(s^3)\right) = -\tfrac{\pi}{2}s + O(s^2).$$

The first-order part is the line

$$(x,y) = s\left\langle -2, -\tfrac{\pi}{2}\right\rangle, \qquad s\in\mathbb{R},$$

which passes through the origin. Eliminate $s$: from $x = -2s$, $s = -x/2$, so

$$y = -\frac{\pi}{2}\left(-\frac{x}{2}\right) = \frac{\pi}{4}x.$$

Comparing with $y = ax + b$:

$$\boxed{a = \frac{\pi}{4}, \qquad b = 0.}$$

(iv) Fix $(x,y,z)$ with $x > 0$ and let $\mathbf{u} = \langle 1,4,1\rangle/\sqrt{18}$ (since $|\langle 1,4,1\rangle| = \sqrt{18}$). Put $\delta = h/\sqrt{18}$, so that

$$(x,y,z) + h\mathbf{u} = (x+\delta,\ y+4\delta,\ z+\delta).$$

Expand each term of $f$:

$$(x+\delta)^3 = x^3 + 3x^2\delta + 3x\delta^2 + \delta^3 = x^3 + 3x^2\delta + O(\delta^2),$$

$$(y+4\delta)^2 = y^2 + 8y\delta + 16\delta^2 = y^2 + 8y\delta + O(\delta^2),$$
$$-4(z+\delta)^2 = -4z^2 - 8z\delta - 4\delta^2 = -4z^2 - 8z\delta + O(\delta^2).$$

For the logarithm, $\ln(x+\delta) = \ln x + \ln\left(1+\frac{\delta}{x}\right) = \ln x + \frac{\delta}{x} + O(\delta^2)$, so

$$(z+\delta)\ln(x+\delta) = z\ln x + \delta\left(\ln x + \frac{z}{x}\right) + O(\delta^2).$$

Adding the four expansions,

$$f\left((x,y,z)+h\mathbf{u}\right) = f(x,y,z) + \delta\left(3x^2 + \frac{z}{x} + 8y - 8z + \ln x\right) + O(\delta^2).$$

By the definition of the directional derivative, with $\delta/h = 1/\sqrt{18}$ and $O(\delta^2)/h \to 0$,

$$D_{\mathbf{u}}f(x,y,z) = \lim_{h\to 0}\frac{f\left((x,y,z)+h\mathbf{u}\right) - f(x,y,z)}{h} = \frac{1}{\sqrt{18}}\left(3x^2 + \frac{z}{x} + 8y - 8z + \ln x\right),$$

that is,

$$\boxed{D_{\mathbf{u}}f(x,y,z) = \frac{3x^2 + z/x + 8y - 8z + \ln x}{\sqrt{18}} \quad (x>0).}$$

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**Method 3: Differentiating the dot product and completing the square**

This method applies to part (iii). Let $\mathbf{n} = \langle 0, 1, 2\rangle$ and

$$g(t) = \mathbf{r}(t) \cdot \mathbf{n} = 0 \cdot \mathrm{e}^{\sin t} \tan t + 1 \cdot \left(t^2 - 10t\right) + 2 \cdot 23 = t^2 - 10t + 46.$$

Since $\mathbf{n}$ is constant, the Product Rule for dot products gives

$$g'(t) = \mathbf{r}'(t) \cdot \mathbf{n} + \mathbf{r}(t) \cdot \mathbf{0} = \mathbf{r}'(t) \cdot \mathbf{n}.$$

So the tangent vector at $t = a$ is orthogonal to $\mathbf{n}$ exactly when $g'(a) = 0$. Complete the square:

$$g(t) = t^2 - 10t + 25 + 21 = (t - 5)^2 + 21.$$

Then

$$g'(t) = 2(t - 5) = 0 \iff t = 5,$$

the vertex of the parabola. The tangent vector at $t = 5$ exists and is non-zero: $\cos 5 \neq 0$ because $\frac{3\pi}{2} < 5 < \frac{5\pi}{2}$, and its first component $\mathrm{e}^{\sin 5}(\sin 5 + \sec^2 5)$ is positive because $\sec^2 5 > 1 \geq -\sin 5$. Hence

$$\boxed{a = 5.}$$

**Method 4: Angle parametrisation of the unit direction**

This method applies to part (v). The partial derivatives $f_x = 2xy + 2$ and $f_y = x^2 - 8y^7$ give

$$\nabla f(6, 1) = \langle 2 \cdot 6 \cdot 1 + 2,\ 6^2 - 8 \cdot 1^7\rangle = \langle 14, 28\rangle.$$

Every unit vector in $\mathbb{R}^2$ is $\mathbf{u} = \langle \cos\theta, \sin\theta\rangle$ for some $\theta \in [0, 2\pi)$. Since $f$ is a polynomial, it is differentiable, and

$$D_{\mathbf{u}} f(6, 1) = 14\cos\theta + 28\sin\theta = 14(\cos\theta + 2\sin\theta).$$

