MH2100 Calculus III

Midterm I (Solutions)

AY 2022/2023, Semester 1

Quantitative Research Society @NTU

September 2022

Unofficial QRS@NTU Solutions

Overview

- **Coverage:** parametric curves and tangent lines; limits and continuity; partial and directional derivatives; tangent planes, the Chain Rule, implicit differentiation, and critical points.
- **Format:** three questions, 50 marks in total, 90 minutes.
- **Current administration (AY2026/2027):** Tuesday 6 October 2026, 16:30–18:20, venue TBA. One double-sided typed A4 sheet is permitted; no attachments or sticky notes are permitted.
- **Source:** MH2100 Midterm I, AY2022/2023 Semester 1. Every question below is transcribed from the official paper. Solutions and marking schemes are unofficial QRS@NTU material.

Instructions to candidates (verbatim)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. ONE two-sided A4 piece of paper with handwritten notes is permitted.

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Unofficial QRS Solutions

Question 1 (17 Marks)

QUESTION 1.

(i) [3 marks] Find parametric equations for the line that is tangent to the curve parametrised by

$$\mathbf{r}(t) = \begin{pmatrix} t^2 - 2t + 1 \\ \sin(t^2 - 1) \end{pmatrix},$$

at the point $(0, 0)$.

(ii) [3 marks] At the points defined by $t = a$ the tangent line to the curve parametrised by

$$\begin{cases} x(t) = 2t, \\ y(t) = 3t^2 + t, \\ z(t) = -3t^2 - t \end{cases}$$

contains the point $(0, -12, 12)$. Find all such values of a.

(iii) [3 marks] Evaluate the following limit or show that the limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{3xy - y}{x^2 - 3x + y^2 + 7y + 1}.$$

(iv) [4 marks] At the point defined by $t = a$ the tangent vector of the curve parametrised by

$$\mathbf{r}(t) = \begin{pmatrix} \cos(3t^2) \\ t \\ t^2 + 2t \end{pmatrix},$$

is a scalar multiple of the gradient vector $\nabla f(1, 2, 3)$ where $f(x, y, z) = x^2 + y^2 - 2x - 2yz$. Find the value of a.

(v) [4 marks] Find the directional derivative of the function

$$f(x, y) = \begin{cases} \dfrac{x^3y^2 - 3y^5}{x^4 + 2x^2y^2 + y^4}; & \text{if } x^2 + y^2 \neq 0, \\ 0; & \text{if } x^2 + y^2 = 0, \end{cases}$$

in the direction of the vector $\langle 1, -1 \rangle$ at the point $(0, 0)$.

Solution

Method 1: Componentwise differentiation, limit laws and the limit definition of $D_{\mathbf{u}}f$

(i) Write $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ with

$$x(t) = t^2 - 2t + 1 = (t - 1)^2, \qquad y(t) = \sin(t^2 - 1).$$

First find the parameter value of the point $(0, 0)$:

$$x(t) = 0 \iff (t - 1)^2 = 0 \iff t = 1,$$

and at $t = 1$

$$y(1) = \sin(1^2 - 1) = \sin 0 = 0.$$

Hence $(0,0) = \mathbf{r}(1)$, and $t = 1$ is the only parameter value giving this point.

Differentiate each component (the second one by the Chain Rule):

$$x'(t) = 2t - 2, \qquad y'(t) = \cos(t^2 - 1) \cdot \frac{d}{dt}(t^2 - 1) = 2t\cos(t^2 - 1).$$

At $t = 1$,

$$\mathbf{r}'(1) = \langle 2(1) - 2,\ 2(1)\cos 0 \rangle = \langle 0, 2 \rangle \neq \mathbf{0},$$

so the tangent line exists and has direction vector $\mathbf{r}'(1)$. The tangent line is $\mathbf{l}(s) = \mathbf{r}(1) + s\,\mathbf{r}'(1)$:

$$\boxed{\mathbf{l}(s) = \langle 0, 0 \rangle + s\langle 0, 2 \rangle,\ s \in \mathbb{R},} \iff \boxed{x = 0,\ y = 2s.}$$

