MH2100 Calculus III

Midterm II (Solutions)

AY 2021/2022, Semester 1

Quantitative Research Society @NTU

October 2021

Overview

- **Historical paper:** Midterm II, October 2021; **90 minutes** were allowed.
- **Original format:** three questions worth 50 marks in total. One two-sided A4 piece of handwritten notes was permitted.
- **Current administration (AY2026/2027):** Tuesday 6 October 2026, 16:30–18:20, venue TBA. One double-sided typed A4 sheet is permitted; no attachments or sticky notes are permitted.
- **2026 scope note:** the current midterm scope ends at double integrals, including polar coordinates. General Jacobian substitutions, cylindrical coordinates, and line integrals are beyond that scope. Question 2(v) is retained as an in-scope volume calculation by a double integral of vertical height.
- **Source:** MH2100 Midterm II, AY2021/2022 Semester 1. The questions and instructions are transcribed from the official paper; solutions and marking schemes are unofficial QRS@NTU material.

Instructions to candidates (verbatim)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. One two-sided A4 piece of paper with handwritten notes is permitted.

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Unofficial Solutions by QRS@NTU

Question 1 (16 Marks)

QUESTION 1.

(i) [5 marks] Find the *maximum* and *minimum* values of $4x + 2y$ on the half disk given by $x^2 + y^2 \leq 16$ and $y \geq 0$. Also give the points (x, y) where the *maximum* and *minimum* values are achieved.

(The set of points (x, y) where $x^2 + y^2 \leq 16$ and $y \geq 0$ is a closed and bounded set so both the maximum and minimum must be achieved.)

(ii) [3 marks] Evaluate the iterated integral.

$$\int_{-1}^{1} \int_{0}^{1} xy \, \mathrm{d}x \, \mathrm{d}y.$$

(iii) [4 marks] By reversing the order of integration, evaluate the integral

$$\int_{0}^{30} \int_{y/6}^{5} \mathrm{e}^{x^2} \, \mathrm{d}x \, \mathrm{d}y.$$

(iv) [4 marks] By reversing the order of integration, evaluate the integral

$$\int_{0}^{4} \int_{y/4}^{\sqrt[3]{y/4}} \sin(2x^2 - x^4) \, \mathrm{d}x \, \mathrm{d}y.$$

Solution

Method 1: Extreme Value Theorem with Lagrange multipliers; iterated integrals and reversal of order by Fubini's Theorem

(i) Let

$$f(x, y) = 4x + 2y, \qquad D = \{(x, y) : x^2 + y^2 \leq 16, \ y \geq 0\}.$$

The set D is closed and bounded, and f is a polynomial, hence continuous on D. By the Extreme Value Theorem, f attains a maximum and a minimum on D. Each extreme value occurs either at a critical point in the interior of D or on the boundary of D. The boundary consists of the semicircular arc $x^2 + y^2 = 16$, $y \geq 0$, and the diameter $y = 0$, $-4 \leq x \leq 4$.

Interior. The partial derivatives are

$$f_x = \frac{\partial}{\partial x}(4x + 2y) = 4, \qquad f_y = \frac{\partial}{\partial y}(4x + 2y) = 2.$$

Since $\nabla f = (4, 2) \neq (0, 0)$ at every point, f has no critical point in the interior of D.

Arc. Let $g(x, y) = x^2 + y^2$, so the arc lies on $g = 16$. The partial derivatives of g are

$$g_x = 2x, \qquad g_y = 2y, \qquad \text{so} \qquad \nabla g = (2x, 2y).$$

$\nabla g = (0, 0)$ only at the origin, which is not on $g = 16$; hence $\nabla g \neq \mathbf{0}$ on the arc and the Lagrange condition $\nabla f = \lambda \nabla g$ applies. The system is

$$4 = 2\lambda x,$$

$$2 = 2\lambda y,$$
$$x^2 + y^2 = 16.$$

If $\lambda = 0$, the first equation reads $4 = 0$, which is impossible. Hence $\lambda \neq 0$ and

$$x = \frac{2}{\lambda}, \qquad y = \frac{1}{\lambda}.$$

Substituting into the constraint,

$$\frac{4}{\lambda^2} + \frac{1}{\lambda^2} = 16 \implies \lambda^2 = \frac{5}{16} \implies \lambda = \frac{\sqrt{5}}{4} \quad \text{or} \quad \lambda = -\frac{\sqrt{5}}{4}.$$

Case $\lambda = \frac{\sqrt{5}}{4}$:

$$x = \frac{2}{\sqrt{5}/4} = \frac{8}{\sqrt{5}}, \qquad y = \frac{1}{\sqrt{5}/4} = \frac{4}{\sqrt{5}} > 0,$$

so $\left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$ lies on the arc and is a candidate.

Case $\lambda = -\frac{\sqrt{5}}{4}$:

$$x = -\frac{8}{\sqrt{5}}, \qquad y = -\frac{4}{\sqrt{5}} < 0,$$

so this point is not in D and is rejected.

The endpoints $(\pm 4, 0)$ of the arc are examined with the diameter.

Diameter. On $y = 0$, $-4 \leq x \leq 4$,

$$h(x) = f(x, 0) = 4x, \qquad h'(x) = 4 \neq 0,$$

so h has no critical point in $(-4, 4)$ and only the endpoints $x = \pm 4$ are candidates.

Comparison. The candidate values are

$$f\left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right) = \frac{32}{\sqrt{5}} + \frac{8}{\sqrt{5}} = \frac{40}{\sqrt{5}} = 8\sqrt{5} \approx 17.89,$$
$$f(4, 0) = 16,$$
$$f(-4, 0) = -16.$$

Since $8\sqrt{5} = \sqrt{320} > \sqrt{256} = 16$, the largest value is $8\sqrt{5}$ and the smallest is -16. Therefore

$$\boxed{\max_D (4x + 2y) = 8\sqrt{5} \text{ at } \left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right), \qquad \min_D (4x + 2y) = -16 \text{ at } (-4, 0).}$$

(ii) The integrand xy is continuous on the rectangle $[0, 1] \times [-1, 1]$, so the iterated integral is evaluated from the inside out. With y fixed, the inner integral is

$$\int_0^1 xy \, \mathrm{d}x = y \left[\frac{x^2}{2}\right]_{x=0}^{x=1} = y\left(\frac{1}{2} - 0\right) = \frac{y}{2}.$$

The outer integral is

$$\int_{-1}^1 \frac{y}{2} \, \mathrm{d}y = \left[\frac{y^2}{4}\right]_{-1}^1 = \frac{1}{4} - \frac{1}{4} = 0.$$

Therefore

$$\int_{-1}^1 \int_0^1 xy \, \mathrm{d}x \, \mathrm{d}y = \boxed{0}$$

(iii) The region of integration is

$$D = \left\{(x, y) : 0 \le y \le 30,\ \frac{y}{6} \le x \le 5\right\}.$$

The function e^{x^2} has no elementary antiderivative in x, so the order is reversed. The left boundary $x = \frac{y}{6}$ is the line $y = 6x$, the right boundary is $x = 5$, and the line $y = 6x$ meets $x = 5$ at $y = 30$. Hence D is the triangle with vertices $(0, 0)$, $(5, 0)$, $(5, 30)$.

For $(x, y) \in D$ we have $0 \le \frac{y}{6} \le x \le 5$, so $0 \le x \le 5$. For fixed $x \in [0, 5]$, the conditions $\frac{y}{6} \le x$ and $y \ge 0$ give $0 \le y \le 6x$, and then $y \le 6x \le 30$ holds automatically. Therefore

$$D = \{(x, y) : 0 \le y \le 30,\ \tfrac{y}{6} \le x \le 5\} = \{(x, y) : 0 \le x \le 5,\ 0 \le y \le 6x\}.$$

Since e^{x^2} is continuous on the closed bounded region D, Fubini's Theorem gives

$$\int_0^{30} \int_{y/6}^{5} \mathrm{e}^{x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{5} \int_0^{6x} \mathrm{e}^{x^2}\,\mathrm{d}y\,\mathrm{d}x.$$

The inner integral is

$$\int_0^{6x} \mathrm{e}^{x^2}\,\mathrm{d}y = \mathrm{e}^{x^2}\big[y\big]_{y=0}^{y=6x} = 6x\mathrm{e}^{x^2}.$$

For the outer integral let $u = x^2$, so $\mathrm{d}u = 2x\,\mathrm{d}x$ and $6x\,\mathrm{d}x = 3\,\mathrm{d}u$; when $x = 0$, $u = 0$, and when $x = 5$, $u = 25$:

$$\int_0^{5} 6x\mathrm{e}^{x^2}\,\mathrm{d}x = \int_0^{25} 3\mathrm{e}^{u}\,\mathrm{d}u = 3\big[\mathrm{e}^{u}\big]_0^{25} = 3(\mathrm{e}^{25} - \mathrm{e}^{0}).$$

Therefore

$$\int_0^{30} \int_{y/6}^{5} \mathrm{e}^{x^2}\,\mathrm{d}x\,\mathrm{d}y = \boxed{3(\mathrm{e}^{25} - 1)}$$

(iv) The region of integration is

$$D = \left\{(x, y) : 0 \le y \le 4,\ \frac{y}{4} \le x \le \sqrt[3]{y/4}\right\}.$$

The boundary $x = \frac{y}{4}$ is the line $y = 4x$, and the boundary $x = \sqrt[3]{y/4}$ is the curve $x^3 = \frac{y}{4}$, i.e. $y = 4x^3$. They meet where

$$4x = 4x^3 \iff 4x(x - 1)(x + 1) = 0,$$

and with $0 \le y \le 4$ (so $x \ge 0$) this gives the points $(0, 0)$ and $(1, 4)$. For $0 \le s \le 1$ we have $s \le \sqrt[3]{s}$, so with $s = \frac{y}{4}$ each horizontal segment $\frac{y}{4} \le x \le \sqrt[3]{y/4}$ is nonempty.