Setting this equal to zero gives $\cos\theta = -2\sin\theta$. If $\sin\theta = 0$ this would force $\cos\theta = 0$ as well, which is impossible; so $\sin\theta \neq 0$, $\cos\theta \neq 0$ and

$$\tan\theta = -\frac{1}{2}.$$

Then

$$\cos^2\theta = \frac{1}{1 + \tan^2\theta} = \frac{1}{1 + \frac{1}{4}} = \frac{4}{5} \implies \cos\theta = \pm\frac{2}{\sqrt{5}}, \qquad \sin\theta = -\frac{\cos\theta}{2} = \mp\frac{1}{\sqrt{5}}.$$

The two angles (one in the fourth quadrant, one in the second) give

$$\boxed{\mathbf{u} = \frac{1}{\sqrt{5}}\langle 2, -1\rangle \quad \text{or} \quad \mathbf{u} = \frac{1}{\sqrt{5}}\langle -2, 1\rangle.}$$

**Method 5: Quarter-turn rotation of the gradient**

This method applies to part (v). As in Method 4, $f_x(6,1) = 2 \cdot 6 \cdot 1 + 2 = 14$ and $f_y(6,1) = 6^2 - 8 \cdot 1^7 = 28$, so $\nabla f(6,1) = \langle 14, 28 \rangle$. For a non-zero vector $\langle p, q \rangle \in \mathbb{R}^2$,

$$\langle p, q \rangle \cdot \langle q, -p \rangle = pq - qp = 0,$$

and the vectors orthogonal to $\langle p, q \rangle$ form a line in $\mathbb{R}^2$, so they are exactly the multiples of $\langle q, -p \rangle$. With $\langle p, q \rangle = \langle 14, 28 \rangle$:

$$\langle 28, -14 \rangle = 14\langle 2, -1 \rangle, \qquad |\langle 2, -1 \rangle| = \sqrt{4+1} = \sqrt{5}.$$

The unit vectors on this line are $\pm\langle 2, -1 \rangle / \sqrt{5}$, and $D_{\mathbf{u}} f(6,1) = \nabla f(6,1) \cdot \mathbf{u} = 0$ exactly for these:

$$\boxed{\mathbf{u} = \frac{1}{\sqrt{5}}\langle 2, -1 \rangle \quad \text{or} \quad \mathbf{u} = \frac{1}{\sqrt{5}}\langle -2, 1 \rangle.}$$

**Unofficial QRS marking scheme (17 marks)**

(i) **(3 marks)** 1: identifies $t = 0$; 1: obtains $\mathbf{r}'(0) = \langle 5, 0, 12 \rangle$; 1: gives the line through $(0, 0, 4)$.

(ii) **(3 marks)** 1: finds the point $(0, 0)$; 1: computes slope $\pi/4$; 1: gives $a = \pi/4$, $b = 0$.

(iii) **(3 marks)** 1: differentiates the relevant components; 1: imposes orthogonality; 1: obtains $a = 5$.

(iv) **(3 marks)** 1: computes the gradient; 1: normalizes the direction; 1: takes the correct dot product.

(v) **(5 marks)** 2: computes $\nabla f(6,1) = \langle 14, 28 \rangle$; 1: states orthogonality; 1: obtains one correct unit direction; 1: obtains the opposite direction.

Unofficial QRS marking scheme

# Question 2 (16 Marks)

**QUESTION 2.**

(i) [8 marks] Evaluate each of the following limits or show that the limit does not exist. Justify your answers.

(a)
$$\lim_{(x,y)\to(5,0)} \frac{5x^2y+10}{\sqrt{x^2+y^2}}$$

(b)
$$\lim_{(x,y)\to(0,0)} \frac{3y}{\sqrt{x^2+y^2}}$$

(c)
$$\lim_{(x,y)\to(0,0)} \frac{3x^2+x^2y+xy^3+3y^2}{x^2+y^2}$$

(ii) [8 marks] Define
$$f(x,y)=\begin{cases}\dfrac{x^3y+y^3}{x^6+y^2}; & \text{if } x^2+y^2\neq 0,\\ 0; & \text{if } x^2+y^2=0.\end{cases}$$

(a) Is $f$ continuous at $(0,0)$? Justify your answer.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

(c) Is $f$ differentiable at $(0,0)$? Justify your answer.

