NANYANG TECHNOLOGICAL UNIVERSITY

MIDTERM II

MH2100 – Calculus III

October 2022 TIME ALLOWED: 90 minutes

INSTRUCTIONS TO CANDIDATES

1. **DO NOT TURN OVER PAPER UNTIL INSTRUCTED.**
2. This midterm paper contains **THREE (3)** questions.
3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
4. Candidates should write their answers in the separate answer book provided.
5. **ONE** two-sided A4 piece of paper with handwritten notes is permitted.

QUESTION 1. **(16 marks)**

(i) [5 marks] Find the *maximum* and *minimum* values of xy satisfying the condition that $3x^2 + 2y^2 = 36$. Also give the points (x, y) where the *maximum* and *minimum* values are achieved.

(ii) [3 marks] Evaluate the iterated integral.

$$\int_{-1}^{1} \int_{0}^{1} 2ye^x \, \mathrm{d}x \, \mathrm{d}y.$$

(iii) [4 marks] By reversing the order of integration, evaluate the integral

$$\int_{0}^{\sqrt{\pi/2}} \int_{y}^{\sqrt{\pi/2}} 2\cos(x^2) \, \mathrm{d}x \, \mathrm{d}y.$$

(iv) [4 marks] Find the area of the region D (in the xy-plane) bounded by the cardioid expressed in polar coordinates as

$$r = 1 - \sin\theta; \quad \text{and} \quad 0 \leq \theta \leq 2\pi.$$

Hint: the formula $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$ may be useful.

QUESTION 2. **(17 marks)**

(i) [4 marks] Use cylindrical coordinates to evaluate

$$\iiint_E \sqrt{x^2+y^2}\mathrm{d}V,$$

where E is the solid bounded by the cylinder $x^2+y^2=1$ and the planes $z=1$ and $z=2$.

(ii) Consider the transformation $u=x+4y$ and $v=y-x$.

(a) [3 marks] Calculate the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$.

(b) [5 marks] Use the transformation to evaluate the iterated integral

$$\int_0^1 \int_y^{5-4y} (x+4y)e^{y-x}\mathrm{d}x\mathrm{d}y.$$

Hint: you can use $\displaystyle\int xe^{ax}\mathrm{d}x = \frac{e^{ax}(ax-1)}{a^2}+C.$

(iii) [5 marks] Use spherical coordinates to find the volume of the region inside the sphere $x^2+y^2+z^2=4$ and outside the cone $z^2=x^2+y^2$. That is, the volume of the region E given by

$$E=\{(x,y,z)\mid x^2+y^2+z^2\leq 4 \text{ and } z^2\leq x^2+y^2\}.$$

QUESTION 3. **(17 marks)**

(i) [4 marks] Let C be the curve parametrised by $x = 3t$ and $y = t^3$ where $0 \leq t \leq 1$. Evaluate

$$\int_C (x^3 + y)\mathrm{d}s.$$

(ii) [4 marks] Find the work done by the vector field $\mathbf{F} = \langle x^3 - y^3, xy^2 \rangle$ along the curve parametrised by $x = t^2$ and $y = t^3$ from $t = -1$ to $t = 0$.

(iii) [4 marks] Determine if the vector field $\mathbf{F} = \langle 3x^2y^2 + e^x, 2x^3y + \sin(y) \rangle$ is conservative. If so, find a potential function f for $\mathbf{F}$.

(iv) [5 marks] Let C be part of the curve (in the xy-plane) described by polar coordinates as $r = \sin(\theta)$ where $0 \leq \theta \leq \pi$. Find the length of C.