For $(x, y) \in D$ we have $0 \le \frac{y}{4} \le x \le \sqrt[3]{y/4} \le 1$, so $0 \le x \le 1$. For fixed $x \in [0, 1]$:

$$x \ge \frac{y}{4} \iff y \le 4x, \qquad x \le \sqrt[3]{y/4} \iff x^3 \le \frac{y}{4} \iff y \ge 4x^3,$$

where the middle step uses that cubing is increasing. Also $4x \le 4$ and $4x^3 \ge 0$, so the conditions $0 \le y \le 4$ hold automatically. Therefore

$$D = \{(x, y) : 0 \le y \le 4,\ \tfrac{y}{4} \le x \le \sqrt[3]{y/4}\} = \{(x, y) : 0 \le x \le 1,\ 4x^3 \le y \le 4x\}.$$

Since $\sin(2x^2 - x^4)$ is continuous on the closed bounded region D, Fubini's Theorem gives

$$\int_0^4 \int_{y/4}^{\sqrt[3]{y/4}} \sin(2x^2 - x^4)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_{4x^3}^{4x} \sin(2x^2 - x^4)\,\mathrm{d}y\,\mathrm{d}x.$$

The inner integral is

$$\int_{4x^3}^{4x} \sin(2x^2 - x^4)\,\mathrm{d}y = \sin(2x^2 - x^4)\big[y\big]_{y=4x^3}^{y=4x} = (4x - 4x^3)\sin(2x^2 - x^4).$$

For the outer integral let $u = 2x^2 - x^4$, so $\mathrm{d}u = (4x - 4x^3)\,\mathrm{d}x$; when $x = 0$, $u = 0$, and when $x = 1$, $u = 2 - 1 = 1$:

$$\begin{aligned}\int_0^1 (4x - 4x^3)\sin(2x^2 - x^4)\,\mathrm{d}x &= \int_0^1 \sin u\,\mathrm{d}u \\ &= \big[-\cos u\big]_0^1 \\ &= -\cos 1 + \cos 0.\end{aligned}$$

Therefore

$$\int_0^4 \int_{y/4}^{\sqrt[3]{y/4}} \sin(2x^2 - x^4)\,\mathrm{d}x\,\mathrm{d}y = \boxed{1 - \cos 1}$$

Unofficial QRS@NTU Solutions

Method 2: Parametrising the constraint; Fubini in the other order; integration by parts without reversing the order

(i) Let $f(x,y) = 4x + 2y$ and $D = \{(x,y) : x^2 + y^2 \le 16,\ y \ge 0\}$. As in Method 1, D is closed and bounded and f is continuous, so by the Extreme Value Theorem both extreme values are attained, in the interior or on the boundary.

Interior. $f_x = 4$ and $f_y = 2$ are never both zero, so there is no interior critical point.

Arc. Parametrise the arc by

$$x = 4\cos t, \qquad y = 4\sin t, \qquad 0 \le t \le \pi,$$

where $0 \le t \le \pi$ is exactly the condition $y = 4\sin t \ge 0$. Along the arc,

$$h(t) = f(4\cos t, 4\sin t) = 16\cos t + 8\sin t, \qquad h'(t) = -16\sin t + 8\cos t.$$

Setting $h'(t) = 0$ gives $8\cos t = 16\sin t$. If $\cos t = 0$ this would force $\sin t = 0$, which is impossible, so we may divide:

$$\tan t = \frac{1}{2}.$$

On $[0,\pi]$, $\tan t > 0$ only for $0 < t < \frac{\pi}{2}$, so the only solution is $t_0 = \arctan\frac{1}{2}$. In a right triangle with legs 1 and 2 the hypotenuse is $\sqrt{5}$, so

$$\cos t_0 = \frac{2}{\sqrt{5}}, \qquad \sin t_0 = \frac{1}{\sqrt{5}}, \qquad (x,y) = \left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right),$$

$$h(t_0) = 16 \cdot \frac{2}{\sqrt{5}} + 8 \cdot \frac{1}{\sqrt{5}} = \frac{40}{\sqrt{5}} = 8\sqrt{5}.$$

The endpoints of the parameter interval give

$$h(0) = 16 \ \text{ at } (4,0), \qquad h(\pi) = -16 \ \text{ at } (-4,0).$$

In amplitude–phase form,

$$h(t) = 8\sqrt{5}\left(\frac{2}{\sqrt{5}}\cos t + \frac{1}{\sqrt{5}}\sin t\right) = 8\sqrt{5}\cos(t - t_0),$$

which equals $8\sqrt{5}$ exactly when $t = t_0$ and is smallest on $[0,\pi]$ at the endpoint $t = \pi$, where $8\sqrt{5}\cos(\pi - t_0) = -8\sqrt{5}\cdot\frac{2}{\sqrt{5}} = -16$.

Diameter. On $y = 0$, $f(x,0) = 4x$ is increasing on $[-4,4]$, with values -16 at $x = -4$ and 16 at $x = 4$.

Comparison. The candidates are $8\sqrt{5} \approx 17.89$, 16 and -16. Therefore

$$\boxed{\max_D(4x + 2y) = 8\sqrt{5} \text{ at } \left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right), \qquad \min_D(4x + 2y) = -16 \text{ at } (-4,0).}$$

(ii) The integrand xy is continuous on the rectangle $[0,1] \times [-1,1]$, so by Fubini's Theorem the integral may be computed with y innermost:

$$\int_{-1}^{1}\int_{0}^{1} xy \,\mathrm{d}x\,\mathrm{d}y = \int_{0}^{1}\int_{-1}^{1} xy \,\mathrm{d}y\,\mathrm{d}x.$$

With x fixed, the inner integral is

$$\int_{-1}^{1} xy \,\mathrm{d}y = x\left[\frac{y^2}{2}\right]_{y=-1}^{y=1} = x\left(\frac{1}{2} - \frac{1}{2}\right) = 0.$$

The outer integral is

$$\int_{0}^{1} 0 \,\mathrm{d}x = 0.$$

Therefore

$$\int_{-1}^{1}\int_{0}^{1} xy \,\mathrm{d}x\,\mathrm{d}y = \boxed{0}$$

(iii) (does not reverse the order of integration; this method should be used only if the course allows this interpretation)

Here the order is kept and the inner integral is handled through an antiderivative function. Let

$$H(t) = \int_{0}^{t} \mathrm{e}^{x^2} \,\mathrm{d}x.$$

Since e^{x^2} is continuous, the Fundamental Theorem of Calculus gives $H'(t) = \mathrm{e}^{t^2}$. The inner integral is

$$\int_{y/6}^{5} \mathrm{e}^{x^2} \,\mathrm{d}x = H(5) - H\left(\frac{y}{6}\right),$$

so

$$I = \int_{0}^{30} \left[H(5) - H\left(\frac{y}{6}\right)\right] \mathrm{d}y = 30H(5) - \int_{0}^{30} H\left(\frac{y}{6}\right) \mathrm{d}y.$$

In the last integral let $y = 6s$, so $\mathrm{d}y = 6\,\mathrm{d}s$; when $y = 0$, $s = 0$, and when $y = 30$, $s = 5$:

$$\int_{0}^{30} H\left(\frac{y}{6}\right) \mathrm{d}y = 6\int_{0}^{5} H(s) \,\mathrm{d}s.$$

Integrate by parts with $U = H(s)$, $\mathrm{d}V = \mathrm{d}s$, so $\mathrm{d}U = \mathrm{e}^{s^2}\,\mathrm{d}s$ and $V = s$:

$$\int_{0}^{5} H(s) \,\mathrm{d}s = \big[sH(s)\big]_{0}^{5} - \int_{0}^{5} s\mathrm{e}^{s^2} \,\mathrm{d}s = 5H(5) - \int_{0}^{5} s\mathrm{e}^{s^2} \,\mathrm{d}s.$$