## Solution

**Method 1: Continuity of quotients, path test and squeeze bounds**

(i) (a) The numerator $N(x,y)=5x^2y+10$ is a polynomial, so it is continuous on $\mathbb{R}^2$. The denominator $D(x,y)=\sqrt{x^2+y^2}$ is the composition of the polynomial $x^2+y^2$ with the square root, so it is continuous on $\mathbb{R}^2$. At $(5,0)$,
$$N(5,0)=5\cdot 5^2\cdot 0+10=10, \qquad D(5,0)=\sqrt{5^2+0^2}=5\neq 0.$$
A quotient of continuous functions is continuous wherever the denominator is nonzero, so $N/D$ is continuous at $(5,0)$ and its limit there equals its value:
$$\boxed{\lim_{(x,y)\to(5,0)} \frac{5x^2y+10}{\sqrt{x^2+y^2}}=\frac{10}{5}=2.}$$

(b) Let $g(x,y)=\dfrac{3y}{\sqrt{x^2+y^2}}$ for $(x,y)\neq(0,0)$. Approach $(0,0)$ along two paths.
Path 1, the $x$-axis ($y=0$, $x\neq 0$):
$$g(x,0)=\frac{3\cdot 0}{\sqrt{x^2}}=0 \quad \text{for every } x\neq 0, \qquad \text{so} \qquad \lim_{x\to 0} g(x,0)=0.$$
Path 2, the positive $y$-axis ($x=0$, $y>0$), where $\sqrt{y^2}=|y|=y$:
$$g(0,y)=\frac{3y}{\sqrt{y^2}}=\frac{3y}{y}=3 \quad \text{for every } y>0, \qquad \text{so} \qquad \lim_{y\to 0^+} g(0,y)=3.$$

(On the negative $y$-axis, $g(0,y) = 3y/(-y) = -3$.) Two paths give the different limits 0 and 3, so

$$\boxed{\text{the limit does not exist.}}$$

(c) Split the numerator as $3x^2 + 3y^2 = 3(x^2+y^2)$ plus the remaining two terms. For $(x,y) \neq (0,0)$,

$$\frac{3x^2 + x^2y + xy^3 + 3y^2}{x^2+y^2} = \frac{3(x^2+y^2)}{x^2+y^2} + \frac{x^2y}{x^2+y^2} + \frac{xy^3}{x^2+y^2} = 3 + \frac{x^2y}{x^2+y^2} + \frac{xy^3}{x^2+y^2}.$$

For the first remaining term, $x^2 \leq x^2+y^2$ gives $\dfrac{x^2}{x^2+y^2} \leq 1$, so

$$\left|\frac{x^2y}{x^2+y^2}\right| = \frac{x^2}{x^2+y^2}\,|y| \leq |y|.$$

For the second, let $r = \sqrt{x^2+y^2}$. Then $|x| \leq r$ and $|y| \leq r$, so $|xy^3| \leq r \cdot r^3 = r^4$ and

$$\left|\frac{xy^3}{x^2+y^2}\right| \leq \frac{r^4}{r^2} = r^2.$$

As $(x,y) \to (0,0)$, both $|y| \to 0$ and $r^2 \to 0$, so by the Squeeze Theorem both terms tend to 0. Therefore the limit is $3+0+0$:

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{3x^2 + x^2y + xy^3 + 3y^2}{x^2+y^2} = 3.}$$

(ii) (a) By definition $f(0,0) = 0$, so $f$ is continuous at $(0,0)$ only if $f(x,y) \to 0$ as $(x,y) \to (0,0)$. Test two paths.
Path 1, the $y$-axis ($x = 0$, $y \neq 0$):

$$f(0,y) = \frac{0^3 \cdot y + y^3}{0^6 + y^2} = \frac{y^3}{y^2} = y \to 0 \quad (y \to 0).$$

Path 2, the curve $y = x^3$ ($x \neq 0$). The numerator is $x^3 \cdot x^3 + (x^3)^3 = x^6 + x^9$ and the denominator is $x^6 + (x^3)^2 = 2x^6$, so

$$f(x,x^3) = \frac{x^6 + x^9}{2x^6} = \frac{1+x^3}{2} \to \frac{1}{2} \quad (x \to 0).$$

The two path limits 0 and $\frac{1}{2}$ differ, so $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist; in particular it is not $f(0,0) = 0$. Hence

$$\boxed{f \text{ is not continuous at } (0,0).}$$

(b) For $h \neq 0$, $f(h,0) = \dfrac{h^3 \cdot 0 + 0^3}{h^6 + 0^2} = 0$. By definition,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

For $k \neq 0$, $f(0,k) = \dfrac{0+k^3}{0+k^2} = k$. By definition,

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k-0}{k} = \lim_{k\to 0} 1 = 1.$$

Hence

$$\boxed{f_x(0,0) = 0, \qquad f_y(0,0) = 1.}$$

(c) A function that is differentiable at a point is continuous at that point. By the contrapositive, since part (a) shows that $f$ is not continuous at $(0,0)$,

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

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**Method 2: Polar coordinates centred at the limit point**

This method applies to part (i).

(i) (a) Write $x = 5 + r\cos\theta$, $y = r\sin\theta$ with $r > 0$, so that $(x, y) \to (5, 0)$ exactly when $r \to 0$, whatever $\theta$ does. Let $\rho = \sqrt{x^2 + y^2}$. By the triangle inequality,

$$|\rho - 5| = \big|\,|(x, y)| - |(5, 0)|\,\big| \le |(x, y) - (5, 0)| = r.$$

For $r < 1$ this gives

$$\rho \ge 5 - r > 4, \qquad |x| \le 5 + r < 6, \qquad |y| = r|\sin\theta| \le r.$$

Now

$$\frac{5x^2y + 10}{\rho} - 2 = \frac{5x^2y + 10 - 2\rho}{\rho} = \frac{5x^2y + 2(5 - \rho)}{\rho},$$

so, for $r < 1$,

$$\left|\frac{5x^2y + 10}{\rho} - 2\right| \le \frac{5 \cdot 36 \cdot r + 2r}{4} = \frac{182r}{4} = \frac{91r}{2}.$$

The bound does not depend on $\theta$ and tends to 0 as $r \to 0$, so by the Squeeze Theorem

$$\boxed{\lim_{(x,y)\to(5,0)} \frac{5x^2y + 10}{\sqrt{x^2 + y^2}} = 2.}$$

(b) Write $x = r\cos\theta$, $y = r\sin\theta$ with $r > 0$. Then $\sqrt{x^2 + y^2} = r$ and

$$\frac{3y}{\sqrt{x^2 + y^2}} = \frac{3r\sin\theta}{r} = 3\sin\theta.$$

The value depends on the direction only. Along $\theta = 0$ (positive $x$-axis) it equals $3\sin 0 = 0$ for every $r$, and along $\theta = \pi/2$ (positive $y$-axis) it equals $3\sin\frac{\pi}{2} = 3$ for every $r$. Letting $r \to 0$ gives 0 and 3 respectively, so

$$\boxed{\text{the limit does not exist.}}$$

(c) Write $x = r\cos\theta$, $y = r\sin\theta$ with $r > 0$. Then $x^2 + y^2 = r^2$ and

$$3x^2 + 3y^2 = 3r^2, \qquad x^2y = r^3\cos^2\theta\sin\theta, \qquad xy^3 = r^4\cos\theta\sin^3\theta.$$

Hence

$$\begin{aligned}\frac{3x^2 + x^2y + xy^3 + 3y^2}{x^2 + y^2} &= \frac{3r^2 + r^3\cos^2\theta\sin\theta + r^4\cos\theta\sin^3\theta}{r^2} \\ &= 3 + r\cos^2\theta\sin\theta + r^2\cos\theta\sin^3\theta.\end{aligned}$$

Each trigonometric product has absolute value at most 1, so

$$\left|\frac{3x^2 + x^2y + xy^3 + 3y^2}{x^2 + y^2} - 3\right| \le r + r^2.$$

The bound does not depend on $\theta$ and tends to 0 as $r \to 0$, so by the Squeeze Theorem

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{3x^2 + x^2y + xy^3 + 3y^2}{x^2 + y^2} = 3.}$$

**Method 3: Splitting off a squeezable term, a one-parameter curve family, and axis restrictions**

This method applies to parts (ii)(a) and (ii)(b).

(ii) (a) For $(x, y) \neq (0, 0)$ split

$$f(x,y) = \underbrace{\frac{x^3 y}{x^6 + y^2}}_{g(x,y)} + \frac{y^3}{x^6 + y^2}.$$

Since $y^2 \leq x^6 + y^2$,

$$\left| \frac{y^3}{x^6 + y^2} \right| = \frac{y^2}{x^6 + y^2} |y| \leq |y| \to 0,$$

so the second term tends to 0 by the Squeeze Theorem. Hence if $f$ had a limit $L$ at $(0, 0)$, then $g = f - \frac{y^3}{x^6+y^2}$ would have the limit $L$ as well. On the curve $y = mx^3$ ($x \neq 0$, $m \in \mathbb{R}$ fixed),

$$g(x, mx^3) = \frac{x^3 \cdot mx^3}{x^6 + m^2 x^6} = \frac{mx^6}{(1 + m^2)x^6} = \frac{m}{1 + m^2},$$

a constant along each curve. For $m = 0$ the value is 0, and for $m = 1$ it is $\frac{1}{2}$. So $g$ has no limit at $(0, 0)$, hence neither has $f$, and $f(x, y) \not\to f(0, 0)$:

$$\boxed{f \text{ is not continuous at } (0, 0).}$$

(b) Restrict $f$ to the axes. On the $x$-axis, $\varphi(x) = f(x, 0) = \dfrac{0}{x^6} = 0$ for $x \neq 0$ and $\varphi(0) = f(0, 0) = 0$, so $\varphi \equiv 0$ and