Hence

$$I = 30H(5) - 30H(5) + 6\int_{0}^{5} s\mathrm{e}^{s^2} \,\mathrm{d}s = 6\int_{0}^{5} s\mathrm{e}^{s^2} \,\mathrm{d}s.$$

Let $w = s^2$, so $\mathrm{d}w = 2s\,\mathrm{d}s$ and $6s\,\mathrm{d}s = 3\,\mathrm{d}w$; when $s = 0$, $w = 0$, and when $s = 5$, $w = 25$:

$$I = \int_{0}^{25} 3\mathrm{e}^{w} \,\mathrm{d}w = 3\big[\mathrm{e}^{w}\big]_{0}^{25} = 3(\mathrm{e}^{25} - 1).$$

Therefore

$$\int_{0}^{30}\int_{y/6}^{5} \mathrm{e}^{x^2} \,\mathrm{d}x\,\mathrm{d}y = \boxed{3(\mathrm{e}^{25} - 1)}$$

(iv) (does not reverse the order of integration; this method should be used only if the course allows this interpretation)

Again the order is kept. Let

$$K(t) = \int_{0}^{t} \sin(2x^2 - x^4) \,\mathrm{d}x,$$

so $K'(t) = \sin(2t^2 - t^4)$ by the Fundamental Theorem of Calculus. The inner integral is $K\big(\sqrt[3]{y/4}\big) - K\big(\frac{y}{4}\big)$, hence

$$I = I_1 - I_2, \qquad I_1 = \int_0^4 K\big(\sqrt[3]{y/4}\big)\,\mathrm{d}y, \qquad I_2 = \int_0^4 K\Big(\frac{y}{4}\Big)\,\mathrm{d}y.$$

For I_2 let $y = 4s$, so $\mathrm{d}y = 4\,\mathrm{d}s$, with s from 0 to 1. Integrating by parts with $U = K(s)$, $V = s$:

$$\begin{aligned} I_2 &= 4\int_0^1 K(s)\,\mathrm{d}s \\ &= 4\Big(\big[sK(s)\big]_0^1 - \int_0^1 sK'(s)\,\mathrm{d}s\Big) \\ &= 4K(1) - \int_0^1 4s\sin(2s^2 - s^4)\,\mathrm{d}s. \end{aligned}$$

For I_1 let $y = 4s^3$ (so $s = \sqrt[3]{y/4}$; the map $s \mapsto 4s^3$ is increasing and continuously differentiable on $[0,1]$), so $\mathrm{d}y = 12s^2\,\mathrm{d}s$, with s from 0 to 1. Integrating by parts with $U = K(s)$, $\mathrm{d}V = 12s^2\,\mathrm{d}s$, $V = 4s^3$:

$$\begin{aligned} I_1 &= \int_0^1 K(s)\,12s^2\,\mathrm{d}s \\ &= \big[4s^3K(s)\big]_0^1 - \int_0^1 4s^3K'(s)\,\mathrm{d}s \\ &= 4K(1) - \int_0^1 4s^3\sin(2s^2 - s^4)\,\mathrm{d}s. \end{aligned}$$

Subtracting, the $4K(1)$ terms cancel:

$$I = I_1 - I_2 = \int_0^1 (4s - 4s^3)\sin(2s^2 - s^4)\,\mathrm{d}s.$$

Let $u = 2s^2 - s^4$, so $\mathrm{d}u = (4s - 4s^3)\,\mathrm{d}s$; when $s = 0$, $u = 0$, and when $s = 1$, $u = 1$:

$$I = \int_0^1 \sin u\,\mathrm{d}u = \big[-\cos u\big]_0^1 = 1 - \cos 1.$$

Therefore

$$\int_0^4 \int_{y/4}^{\sqrt[3]{y/4}} \sin(2x^2 - x^4)\,\mathrm{d}x\,\mathrm{d}y = \boxed{1 - \cos 1}$$

Method 3: Cauchy–Schwarz bound; odd symmetry in y

(i) Write $4x + 2y$ as the dot product of $\mathbf{a} = (4, 2)$ and $\mathbf{p} = (x, y)$. Here

$$|\mathbf{a}| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}, \qquad |\mathbf{p}| = \sqrt{x^2 + y^2} \le 4 \text{ on } D.$$

By the Cauchy–Schwarz inequality $\mathbf{a} \cdot \mathbf{p} \le |\mathbf{a}|\,|\mathbf{p}|$,

$$4x + 2y \le 2\sqrt{5} \cdot 4 = 8\sqrt{5} \qquad \text{for all } (x, y) \in D.$$

Equality requires $\mathbf{p} = c\,\mathbf{a}$ with $c \ge 0$ and $|\mathbf{p}| = 4$, i.e. $c\sqrt{20} = 4$, $c = \frac{2}{\sqrt{5}}$:

$$(x, y) = \frac{2}{\sqrt{5}}(4, 2) = \left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right),$$

and $y = \frac{4}{\sqrt{5}} \ge 0$, so this point is in D and the bound $8\sqrt{5}$ is attained.

For the lower bound, on D we have $x^2 \le x^2 + y^2 \le 16$, so $x \ge -4$, and $y \ge 0$. Hence

$$4x + 2y \ge 4(-4) + 2(0) = -16,$$

with equality exactly when $x = -4$ and $y = 0$; the point $(-4, 0)$ lies in D. Therefore

$$\boxed{\max_D(4x + 2y) = 8\sqrt{5} \text{ at } \left(\frac{8}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right), \qquad \min_D(4x + 2y) = -16 \text{ at } (-4, 0).}$$

(ii) Let

$$F(y) = \int_0^1 xy\,\mathrm{d}x, \qquad \text{so} \qquad \int_{-1}^1 \int_0^1 xy\,\mathrm{d}x\,\mathrm{d}y = \int_{-1}^1 F(y)\,\mathrm{d}y.$$

For every y,

$$F(-y) = \int_0^1 x(-y)\,\mathrm{d}x = -\int_0^1 xy\,\mathrm{d}x = -F(y),$$

so F is odd. In the integral over $[-1, 0]$ substitute $y = -w$, $\mathrm{d}y = -\,\mathrm{d}w$:

$$\int_{-1}^0 F(y)\,\mathrm{d}y = \int_1^0 F(-w)(-\,\mathrm{d}w) = \int_0^1 F(-w)\,\mathrm{d}w = -\int_0^1 F(w)\,\mathrm{d}w.$$

Hence

$$\int_{-1}^1 F(y)\,\mathrm{d}y = \int_{-1}^0 F(y)\,\mathrm{d}y + \int_0^1 F(y)\,\mathrm{d}y = -\int_0^1 F(w)\,\mathrm{d}w + \int_0^1 F(y)\,\mathrm{d}y = 0.$$

Therefore

$$\int_{-1}^1 \int_0^1 xy\,\mathrm{d}x\,\mathrm{d}y = \boxed{0}$$

Marking scheme (16 marks)

(i) **(5 marks)**

- 1 mark: uses the compactness/continuity remark or otherwise identifies that extrema are attained.
- 2 marks: correctly finds the semicircular candidate by Lagrange multipliers or an equivalent argument.

- 1 mark: checks the diameter endpoints.
- 1 mark: gives both extremal values and their points.

(ii) **(3 marks)**

- 1 mark: evaluates the inner integral correctly.
- 1 mark: evaluates the outer integral or uses oddness correctly.
- 1 mark: obtains 0.

(iii) **(4 marks)**

- 2 marks: gives the reversed bounds $0 \leq x \leq 5$, $0 \leq y \leq 6x$.
- 1 mark: integrates the y variable and makes the correct substitution.
- 1 mark: obtains $3(\mathrm{e}^{25} - 1)$.

(iv) **(4 marks)**

- 2 marks: gives the reversed bounds $0 \leq x \leq 1$, $4x^3 \leq y \leq 4x$.
- 1 mark: obtains $4x(1 - x^2)\sin(2x^2 - x^4)$ and substitutes correctly.
- 1 mark: obtains $1 - \cos 1$.

Unofficial QRS@NTU Question Bank

Question 2 (17 Marks)

QUESTION 2.

(i) [3 marks] Find the area of the region bounded by the lines $y = x^2$ and $y = 2x + 3$.

(ii) *(beyond 2026 midterm scope)* [2 marks] Find the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)}$ using the equations $x = -2u + 5$, $y = 5v + 4$, and $z = 3w - 4$.

(iii) *(beyond 2026 midterm scope)* [4 marks] Use the transformation $u = x+y$ and $v = -2x+y$ to evaluate the integral

$$\iint_R 3x \, \mathrm{d}A,$$

where R is the parallelogram bounded by the lines

$$y = -x + 1, \quad y = -x + 4, \quad y = 2x + 2 \quad \text{and} \quad y = 2x + 5.$$

(iv) *(beyond 2026 midterm scope)* [4 marks] Use cylindrical coordinates to evaluate

$$\iiint_E x^2 y \, \mathrm{d}V,$$

where E is the solid that lies between the cylinders $x^2 + y^2 = 2$ and $x^2 + y^2 = 4$, above the xy-plane and below the plane $z = y$.