$$f_x(0, 0) = \varphi'(0) = 0.$$

On the $y$-axis, $\psi(y) = f(0, y) = \dfrac{y^3}{y^2} = y$ for $y \neq 0$ and $\psi(0) = f(0, 0) = 0$, so $\psi(y) = y$ for every $y$ and

$$f_y(0, 0) = \psi'(0) = 1.$$

Hence

$$\boxed{f_x(0, 0) = 0, \qquad f_y(0, 0) = 1.}$$

**Method 4: Linearisation error from the definition of differentiability**

This method applies to part (ii)(c). By definition, $f$ is differentiable at $(0, 0)$ if and only if

$$\lim_{(x,y)\to(0,0)} \frac{f(x, y) - f(0, 0) - f_x(0, 0)\, x - f_y(0, 0)\, y}{\sqrt{x^2 + y^2}} = 0.$$

With $f(0, 0) = 0$, $f_x(0, 0) = 0$ and $f_y(0, 0) = 1$ from part (b), the numerator is

$$E(x, y) = f(x, y) - 0 - 0 \cdot x - 1 \cdot y = f(x, y) - y.$$

Along $y = x^3$ with $x > 0$, $f(x, x^3) = \dfrac{x^6 + x^9}{2x^6} = \dfrac{1 + x^3}{2}$, so

$$E(x, x^3) = \frac{1 + x^3}{2} - x^3 = \frac{1 - x^3}{2}, \qquad \sqrt{x^2 + x^6} = x\sqrt{1 + x^4}.$$

Therefore

$$\frac{E(x,x^3)}{\sqrt{x^2+x^6}} = \frac{1-x^3}{2x\sqrt{1+x^4}} \to +\infty \quad (x \to 0^+),$$

which is not 0. Hence

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

**Unofficial QRS marking scheme (16 marks)**

(i) **(8 marks)**

(a) **(2 marks)** 1: continuity/substitution justification; 1: final value 2.

(b) **(3 marks)** 1: computes the $y = 0$ path; 1: uses $x = 0$ to obtain distinct one-sided values; 1: concludes nonexistence.

(c) **(3 marks)** 1: separates the constant term; 1: bounds both remaining terms; 1: concludes 3.

(ii) **(8 marks)**

(a) **(3 marks)** 1: path $x = 0$; 1: path $y = x^3$; 1: concludes non-continuity.

(b) **(3 marks)** 1: correct difference quotient for $f_x$; 1: correct difference quotient for $f_y$; 1: states both values.

(c) **(2 marks)**

- 1 mark: invokes differentiability implies continuity.
- 1 mark: concludes non-differentiability.

Unofficial QRS@NTU solutions

# Question 3 (17 Marks)

**QUESTION 3.**

(i) [3 marks] Find the directional derivative of the function

$$f(x,y) = \begin{cases} \dfrac{xy+y^2}{x^2+y^2}; & \text{if } x^2+y^2 \neq 0, \\ 0; & \text{if } x^2+y^2 = 0, \end{cases}$$

in the direction of the vector $\langle 3,4\rangle$ at the point $(0,0)$.

(ii) [3 marks] Find an equation for the tangent plane to the surface given by the equation $z = 5x^2 - 10y^2$ at the point $(2,1,10)$.

(iii) [3 marks] Use the Chain Rule to find $\dfrac{\partial z}{\partial u}$ when $u = 0$ and $v = 2$ for

$$z = \sin(xy) + \sin(y), \qquad x = 2u^2 + 3v^2, \qquad y = 2uv.$$

(iv) [3 marks] Find $\dfrac{\partial z}{\partial y}$ at the point $(\pi, 2\pi, \pi/2)$ where $z$ is defined implicitly by

$$3\cos(x+2y) + 3\cos(x+2z) + \cos(2y+z) = 0.$$

(v) [5 marks] Let

$$f(x,y) = 27x^3 - 81x + 2y^2 + 8y - 17.$$

Find and classify all critical points of $f$. (For each critical point, determine whether or not it is a local max/min or a saddle point.)

## Solution

### Method 1: Definitions and standard formulas

(i) Normalise the direction:

$$|\langle 3,4\rangle| = \sqrt{9+16} = 5, \qquad \mathbf{u} = \left\langle \tfrac{3}{5}, \tfrac{4}{5} \right\rangle.$$

The formula $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ requires $f$ to be differentiable at $(0,0)$, which is not known here, so use the definition

$$D_{\mathbf{u}} f(0,0) = \lim_{t \to 0} \frac{f(t\mathbf{u}) - f(0,0)}{t}.$$

For $t \neq 0$, the point $t\mathbf{u} = \left(\frac{3t}{5}, \frac{4t}{5}\right)$ is not the origin, and

$$xy = \frac{3t}{5} \cdot \frac{4t}{5} = \frac{12t^2}{25}, \qquad y^2 = \frac{16t^2}{25}, \qquad x^2 + y^2 = \frac{9t^2}{25} + \frac{16t^2}{25} = t^2.$$

Hence

$$f(t\mathbf{u}) = \frac{\frac{12t^2}{25} + \frac{16t^2}{25}}{t^2} = \frac{28}{25} \qquad (t \neq 0),$$

and the difference quotient is

$$\frac{f(t\mathbf{u}) - f(0,0)}{t} = \frac{\frac{28}{25} - 0}{t} = \frac{28}{25t}.$$

As $t \to 0^+$ this tends to $+\infty$, and as $t \to 0^-$ it tends to $-\infty$; it has no finite limit. Therefore

$$\boxed{D_{\mathbf{u}} f(0,0) \text{ does not exist.}}$$

(ii) The surface is the graph $z = g(x, y)$ with $g(x, y) = 5x^2 - 10y^2$. The point lies on it, since $g(2, 1) = 5 \cdot 4 - 10 \cdot 1 = 10$. The function $g$ is a polynomial, hence differentiable, so the tangent plane is

$$z - z_0 = g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0).$$

The partial derivatives are

$$g_x = 10x, \qquad g_y = -20y \implies g_x(2, 1) = 20, \qquad g_y(2, 1) = -20.$$

Substituting $(x_0, y_0, z_0) = (2, 1, 10)$:

$$z - 10 = 20(x - 2) - 20(y - 1) = 20x - 40 - 20y + 20 = 20x - 20y - 20.$$

So $z = 20x - 20y - 10$, that is,

$$\boxed{20x - 20y - z = 10.}$$

(iii) The function $z = \sin(xy) + \sin y$ has continuous partial derivatives in $(x, y)$, and $x = 2u^2 + 3v^2$, $y = 2uv$ are polynomials in $(u, v)$, so the Chain Rule applies:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}.$$

The four partial derivatives are

$$z_x = y\cos(xy), \qquad z_y = x\cos(xy) + \cos y, \qquad x_u = 4u, \qquad y_u = 2v.$$

At $(u, v) = (0, 2)$ the intermediate variables are

$$x = 2 \cdot 0^2 + 3 \cdot 2^2 = 12, \qquad y = 2 \cdot 0 \cdot 2 = 0, \qquad xy = 0,$$

so

$$z_x = 0 \cdot \cos 0 = 0, \qquad z_y = 12\cos 0 + \cos 0 = 13, \qquad x_u = 4 \cdot 0 = 0, \qquad y_u = 2 \cdot 2 = 4.$$

Hence

$$\boxed{\left.\frac{\partial z}{\partial u}\right|_{(u,v)=(0,2)} = 0 \cdot 0 + 13 \cdot 4 = 52.}$$

(iv) Let $F(x, y, z) = 3\cos(x + 2y) + 3\cos(x + 2z) + \cos(2y + z)$. The point lies on the surface $F = 0$:

$$F\left(\pi, 2\pi, \tfrac{\pi}{2}\right) = 3\cos 5\pi + 3\cos 2\pi + \cos\tfrac{9\pi}{2} = -3 + 3 + 0 = 0.$$

Compute the partial derivatives (the middle term does not contain $y$):

$$F_y = -3\sin(x + 2y) \cdot 2 - \sin(2y + z) \cdot 2 = -6\sin(x + 2y) - 2\sin(2y + z),$$
$$F_z = -3\sin(x + 2z) \cdot 2 - \sin(2y + z) \cdot 1 = -6\sin(x + 2z) - \sin(2y + z).$$

At $(\pi, 2\pi, \pi/2)$: $x + 2y = 5\pi$, $x + 2z = 2\pi$ and $2y + z = \frac{9\pi}{2} = 4\pi + \frac{\pi}{2}$, so

$$\sin 5\pi = 0, \qquad \sin 2\pi = 0, \qquad \sin\tfrac{9\pi}{2} = \sin\tfrac{\pi}{2} = 1,$$

and

$$F_y = -6 \cdot 0 - 2 \cdot 1 = -2, \qquad F_z = -6 \cdot 0 - 1 = -1.$$

Implicit Function Theorem: $F$ has continuous partial derivatives, $F(\pi, 2\pi, \pi/2) = 0$ and $F_z = -1 \neq 0$ there, so near this point $z$ is a differentiable function of $(x, y)$ with $z_y = -F_y/F_z$:

$$\boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -2.}$$

(v) Compute the first partial derivatives:

$$f_x = 81x^2 - 81 = 81(x^2 - 1), \qquad f_y = 4y + 8.$$

Since $f$ is a polynomial, the critical points are the solutions of $f_x = f_y = 0$:

$$81(x^2 - 1) = 0 \iff x = \pm 1, \qquad 4y + 8 = 0 \iff y = -2.$$

The critical points are $(1, -2)$ and $(-1, -2)$. The second partial derivatives are

$$f_{xx} = 162x, \qquad f_{xy} = f_{yx} = 0, \qquad f_{yy} = 4,$$

so

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 162x \cdot 4 - 0^2 = 648x.$$

The function $f$ has continuous second partial derivatives, so the Second Derivative Test applies.