(v) [4 marks] Find the volume of the region bounded above by the sphere

$$x^2 + y^2 + z^2 = 9$$

and below by the cone $z = \sqrt{x^2 + y^2}$.

Solution

Method 1: Type I area integral, Jacobian determinant, change of variables, cylindrical and polar coordinates

(i) The curves meet where

$$x^2 = 2x + 3 \iff x^2 - 2x - 3 = 0 \iff (x - 3)(x + 1) = 0 \iff x = -1 \text{ or } x = 3,$$

i.e. at $(-1, 1)$ and $(3, 9)$. For $-1 < x < 3$,

$$(2x + 3) - x^2 = -(x - 3)(x + 1) = (3 - x)(x + 1) > 0,$$

so the line lies above the parabola. The region is $D = \{(x, y) : -1 \le x \le 3,\ x^2 \le y \le 2x + 3\}$ and

$$A = \iint_D 1 \, \mathrm{d}A = \int_{-1}^{3} \int_{x^2}^{2x+3} \mathrm{d}y \, \mathrm{d}x = \int_{-1}^{3} \left[(2x + 3) - x^2\right] \mathrm{d}x.$$

An antiderivative is $x^2 + 3x - \frac{x^3}{3}$, and

$$\left[x^2 + 3x - \frac{x^3}{3}\right]_{x=3} = 9 + 9 - 9 = 9,$$
$$\left[x^2 + 3x - \frac{x^3}{3}\right]_{x=-1} = 1 - 3 + \frac{1}{3} = -\frac{5}{3}.$$

Therefore

$$A = 9 - \left(-\frac{5}{3}\right) = \boxed{\frac{32}{3}}$$

(ii) From $x = -2u + 5$, $y = 5v + 4$, $z = 3w - 4$, the partial derivatives are

$$\begin{aligned} x_u &= -2, & x_v &= 0, & x_w &= 0, \\ y_u &= 0, & y_v &= 5, & y_w &= 0, \\ z_u &= 0, & z_v &= 0, & z_w &= 3. \end{aligned}$$

Expanding the determinant along the first row,

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix} = -2\begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0\begin{vmatrix} 0 & 5 \\ 0 & 0 \end{vmatrix}$$
$$= -2(5 \cdot 3 - 0 \cdot 0) = \boxed{-30}$$

(iii) Solve $u = x + y$, $v = -2x + y$ for x and y. Subtracting,

$$u - v = (x + y) - (-2x + y) = 3x \implies x = \frac{u - v}{3},$$

and then

$$y = u - x = u - \frac{u - v}{3} = \frac{2u + v}{3}.$$

The partial derivatives are

$$x_u = \frac{1}{3}, \qquad x_v = -\frac{1}{3}, \qquad y_u = \frac{2}{3}, \qquad y_v = \frac{1}{3},$$

so

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3} \cdot \frac{1}{3} - \left(-\frac{1}{3}\right) \cdot \frac{2}{3} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}, \qquad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3}.$$

The map is linear with nonzero determinant, hence one-to-one. The boundary lines become

$$\begin{aligned} y = -x + 1 &\iff x + y = 1 \iff u = 1, & y = -x + 4 &\iff u = 4, \\ y = 2x + 2 &\iff -2x + y = 2 \iff v = 2, & y = 2x + 5 &\iff v = 5. \end{aligned}$$

R lies between each pair of parallel lines, so its image is the rectangle $S = \{(u, v) : 1 \le u \le 4,\ 2 \le v \le 5\}$. The integrand is $3x = u - v$. By the change of variables formula,

$$\iint_R 3x \, \mathrm{d}A = \iint_S (u - v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u \, \mathrm{d}v = \int_1^4 \int_2^5 (u - v) \cdot \frac{1}{3} \, \mathrm{d}v \, \mathrm{d}u.$$

The inner integral is

$$\int_2^5 (u-v)\,\mathrm{d}v = \left[uv - \frac{v^2}{2}\right]_{v=2}^{v=5} = \left(5u - \frac{25}{2}\right) - (2u-2) = 3u - \frac{21}{2},$$

so $\frac{1}{3}\int_2^5 (u-v)\,\mathrm{d}v = u - \frac{7}{2}$. The outer integral is

$$\int_1^4 \left(u - \frac{7}{2}\right)\,\mathrm{d}u = \left[\frac{u^2}{2} - \frac{7u}{2}\right]_1^4 = (8-14) - \left(\frac{1}{2} - \frac{7}{2}\right) = -6+3.$$

Therefore

$$\iint_R 3x\,\mathrm{d}A = \boxed{-3}$$

(iv) Use cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, with $\mathrm{d}V = r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$. Between the cylinders, $2 \le r^2 \le 4$, i.e. $\sqrt{2} \le r \le 2$. Above the xy-plane and below $z = y$ means $0 \le z \le y = r\sin\theta$, which is nonempty only when $\sin\theta \ge 0$, i.e. $0 \le \theta \le \pi$. Hence

$$E = \{(r,\theta,z) : 0 \le \theta \le \pi,\ \sqrt{2} \le r \le 2,\ 0 \le z \le r\sin\theta\}.$$

The integrand is $x^2y = r^2\cos^2\theta \cdot r\sin\theta = r^3\cos^2\theta\sin\theta$, and with the factor r from $\mathrm{d}V$,

$$\iiint_E x^2y\,\mathrm{d}V = \int_0^\pi \int_{\sqrt{2}}^2 \int_0^{r\sin\theta} r^4\cos^2\theta\sin\theta\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

The z-integral is

$$\int_0^{r\sin\theta} r^4\cos^2\theta\sin\theta\,\mathrm{d}z = r^4\cos^2\theta\sin\theta \cdot r\sin\theta = r^5\cos^2\theta\sin^2\theta.$$

The remaining integrand is a product of a function of θ and a function of r over a rectangle, so

$$\iiint_E x^2y\,\mathrm{d}V = \left(\int_0^\pi \cos^2\theta\sin^2\theta\,\mathrm{d}\theta\right)\left(\int_{\sqrt{2}}^2 r^5\,\mathrm{d}r\right).$$

For the θ-integral, $\cos\theta\sin\theta = \frac{1}{2}\sin 2\theta$ and $\sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)$, so

$$\int_0^\pi \cos^2\theta\sin^2\theta\,\mathrm{d}\theta = \int_0^\pi \frac{1}{8}(1-\cos 4\theta)\,\mathrm{d}\theta = \frac{1}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^\pi = \frac{\pi}{8}.$$

For the r-integral,

$$\int_{\sqrt{2}}^2 r^5\,\mathrm{d}r = \left[\frac{r^6}{6}\right]_{\sqrt{2}}^2 = \frac{64-8}{6} = \frac{28}{3}.$$

Therefore

$$\iiint_E x^2y\,\mathrm{d}V = \frac{\pi}{8}\cdot\frac{28}{3} = \boxed{\frac{7\pi}{6}}$$

(v) In polar coordinates the upper hemisphere is $z = \sqrt{9-x^2-y^2} = \sqrt{9-r^2}$ and the cone is $z = \sqrt{x^2+y^2} = r$. They meet where

$$r = \sqrt{9-r^2} \implies r^2 = 9-r^2 \implies r^2 = \frac{9}{2} \implies r = \frac{3}{\sqrt{2}}.$$

For $0 \le r \le \frac{3}{\sqrt{2}}$ we have $9 - r^2 \ge r^2$, so $\sqrt{9-r^2} \ge r$ and the sphere is above the cone. The solid lies over the disk $D = \{r \le \frac{3}{\sqrt{2}}\}$ with vertical height $\sqrt{9-r^2} - r$, so

$$V = \iint_D \left(\sqrt{9-r^2} - r\right) \mathrm{d}A = \int_0^{2\pi} \int_0^{3/\sqrt{2}} \left(\sqrt{9-r^2} - r\right) r \, \mathrm{d}r \, \mathrm{d}\theta = 2\pi(J_1 - J_2),$$

where

$$J_1 = \int_0^{3/\sqrt{2}} r\sqrt{9-r^2} \, \mathrm{d}r, \qquad J_2 = \int_0^{3/\sqrt{2}} r^2 \, \mathrm{d}r.$$

For J_1 let $u = 9 - r^2$, so $\mathrm{d}u = -2r \, \mathrm{d}r$ and $r \, \mathrm{d}r = -\frac{1}{2} \mathrm{d}u$; when $r = 0$, $u = 9$, and when $r = \frac{3}{\sqrt{2}}$, $u = \frac{9}{2}$:

$$\begin{aligned} J_1 &= \int_9^{9/2} u^{1/2} \left(-\frac{1}{2}\right) \mathrm{d}u = \frac{1}{2} \int_{9/2}^{9} u^{1/2} \, \mathrm{d}u = \frac{1}{2} \cdot \frac{2}{3} \left[u^{3/2}\right]_{9/2}^{9} \\ &= \frac{1}{3} \left(27 - \frac{27}{2\sqrt{2}}\right) = 9 - \frac{9}{2\sqrt{2}}, \end{aligned}$$

using $\left(\frac{9}{2}\right)^{3/2} = \frac{27}{2\sqrt{2}}$. For J_2,

$$J_2 = \left[\frac{r^3}{3}\right]_0^{3/\sqrt{2}} = \frac{1}{3} \cdot \frac{27}{2\sqrt{2}} = \frac{9}{2\sqrt{2}}.$$

Hence

$$V = 2\pi \left(9 - \frac{9}{2\sqrt{2}} - \frac{9}{2\sqrt{2}}\right) = 2\pi \left(9 - \frac{9}{\sqrt{2}}\right) = 18\pi - 9\sqrt{2}\,\pi.$$

Therefore

$$V = \boxed{9\pi(2 - \sqrt{2})}$$

Unofficial Question Solutions

Method 2: Horizontal strips; inverse Jacobian; Cartesian vertical strips; reduction to a polar double integral; spherical coordinates

(i) Describe the region with y as the outer variable. For fixed y, the condition $x^2 \le y$ gives $-\sqrt{y} \le x \le \sqrt{y}$, and $y \le 2x+3$ gives $x \ge \frac{y-3}{2}$. The region spans $0 \le y \le 9$. To decide which left boundary is active, compare

$$-\sqrt{y} - \frac{y-3}{2} = -\frac{y+2\sqrt{y}-3}{2} = -\frac{(\sqrt{y}+3)(\sqrt{y}-1)}{2},$$

which is ≥ 0 for $0 \le y \le 1$ and ≤ 0 for $1 \le y \le 9$. On the right, $\sqrt{y} - \frac{y-3}{2} = -\frac{(\sqrt{y}-3)(\sqrt{y}+1)}{2} \ge 0$ for $0 \le y \le 9$, so $\sqrt{y}$ is the right boundary. Hence

$$D = \{0 \le y \le 1,\ -\sqrt{y} \le x \le \sqrt{y}\} \cup \left\{1 \le y \le 9,\ \tfrac{y-3}{2} \le x \le \sqrt{y}\right\},$$

$$A = \int_0^1 \left(\sqrt{y} - (-\sqrt{y})\right) \mathrm{d}y + \int_1^9 \left(\sqrt{y} - \frac{y-3}{2}\right) \mathrm{d}y.$$

The first integral is

$$\int_0^1 2\sqrt{y}\,\mathrm{d}y = \left[\frac{4}{3}y^{3/2}\right]_0^1 = \frac{4}{3}.$$

For the second, an antiderivative is $\frac{2}{3}y^{3/2} - \frac{y^2}{4} + \frac{3y}{2}$, and

$$\left[\tfrac{2}{3}y^{3/2} - \tfrac{y^2}{4} + \tfrac{3y}{2}\right]_{y=9} = 18 - \frac{81}{4} + \frac{27}{2} = \frac{45}{4},$$
$$\left[\tfrac{2}{3}y^{3/2} - \tfrac{y^2}{4} + \tfrac{3y}{2}\right]_{y=1} = \frac{2}{3} - \frac{1}{4} + \frac{3}{2} = \frac{23}{12},$$

so the second integral is $\frac{135}{12} - \frac{23}{12} = \frac{112}{12} = \frac{28}{3}$. Therefore

$$A = \frac{4}{3} + \frac{28}{3} = \boxed{\frac{32}{3}}$$

(ii) Invert the transformation:

$$u = \frac{5-x}{2}, \qquad v = \frac{y-4}{5}, \qquad w = \frac{z+4}{3}.$$

Its partial derivatives are $u_x = -\frac{1}{2}$, $v_y = \frac{1}{5}$, $w_z = \frac{1}{3}$ and all others 0, so, expanding along the first row,

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = -\frac{1}{2}\cdot\frac{1}{5}\cdot\frac{1}{3} = -\frac{1}{30}.$$

By the Chain Rule, the Jacobian matrices of a map and its inverse multiply to the identity matrix, so their determinants multiply to 1:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1 \implies \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{-1/30} = \boxed{-30}$$

(iii) (does not use the prescribed transformation; this method should be used only if the course allows this interpretation)

Work directly in x, y. The region is $1 \le x+y \le 4$, $2 \le y-2x \le 5$. Its vertices are the intersections of adjacent sides:

$$\begin{aligned} x+y=1,\ y=2x+5 &\implies 3x=-4: & A &= \left(-\tfrac{4}{3}, \tfrac{7}{3}\right), \\ x+y=1,\ y=2x+2 &\implies 3x=-1: & B &= \left(-\tfrac{1}{3}, \tfrac{4}{3}\right), \\ x+y=4,\ y=2x+2 &\implies 3x=2: & C &= \left(\tfrac{2}{3}, \tfrac{10}{3}\right), \\ x+y=4,\ y=2x+5 &\implies 3x=-1: & D &= \left(-\tfrac{1}{3}, \tfrac{13}{3}\right). \end{aligned}$$

For fixed x, $y \ge \max(1-x,\, 2x+2)$ and $y \le \min(4-x,\, 2x+5)$. Since $1-x \ge 2x+2 \iff x \le -\tfrac{1}{3}$ and $4-x \le 2x+5 \iff x \ge -\tfrac{1}{3}$, the strips split at $x=-\tfrac{1}{3}$:

$$R = \left\{-\tfrac{4}{3} \le x \le -\tfrac{1}{3},\ 1-x \le y \le 2x+5\right\} \cup \left\{-\tfrac{1}{3} \le x \le \tfrac{2}{3},\ 2x+2 \le y \le 4-x\right\}.$$

The strip lengths are $(2x+5)-(1-x) = 3x+4$ and $(4-x)-(2x+2) = 2-3x$. Hence

$$\iint_R 3x \,\mathrm{d}A = \int_{-4/3}^{-1/3} 3x(3x+4)\,\mathrm{d}x + \int_{-1/3}^{2/3} 3x(2-3x)\,\mathrm{d}x.$$

The first integral is

$$\int_{-4/3}^{-1/3} (9x^2+12x)\,\mathrm{d}x = \Big[3x^3+6x^2\Big]_{-4/3}^{-1/3} = \left(-\frac{1}{9}+\frac{2}{3}\right) - \left(-\frac{64}{9}+\frac{96}{9}\right) = \frac{5}{9} - \frac{32}{9} = -3.$$

The second integral is

$$\int_{-1/3}^{2/3} (6x-9x^2)\,\mathrm{d}x = \Big[3x^2-3x^3\Big]_{-1/3}^{2/3} = \left(\frac{4}{3}-\frac{8}{9}\right) - \left(\frac{1}{3}+\frac{1}{9}\right) = \frac{4}{9} - \frac{4}{9} = 0.$$

Therefore

$$\iint_R 3x \,\mathrm{d}A = -3 + 0 = \boxed{-3}$$

(iv) (integrates in z first and then uses polar coordinates in the xy-plane; this method should be used only if the course accepts it as the cylindrical-coordinates computation)

This route stays within double integrals. The solid is $E = \{(x,y,z) : (x,y) \in D,\ 0 \le z \le y\}$ over the half-annulus

$$D = \{(x,y) : 2 \le x^2+y^2 \le 4,\ y \ge 0\},$$

where $y \ge 0$ is needed for $0 \le z \le y$ to be nonempty. Integrating in z first,

$$\iiint_E x^2 y \,\mathrm{d}V = \iint_D \left(\int_0^y x^2 y \,\mathrm{d}z\right) \mathrm{d}A = \iint_D x^2 y \cdot y \,\mathrm{d}A = \iint_D x^2 y^2 \,\mathrm{d}A.$$

In polar coordinates $D = \{\sqrt{2} \le r \le 2,\ 0 \le \theta \le \pi\}$, $x^2y^2 = r^4 \cos^2\theta \sin^2\theta$ and $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$:

$$\begin{aligned} \iint_D x^2 y^2 \,\mathrm{d}A &= \int_0^{\pi} \int_{\sqrt{2}}^{2} r^5 \cos^2\theta \sin^2\theta \,\mathrm{d}r\,\mathrm{d}\theta \\ &= \left[\frac{r^6}{6}\right]_{\sqrt{2}}^{2} \int_0^{\pi} \cos^2\theta \sin^2\theta \,\mathrm{d}\theta \\ &= \frac{28}{3} \int_0^{\pi} \left(\sin^2\theta - \sin^4\theta\right) \mathrm{d}\theta, \end{aligned}$$

using $\cos^2\theta = 1 - \sin^2\theta$. Using $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ and $\sin^4\theta = \frac{3-4\cos 2\theta+\cos 4\theta}{8}$,

$$\int_0^{\pi} \sin^2\theta\,\mathrm{d}\theta = \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi} = \frac{\pi}{2}, \qquad \int_0^{\pi} \sin^4\theta\,\mathrm{d}\theta = \left[\frac{3\theta}{8} - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\right]_0^{\pi} = \frac{3\pi}{8}.$$