- At $(1, -2)$: $D = 648 > 0$ and $f_{xx} = 162 > 0$, so $(1, -2)$ is a local minimum, with $f(1, -2) = 27 - 81 + 8 - 16 - 17 = -79$.
- At $(-1, -2)$: $D = -648 < 0$, so $(-1, -2)$ is a saddle point, with $f(-1, -2) = -27 + 81 + 8 - 16 - 17 = 29$.

Therefore

$$\boxed{(1, -2) \text{ is a local minimum; } (-1, -2) \text{ is a saddle point.}}$$

Unofficial QRS@NTU Solutions

**Method 2: Substitute first, then differentiate**

This method applies to parts (i), (iii) and (iv).

(i) With $\mathbf{u} = \langle 3,4\rangle/5 = \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle$, define the one-variable function $g(t) = f\left(\frac{3t}{5}, \frac{4t}{5}\right)$. By the definition of the directional derivative, $D_{\mathbf{u}}f(0,0) = g'(0)$. For $t \neq 0$,

$$g(t) = \frac{\frac{3t}{5}\cdot\frac{4t}{5} + \left(\frac{4t}{5}\right)^2}{\left(\frac{3t}{5}\right)^2 + \left(\frac{4t}{5}\right)^2} = \frac{\frac{12+16}{25}t^2}{\frac{9+16}{25}t^2} = \frac{28}{25},$$

while $g(0) = f(0,0) = 0$. So

$$\lim_{t\to 0} g(t) = \frac{28}{25} \neq 0 = g(0),$$

and $g$ is not continuous at 0. A function that is differentiable at a point is continuous there, so $g'(0)$ does not exist:

$$\boxed{D_{\mathbf{u}}f(0,0) \text{ does not exist.}}$$

(iii) The question asks for the multivariable Chain Rule; this method should be used only if the course allows this interpretation.

Substitute $x = 2u^2 + 3v^2$ and $y = 2uv$ into $z$ before differentiating:

$$xy = (2u^2+3v^2)(2uv) = 4u^3v + 6uv^3, \qquad z(u,v) = \sin\left(4u^3v + 6uv^3\right) + \sin(2uv).$$

Differentiate with respect to $u$, holding $v$ fixed (one-variable Chain Rule):

$$\frac{\partial z}{\partial u} = \cos\left(4u^3v + 6uv^3\right)\cdot\left(12u^2v + 6v^3\right) + \cos(2uv)\cdot 2v.$$

At $(u,v) = (0,2)$: $4u^3v + 6uv^3 = 0$, $12u^2v + 6v^3 = 0 + 6\cdot 8 = 48$ and $2uv = 0$, so

$$\frac{\partial z}{\partial u} = \cos 0 \cdot 48 + \cos 0 \cdot 2 \cdot 2,$$

that is,

$$\boxed{\left.\frac{\partial z}{\partial u}\right|_{(u,v)=(0,2)} = 48 + 4 = 52.}$$

(iv) The Implicit Function Theorem applies at $(\pi, 2\pi, \pi/2)$: the left-hand side $F$ has continuous partial derivatives, $F(\pi, 2\pi, \pi/2) = 3\cos 5\pi + 3\cos 2\pi + \cos\frac{9\pi}{2} = -3 + 3 + 0 = 0$, and $F_z = -6\sin(x+2z) - \sin(2y+z) = -6\sin 2\pi - \sin\frac{9\pi}{2} = -1 \neq 0$. So $z = z(x,y)$ near this point, and

$$3\cos(x+2y) + 3\cos\left(x + 2z(x,y)\right) + \cos\left(2y + z(x,y)\right) = 0$$

holds identically. Differentiate this identity with respect to $y$, holding $x$ fixed:

$$-3\sin(x+2y)\cdot 2 - 3\sin(x+2z)\cdot 2z_y - \sin(2y+z)\cdot(2+z_y) = 0.$$

At $(\pi, 2\pi, \pi/2)$, $\sin 5\pi = 0$, $\sin 2\pi = 0$ and $\sin\frac{9\pi}{2} = 1$, so

$$-6\cdot 0 - 6\cdot 0\cdot z_y - 1\cdot(2+z_y) = 0 \implies 2 + z_y = 0.$$

Hence

$$\boxed{\frac{\partial z}{\partial y} = -2.}$$

**Method 3: Increment expansion about the point**

This method applies to parts (ii) and (v).