Hence the θ-integral is $\frac{\pi}{2} - \frac{3\pi}{8} = \frac{\pi}{8}$, and

$$\iiint_E x^2 y\,\mathrm{d}V = \frac{28}{3}\cdot\frac{\pi}{8} = \boxed{\frac{7\pi}{6}}$$

(v) (uses spherical coordinates / beyond the 2026 midterm scope)

In spherical coordinates $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, with $\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$. The sphere is $\rho = 3$. The cone $z = \sqrt{x^2+y^2}$ is $\rho\cos\phi = \rho\sin\phi$, i.e. $\tan\phi = 1$, $\phi = \frac{\pi}{4}$, and the region above it is $0 \le \phi \le \frac{\pi}{4}$. Hence

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{3} \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta = \left(\int_0^{2\pi}\mathrm{d}\theta\right)\left(\int_0^{\pi/4}\sin\phi\,\mathrm{d}\phi\right)\left(\int_0^{3}\rho^2\,\mathrm{d}\rho\right).$$

The three factors are

$$\int_0^{2\pi}\mathrm{d}\theta = 2\pi, \qquad \int_0^{\pi/4}\sin\phi\,\mathrm{d}\phi = \left[-\cos\phi\right]_0^{\pi/4} = 1 - \frac{\sqrt{2}}{2}, \qquad \int_0^{3}\rho^2\,\mathrm{d}\rho = \left[\frac{\rho^3}{3}\right]_0^{3} = 9.$$

Hence $V = 2\pi\cdot 9\left(1 - \frac{\sqrt{2}}{2}\right) = 18\pi - 9\sqrt{2}\,\pi$. Therefore

$$V = \boxed{9\pi(2-\sqrt{2})}$$

Unofficial QRS@NTU

Method 3: Parabola–chord area formula; centroid of the parallelogram; volume by cross-sections

(i) The vertical gap between line and parabola factors through the intersection abscissae $x = -1$ and $x = 3$:

$$(2x + 3) - x^2 = -(x^2 - 2x - 3) = (x + 1)(3 - x).$$

Let $s = x + 1$, so $\mathrm{d}s = \mathrm{d}x$, $3 - x = 4 - s$; when $x = -1$, $s = 0$, and when $x = 3$, $s = 4$:

$$A = \int_{-1}^{3} (x + 1)(3 - x)\,\mathrm{d}x = \int_0^4 s(4 - s)\,\mathrm{d}s = \left[2s^2 - \frac{s^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}.$$

This is the parabola–chord formula $\int_a^b (x - a)(b - x)\,\mathrm{d}x = \frac{(b-a)^3}{6}$ with $a = -1$, $b = 3$: $\frac{4^3}{6} = \frac{32}{3}$. Therefore

$$A = \boxed{\frac{32}{3}}$$

(iii) (does not use the prescribed transformation; this method should be used only if the course allows this interpretation)

With the vertices $A = \left(-\frac{4}{3}, \frac{7}{3}\right)$, $B = \left(-\frac{1}{3}, \frac{4}{3}\right)$, $C = \left(\frac{2}{3}, \frac{10}{3}\right)$, $D = \left(-\frac{1}{3}, \frac{13}{3}\right)$ found in Method 2, the diagonals AC and BD have the common midpoint

$$M = \left(\frac{-\frac{4}{3} + \frac{2}{3}}{2}, \frac{\frac{7}{3} + \frac{10}{3}}{2}\right) = \left(-\frac{1}{3}, \frac{17}{6}\right) = \left(\frac{-\frac{1}{3} - \frac{1}{3}}{2}, \frac{\frac{4}{3} + \frac{13}{3}}{2}\right).$$

The point reflection $(x, y) \mapsto \left(-\frac{2}{3} - x,\ \frac{17}{3} - y\right)$ through M sends $x + y$ to $5 - (x + y)$ and $y - 2x$ to $7 - (y - 2x)$, so it swaps the lines $x + y = 1$, $x + y = 4$ and swaps the lines $y - 2x = 2$, $y - 2x = 5$. Hence it maps the parallelogram R onto itself, and it sends $x + \frac{1}{3}$ to $-(x + \frac{1}{3})$, so

$$\iint_R \left(x + \frac{1}{3}\right)\,\mathrm{d}A = 0.$$

Since $3x = 3\left(x + \frac{1}{3}\right) - 1$,

$$\iint_R 3x\,\mathrm{d}A = 3\iint_R \left(x + \frac{1}{3}\right)\,\mathrm{d}A - \iint_R 1\,\mathrm{d}A = -\operatorname{Area}(R).$$

The edge vectors from A are $\overrightarrow{AB} = (1, -1)$ and $\overrightarrow{AD} = (1, 2)$, so

$$\operatorname{Area}(R) = \left|\det \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}\right| = |1 \cdot 2 - 1 \cdot (-1)| = 3.$$

Therefore

$$\iint_R 3x\,\mathrm{d}A = -\operatorname{Area}(R) = \boxed{-3}$$

(v) Slice the solid by horizontal planes (Cavalieri's principle, a single-variable integral of cross-sectional area). At height $z \geq 0$ the cross-section consists of the points with $r \leq z$ (on or above the cone) and $r^2 \leq 9 - z^2$ (inside the sphere), a disk of radius $\min\left(z, \sqrt{9 - z^2}\right)$. Since $z \leq \sqrt{9 - z^2} \iff z^2 \leq \frac{9}{2} \iff z \leq \frac{3}{\sqrt{2}}$, the cross-sectional area is

$$A(z) = \begin{cases} \pi z^2, & 0 \leq z \leq \frac{3}{\sqrt{2}}, \\ \pi(9 - z^2), & \frac{3}{\sqrt{2}} \leq z \leq 3. \end{cases}$$

The first piece gives

$$\int_0^{3/\sqrt{2}} \pi z^2 \, \mathrm{d}z = \pi \left[\frac{z^3}{3}\right]_0^{3/\sqrt{2}} = \frac{\pi}{3} \cdot \frac{27}{2\sqrt{2}} = \frac{9\pi}{2\sqrt{2}}.$$

The second piece gives

$$\begin{aligned}\int_{3/\sqrt{2}}^{3} \pi(9 - z^2) \, \mathrm{d}z &= \pi \left[9z - \frac{z^3}{3}\right]_{3/\sqrt{2}}^{3} = \pi \left[(27 - 9) - \left(\frac{27}{\sqrt{2}} - \frac{9}{2\sqrt{2}}\right)\right] \\ &= \pi \left(18 - \frac{45}{2\sqrt{2}}\right).\end{aligned}$$

Adding,

$$V = \frac{9\pi}{2\sqrt{2}} + 18\pi - \frac{45\pi}{2\sqrt{2}} = 18\pi - \frac{36\pi}{2\sqrt{2}} = 18\pi - \frac{18\pi}{\sqrt{2}} = 18\pi - 9\sqrt{2}\,\pi.$$

Therefore

$$V = \boxed{9\pi(2 - \sqrt{2})}$$

Marking scheme (17 marks)

(i) **(3 marks)** 1 mark: finds both intersections; 1 mark: supplies the correct area integral; 1 mark: obtains 32/3.

(ii) **(2 marks)** 1 mark: forms the Jacobian matrix; 1 mark: obtains -30.

(iii) **(4 marks)**

- 1 mark: solves for x, y and finds the absolute Jacobian $1/3$.
- 2 marks: maps R to $1 \le u \le 4$, $2 \le v \le 5$ and transforms the integrand.
- 1 mark: evaluates the integral as -3.

(iv) **(4 marks)**

- 1 mark: gives correct cylindrical bounds, including $0 \le \theta \le \pi$.
- 1 mark: transforms the integrand and volume element correctly.
- 1 mark: integrates correctly.
- 1 mark: obtains $7\pi/6$.

(v) **(4 marks)**

- 1 mark: finds the intersection radius $3/\sqrt{2}$.
- 2 marks: sets up and evaluates the polar double integral of vertical height.
- 1 mark: obtains $9\pi(2 - \sqrt{2})$.

Question 3 (17 Marks)

(beyond 2026 midterm scope)

QUESTION 3.

(i) *(beyond 2026 midterm scope)* [4 marks] Evaluate the line integral

$$\int_C 3y \, \mathrm{d}s,$$

where C is the part of the curve $y = x^3/3$ where $0 \leq x \leq 2$.

(ii) *(beyond 2026 midterm scope)* [4 marks] Find the work done by the force field $\mathbf{F}(x, y) = (y - x)\mathbf{i} + x^2\mathbf{j}$ on a particle that moves along the curve given by $x = 3t$, $y = t^2 - 1$ from $(0, -1)$ to $(6, 3)$.