(ii) Write $x = 2 + h$, $y = 1 + k$. Then

$$5(2+h)^2 = 5\left(4 + 4h + h^2\right) = 20 + 20h + 5h^2,$$
$$10(1+k)^2 = 10\left(1 + 2k + k^2\right) = 10 + 20k + 10k^2,$$

so

$$z = 5x^2 - 10y^2 = 10 + 20h - 20k + \left(5h^2 - 10k^2\right).$$

The remainder satisfies

$$\frac{|5h^2 - 10k^2|}{\sqrt{h^2+k^2}} \le \frac{10\left(h^2+k^2\right)}{\sqrt{h^2+k^2}} = 10\sqrt{h^2+k^2} \to 0,$$

so the tangent plane is the constant-plus-linear part $z = 10 + 20h - 20k$. Substituting $h = x - 2$, $k = y - 1$:

$$z = 10 + 20(x-2) - 20(y-1) = 20x - 20y - 10.$$

Hence

$$\boxed{20x - 20y - z = 10.}$$

(v) Complete the square in $y$:

$$2y^2 + 8y - 17 = 2\left(y^2 + 4y\right) - 17 = 2(y+2)^2 - 8 - 17 = 2(y+2)^2 - 25,$$

so

$$f(x,y) = 27x^3 - 81x + 2(y+2)^2 - 25.$$

The critical points come from $f_x = 81x^2 - 81 = 0$ and $f_y = 4(y+2) = 0$, giving $x = \pm 1$, $y = -2$: the points $(1,-2)$ and $(-1,-2)$.

Near $(1,-2)$ write $x = 1 + h$, $y = -2 + k$. Expand:

$$27(1+h)^3 - 81(1+h) = \left(27 + 81h + 81h^2 + 27h^3\right) - 81 - 81h = -54 + 81h^2 + 27h^3.$$

Hence

$$f(1+h,-2+k) = -54 + 81h^2 + 27h^3 + 2k^2 - 25 = -79 + 27h^2(3+h) + 2k^2.$$

For $|h| < 3$, $3 + h > 0$, so $27h^2(3+h) \ge 0$; also $2k^2 \ge 0$. Therefore $f(1+h,-2+k) \ge -79 = f(1,-2)$ for all $(h,k)$ with $|h| < 3$, and $(1,-2)$ is a local minimum.

Near $(-1,-2)$ write $x = -1 + h$, $y = -2 + k$. Expand:

$$27(-1+h)^3 - 81(-1+h) = \left(-27 + 81h - 81h^2 + 27h^3\right) + 81 - 81h = 54 - 81h^2 + 27h^3.$$

Hence

$$f(-1+h,-2+k) = 54 - 81h^2 + 27h^3 + 2k^2 - 25 = 29 - 27h^2(3-h) + 2k^2.$$

Along $k = 0$ with $0 < |h| < 3$, $f = 29 - 27h^2(3-h) < 29$. Along $h = 0$ with $k \ne 0$, $f = 29 + 2k^2 > 29$. Every neighbourhood of $(-1,-2)$ contains values above and below $f(-1,-2) = 29$, so $(-1,-2)$ is a saddle point. Therefore

$$\boxed{(1,-2) \text{ is a local minimum; } (-1,-2) \text{ is a saddle point.}}$$

**Method 4: Gradient of the level-surface function as a normal vector**

This method applies to part (ii). Write the surface as the level set $F(x, y, z) = 0$ with

$$F(x, y, z) = 5x^2 - 10y^2 - z, \qquad F(2, 1, 10) = 20 - 10 - 10 = 0.$$

The partial derivatives are

$$F_x = 10x, \qquad F_y = -20y, \qquad F_z = -1,$$

so

$$\nabla F(2, 1, 10) = \langle 20, -20, -1 \rangle \neq \mathbf{0}.$$

A non-zero gradient is normal to the level surface, so the tangent plane is

$$20(x - 2) - 20(y - 1) - (z - 10) = 0,$$

which expands to

$$20x - 40 - 20y + 20 - z + 10 = 0 \iff 20x - 20y - z - 10 = 0,$$

that is,

$$\boxed{20x - 20y - z = 10.}$$

**Unofficial QRS marking scheme (17 marks)**

(i) **(3 marks)** 1: normalizes the direction; 1: evaluates the function along that ray; 1: concludes that the directional derivative does not exist.

(ii) **(3 marks)** 1: finds both partial derivatives; 1: evaluates the slopes at $(2, 1)$; 1: gives the correct plane.

(iii) **(3 marks)** 1: writes the Chain Rule; 1: evaluates the intermediate variables and derivatives; 1: obtains 52.

(iv) **(3 marks)** 1: forms $F_y$ and $F_z$; 1: evaluates them at the stated point; 1: obtains $z_y = -2$.

(v) **(5 marks)**

- 2 marks: computes and solves $f_x = f_y = 0$.
- 1 mark: computes the Hessian entries and discriminant.
- 1 mark: classifies $(1, -2)$.
- 1 mark: classifies $(-1, -2)$.

Solutions by QRS@NTU