(iii) *(beyond 2026 midterm scope)* [3 marks] Evaluate the line integral

$$\int_C 8x^2yz \, \mathrm{d}x + 5z \, \mathrm{d}y - 4xy \, \mathrm{d}z,$$

where C has parametrisation $x = t$, $y = t^2$ and $z = t^3$, going from the point $(0, 0, 0)$ to $(1, 1, 1)$.

(iv) *(beyond 2026 midterm scope)* Let $\mathbf{F}(x, y, z) = (4z^2)\mathbf{i} + (\cos z)\mathbf{j} + (8xz - y \sin z + 1)\mathbf{k}$.

(1) [3 marks] Show that $\mathbf{F}$ is conservative by finding a potential function f such that $\mathbf{F} = \nabla f$.

(2) [3 marks] Evaluate the line integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ where C is the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ starting at the point with $t = 0$ and ending at the point with $t = \pi$.

Solution

Method 1: Parametrisation and the Fundamental Theorem for Line Integrals

(i) *Beyond 2026 midterm scope.* Parametrise C by

$$\mathbf{r}(x) = \left\langle x, \frac{x^3}{3} \right\rangle, \qquad 0 \leq x \leq 2.$$

Then

$$\mathbf{r}'(x) = \langle 1, x^2 \rangle, \qquad |\mathbf{r}'(x)| = \sqrt{1 + x^4}, \qquad \mathrm{d}s = \sqrt{1 + x^4} \, \mathrm{d}x,$$

and the integrand is $3y = 3 \cdot \frac{x^3}{3} = x^3$. Hence

$$\int_C 3y \, \mathrm{d}s = \int_0^2 x^3 \sqrt{1 + x^4} \, \mathrm{d}x.$$

Let $u = 1 + x^4$, so $\mathrm{d}u = 4x^3 \, \mathrm{d}x$ and $x^3 \, \mathrm{d}x = \frac{1}{4} \, \mathrm{d}u$; when $x = 0$, $u = 1$, and when $x = 2$, $u = 17$:

$$\int_0^2 x^3 \sqrt{1 + x^4} \, \mathrm{d}x = \frac{1}{4} \int_1^{17} u^{1/2} \, \mathrm{d}u = \frac{1}{4} \cdot \frac{2}{3} \Big[u^{3/2} \Big]_1^{17}$$

$$= \frac{1}{6}\left(17^{3/2} - 1\right) = \frac{1}{6}\left(17\sqrt{17} - 1\right).$$

Therefore

$$\int_C 3y \, \mathrm{d}s = \boxed{\frac{17\sqrt{17} - 1}{6}}$$

(ii) *Beyond 2026 midterm scope.* Let $\mathbf{r}(t) = \langle 3t, t^2 - 1\rangle$. The start $(0, -1)$ corresponds to $3t = 0$, $t = 0$ (and $t^2 - 1 = -1$), and the end $(6, 3)$ to $3t = 6$, $t = 2$ (and $t^2 - 1 = 3$), so $0 \le t \le 2$. Then

$$\mathbf{r}'(t) = \langle 3, 2t\rangle, \qquad \mathbf{F}(\mathbf{r}(t)) = \left\langle (t^2 - 1) - 3t,\ (3t)^2 \right\rangle = \langle t^2 - 3t - 1,\ 9t^2\rangle,$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 3(t^2 - 3t - 1) + 2t \cdot 9t^2 = 18t^3 + 3t^2 - 9t - 3.$$

Hence

$$W = \int_0^2 (18t^3 + 3t^2 - 9t - 3)\, \mathrm{d}t = \left[\frac{9}{2}t^4 + t^3 - \frac{9}{2}t^2 - 3t\right]_0^2$$
$$= \frac{9}{2} \cdot 16 + 8 - \frac{9}{2} \cdot 4 - 6 = 72 + 8 - 18 - 6.$$

Therefore

$$W = \boxed{56}$$

(iii) *Beyond 2026 midterm scope.* With $x = t$, $y = t^2$, $z = t^3$, the point $(0, 0, 0)$ is $t = 0$ and $(1, 1, 1)$ is $t = 1$, and

$$\mathrm{d}x = \mathrm{d}t, \qquad \mathrm{d}y = 2t\, \mathrm{d}t, \qquad \mathrm{d}z = 3t^2\, \mathrm{d}t.$$

Each term becomes

$$8x^2yz\, \mathrm{d}x = 8\, t^2 \cdot t^2 \cdot t^3\, \mathrm{d}t = 8t^7\, \mathrm{d}t,$$
$$5z\, \mathrm{d}y = 5t^3 \cdot 2t\, \mathrm{d}t = 10t^4\, \mathrm{d}t,$$
$$-4xy\, \mathrm{d}z = -4\, t \cdot t^2 \cdot 3t^2\, \mathrm{d}t = -12t^5\, \mathrm{d}t.$$

Hence

$$\int_C 8x^2yz\, \mathrm{d}x + 5z\, \mathrm{d}y - 4xy\, \mathrm{d}z = \int_0^1 (8t^7 + 10t^4 - 12t^5)\, \mathrm{d}t = \left[t^8 + 2t^5 - 2t^6\right]_0^1 = 1 + 2 - 2.$$

Therefore

$$\int_C 8x^2yz\, \mathrm{d}x + 5z\, \mathrm{d}y - 4xy\, \mathrm{d}z = \boxed{1}$$

(iv) *Beyond 2026 midterm scope.*

(1) Write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ with $P = 4z^2$, $Q = \cos z$, $R = 8xz - y\sin z + 1$. The cross-partials are

$$P_y = 0 = Q_x, \qquad P_z = 8z = R_x, \qquad Q_z = -\sin z = R_y,$$

so $\operatorname{curl}\mathbf{F} = \mathbf{0}$ on $\mathbb{R}^3$, which is simply connected; by the curl test $\mathbf{F}$ is conservative. To find f with $\nabla f = \mathbf{F}$, integrate $f_x = 4z^2$ with respect to x:

$$f(x, y, z) = \int 4z^2\, \mathrm{d}x = 4xz^2 + h(y, z).$$

Differentiate in y and compare with Q:

$$f_y = h_y(y,z) = \cos z \implies h(y,z) = y\cos z + k(z).$$

Differentiate $f = 4xz^2 + y\cos z + k(z)$ in z and compare with R:

$$f_z = 8xz - y\sin z + k'(z) = 8xz - y\sin z + 1 \implies k'(z) = 1 \implies k(z) = z + C.$$

Taking $C = 0$ and checking every partial derivative,

$$f_x = 4z^2, \qquad f_y = \cos z, \qquad f_z = 8xz - y\sin z + 1,$$

so $\nabla f = \mathbf{F}$ and $\mathbf{F}$ is conservative, with potential

$$\boxed{f(x,y,z) = 4xz^2 + y\cos z + z}$$

(2) The endpoints of the helix are

$$\mathbf{r}(0) = (\cos 0, \sin 0, 0) = (1,0,0), \qquad \mathbf{r}(\pi) = (\cos\pi, \sin\pi, \pi) = (-1,0,\pi).$$

Since $\mathbf{F} = \nabla f$ with f continuously differentiable on $\mathbb{R}^3$ and C is smooth, the Fundamental Theorem for Line Integrals (FTLI) applies:

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = f(-1,0,\pi) - f(1,0,0).$$

The two values are

$$f(-1,0,\pi) = 4(-1)\pi^2 + 0\cdot\cos\pi + \pi = -4\pi^2 + \pi,$$
$$f(1,0,0) = 4(1)(0)^2 + 0\cdot\cos 0 + 0 = 0.$$

Therefore

$$\int_C \mathbf{F}\cdot \mathrm{d}\mathbf{r} = (-4\pi^2 + \pi) - 0 = \boxed{\pi - 4\pi^2}$$

Unofficial solutions

Method 2: Eliminating the parameter; coordinate-wise parametrisation; integration along a polygonal path; direct parametrisation of the helix

(ii) Eliminate t: from $x = 3t$, $t = \frac{x}{3}$ and $y = \frac{x^2}{9} - 1$. As t runs from 0 to 2, x increases from 0 to 6, so x is an orientation-preserving parameter, and

$$\mathrm{d}y = \frac{2x}{9}\,\mathrm{d}x.$$

The work is $\int_C (y - x)\,\mathrm{d}x + x^2\,\mathrm{d}y$, so

$$\begin{aligned} W &= \int_0^6 \left[\left(\frac{x^2}{9} - 1 - x\right) + x^2 \cdot \frac{2x}{9}\right] \mathrm{d}x = \int_0^6 \left(\frac{2x^3}{9} + \frac{x^2}{9} - x - 1\right) \mathrm{d}x \\ &= \left[\frac{x^4}{18} + \frac{x^3}{27} - \frac{x^2}{2} - x\right]_0^6 = \frac{1296}{18} + \frac{216}{27} - \frac{36}{2} - 6 = 72 + 8 - 18 - 6. \end{aligned}$$

Therefore

$$W = \boxed{56}$$

(iii) Along C the points are (t, t^2, t^3) with $0 \le t \le 1$, and each of x, y, z increases from 0 to 1. Hence each term may be integrated with its own coordinate as an orientation-preserving parameter:

$$C: \ (x, x^2, x^3) = \left(y^{1/2}, y, y^{3/2}\right) = \left(z^{1/3}, z^{2/3}, z\right).$$

The three terms are

$$\begin{aligned} \int_C 8x^2yz\,\mathrm{d}x &= \int_0^1 8x^2 \cdot x^2 \cdot x^3\,\mathrm{d}x = \int_0^1 8x^7\,\mathrm{d}x = \left[x^8\right]_0^1 = 1, \\ \int_C 5z\,\mathrm{d}y &= \int_0^1 5y^{3/2}\,\mathrm{d}y = 5 \cdot \frac{2}{5}\left[y^{5/2}\right]_0^1 = 2, \\ \int_C (-4xy)\,\mathrm{d}z &= \int_0^1 -4z^{1/3}z^{2/3}\,\mathrm{d}z = \int_0^1 -4z\,\mathrm{d}z = \left[-2z^2\right]_0^1 = -2. \end{aligned}$$

Adding,

$$\int_C 8x^2yz\,\mathrm{d}x + 5z\,\mathrm{d}y - 4xy\,\mathrm{d}z = 1 + 2 - 2 = \boxed{1}$$

(iv) (1) With P, Q, R as in Method 1, the cross-partials agree: $P_y = 0 = Q_x$, $P_z = 8z = R_x$, $Q_z = -\sin z = R_y$. Hence $\operatorname{curl}\mathbf{F} = \mathbf{0}$ on $\mathbb{R}^3$, which is simply connected, so $\mathbf{F}$ is conservative and $f(x, y, z) = \int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ along any path from $(0, 0, 0)$ to (x, y, z) is a potential. Take the polygonal path

$$(0, 0, 0) \to (x, 0, 0) \to (x, y, 0) \to (x, y, z).$$

On the first leg $(s, 0, 0)$, $0 \le s \le x$, $\mathrm{d}\mathbf{r} = (1, 0, 0)\,\mathrm{d}s$ and $P(s, 0, 0) = 4 \cdot 0^2 = 0$:

$$\int_0^x 0\,\mathrm{d}s = 0.$$

On the second leg $(x, s, 0)$, $0 \le s \le y$, $\mathrm{d}\mathbf{r} = (0, 1, 0)\,\mathrm{d}s$ and $Q(x, s, 0) = \cos 0 = 1$:

$$\int_0^y 1\,\mathrm{d}s = y.$$

On the third leg (x, y, s), $0 \le s \le z$, $\mathrm{d}\mathbf{r} = (0, 0, 1)\,\mathrm{d}s$ and $R(x, y, s) = 8xs - y\sin s + 1$:

$$\int_0^z (8xs - y\sin s + 1)\,\mathrm{d}s = \Big[4xs^2 + y\cos s + s\Big]_0^z = 4xz^2 + y\cos z + z - y.$$

Adding the three legs,

$$f(x, y, z) = 0 + y + \big(4xz^2 + y\cos z + z - y\big) = 4xz^2 + y\cos z + z.$$

Checking, $f_x = 4z^2$, $f_y = \cos z$, $f_z = 8xz - y\sin z + 1$, so $\nabla f = \mathbf{F}$ and

$$\boxed{f(x, y, z) = 4xz^2 + y\cos z + z}$$

(2) Compute the line integral directly. For $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le \pi$,

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1\rangle, \qquad \mathbf{F}(\mathbf{r}(t)) = \big\langle 4t^2,\ \cos t,\ 8t\cos t - \sin^2 t + 1\big\rangle,$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= -4t^2\sin t + \cos^2 t + 8t\cos t - \sin^2 t + 1\\ &= -4t^2\sin t + 8t\cos t + \cos 2t + 1.\end{aligned}$$

By parts with $U = t$, $\mathrm{d}V = \cos t\,\mathrm{d}t$:

$$\int_0^\pi t\cos t\,\mathrm{d}t = \Big[t\sin t\Big]_0^\pi - \int_0^\pi \sin t\,\mathrm{d}t = 0 - \Big[-\cos t\Big]_0^\pi = -(1 + 1) = -2.$$

By parts with $U = t^2$, $\mathrm{d}V = \sin t\,\mathrm{d}t$:

$$\int_0^\pi t^2\sin t\,\mathrm{d}t = \Big[-t^2\cos t\Big]_0^\pi + \int_0^\pi 2t\cos t\,\mathrm{d}t = \pi^2 + 2(-2) = \pi^2 - 4.$$

Also

$$\int_0^\pi \cos 2t\,\mathrm{d}t = \left[\frac{\sin 2t}{2}\right]_0^\pi = 0, \qquad \int_0^\pi 1\,\mathrm{d}t = \pi.$$

Hence

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = -4(\pi^2 - 4) + 8(-2) + 0 + \pi = -4\pi^2 + 16 - 16 + \pi.$$

Therefore

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \boxed{\pi - 4\pi^2}$$

Method 3: Splitting off a gradient field; path independence along a straight segment

(ii) Let $\varphi(x,y) = xy - \frac{x^2}{2}$. Its partial derivatives are

$$\varphi_x = y - x, \qquad \varphi_y = x,$$

so

$$\mathbf{F} = \nabla\varphi + \mathbf{G}, \qquad \mathbf{G} = \mathbf{F} - \nabla\varphi = \big(0,\ x^2 - x\big).$$

By the Fundamental Theorem for Line Integrals applied to $\nabla\varphi$,

$$W = \varphi(6,3) - \varphi(0,-1) + \int_C (x^2 - x)\,\mathrm{d}y,$$

with

$$\varphi(6,3) = 18 - 18 = 0, \qquad \varphi(0,-1) = 0 - 0 = 0.$$

On C, $x = 3t$ and $\mathrm{d}y = 2t\,\mathrm{d}t$ with $0 \le t \le 2$, so

$$\int_C (x^2 - x)\,\mathrm{d}y = \int_0^2 (9t^2 - 3t)\,2t\,\mathrm{d}t = \int_0^2 (18t^3 - 6t^2)\,\mathrm{d}t = \left[\frac{9}{2}t^4 - 2t^3\right]_0^2 = 72 - 16.$$

Therefore

$$W = 0 - 0 + 56 = \boxed{56}$$

(iv) (2) By part (1), $\mathbf{F} = \nabla f$ on $\mathbb{R}^3$, so $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r}$ depends only on the endpoints $(1,0,0)$ and $(-1,0,\pi)$. Replace the helix by the segment

$$\mathbf{r}(s) = (1 - 2s,\ 0,\ \pi s), \qquad 0 \le s \le 1, \qquad \mathbf{r}'(s) = (-2,0,\pi).$$

Along the segment $x = 1 - 2s$, $y = 0$, $z = \pi s$, so

$$\mathbf{F}(\mathbf{r}(s)) = \big(4\pi^2 s^2,\ \cos\pi s,\ 8(1-2s)\pi s + 1\big),$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(s))\cdot\mathbf{r}'(s) &= -8\pi^2 s^2 + 0 + \pi\big(8\pi s - 16\pi s^2 + 1\big)\\ &= -24\pi^2 s^2 + 8\pi^2 s + \pi.\end{aligned}$$

Hence

$$\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \int_0^1 \big(-24\pi^2 s^2 + 8\pi^2 s + \pi\big)\,\mathrm{d}s = \Big[-8\pi^2 s^3 + 4\pi^2 s^2 + \pi s\Big]_0^1 = -8\pi^2 + 4\pi^2 + \pi.$$

Therefore

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \boxed{\pi - 4\pi^2}$$

Marking scheme (17 marks)

(i) **(4 marks)** 1 mark: parametrises C; 1 mark: obtains $\mathrm{d}s = \sqrt{1+x^4}\,\mathrm{d}x$; 1 mark: makes the correct substitution; 1 mark: obtains $(17\sqrt{17} - 1)/6$.

(ii) **(4 marks)** 1 mark: supplies the parameter interval and derivatives; 2 marks: forms $\mathbf{F}\cdot\mathbf{r}'$ correctly; 1 mark: obtains 56.

(iii) **(3 marks)** 1 mark: obtains $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$; 1 mark: substitutes correctly; 1 mark: obtains 1.

(iv) **(6 marks)**

(1) **(3 marks)** 1 mark: integrates f_x; 1 mark: determines the y dependence; 1 mark: determines the z dependence and supplies a valid potential.

(2) **(3 marks)** 1 mark: finds the endpoints; 1 mark: applies the potential-difference theorem; 1 mark: obtains $\pi - 4\pi^2$.

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