(ii) Write $\mathbf{r}(t) = \langle 2t,\ 3t^2 + t,\ -3t^2 - t \rangle$. Differentiating each component,

$$\mathbf{r}'(t) = \langle 2,\ 6t + 1,\ -6t - 1 \rangle.$$

At $t = a$,

$$\mathbf{r}(a) = \langle 2a,\ 3a^2 + a,\ -3a^2 - a \rangle, \qquad \mathbf{r}'(a) = \langle 2,\ 6a + 1,\ -6a - 1 \rangle \neq \mathbf{0},$$

the last because the first component is 2. The tangent line at $t = a$ is

$$\mathbf{L}(s) = \mathbf{r}(a) + s\,\mathbf{r}'(a) = \langle 2a + 2s,\ 3a^2 + a + s(6a + 1),\ -3a^2 - a - s(6a + 1) \rangle, \qquad s \in \mathbb{R}.$$

The point $(0, -12, 12)$ lies on this line if and only if some s satisfies all three component equations.

The x-component gives

$$0 = 2a + 2s \implies s = -a.$$

Substitute $s = -a$ in the y-component:

$$\begin{aligned} -12 &= 3a^2 + a + (-a)(6a + 1) \\ &= 3a^2 + a - 6a^2 - a \\ &= -3a^2, \end{aligned}$$

hence

$$a^2 = 4 \implies a = 2 \ \text{ or } \ a = -2.$$

Substitute $s = -a$ in the z-component:

$$\begin{aligned} -3a^2 - a - (-a)(6a + 1) &= -3a^2 - a + 6a^2 + a \\ &= 3a^2 = 3 \cdot 4 = 12, \end{aligned}$$

so the z-equation also holds for both values. Directly, for $a = 2$ ($s = -2$):

$$\mathbf{r}(2) - 2\,\mathbf{r}'(2) = \langle 4, 14, -14 \rangle - 2\langle 2, 13, -13 \rangle = \langle 0, -12, 12 \rangle,$$

and for $a = -2$ ($s = 2$):

$$\mathbf{r}(-2) + 2\,\mathbf{r}'(-2) = \langle -4, 10, -10 \rangle + 2\langle 2, -11, 11 \rangle = \langle 0, -12, 12 \rangle.$$

Therefore

$$\boxed{a = -2 \text{ or } a = 2.}$$

(iii) Let

$$N(x,y) = 3xy - y, \qquad D(x,y) = x^2 - 3x + y^2 + 7y + 1.$$

Both are polynomials, hence continuous on $\mathbb{R}^2$, so

$$\lim_{(x,y)\to(0,0)} N(x,y) = N(0,0) = 0, \qquad \lim_{(x,y)\to(0,0)} D(x,y) = D(0,0) = 1.$$

Since the limit of the denominator is $1 \neq 0$, the Quotient Law for limits applies:

$$\lim_{(x,y)\to(0,0)} \frac{3xy - y}{x^2 - 3x + y^2 + 7y + 1} = \frac{0}{1} = \boxed{0}$$

(iv) Differentiate each component of $\mathbf{r}$ (the first by the Chain Rule):

$$\frac{d}{dt}\cos(3t^2) = -\sin(3t^2)\cdot 6t, \qquad \frac{d}{dt}\,t = 1, \qquad \frac{d}{dt}(t^2 + 2t) = 2t + 2,$$

so

$$\mathbf{r}'(t) = \langle -6t\sin(3t^2),\ 1,\ 2t + 2\rangle.$$

The partial derivatives of $f(x,y,z) = x^2 + y^2 - 2x - 2yz$ are

$$f_x = 2x - 2, \qquad f_y = 2y - 2z, \qquad f_z = -2y.$$

At $(1,2,3)$:

$$f_x = 2 - 2 = 0, \qquad f_y = 4 - 6 = -2, \qquad f_z = -4, \qquad \text{so} \qquad \nabla f(1,2,3) = \langle 0, -2, -4\rangle.$$

The condition is $\mathbf{r}'(a) = \lambda \nabla f(1,2,3)$ for some scalar λ, i.e.

$$\begin{aligned} -6a\sin(3a^2) &= 0 \cdot \lambda = 0 && (x)\\ 1 &= -2\lambda && (y)\\ 2a + 2 &= -4\lambda && (z) \end{aligned}$$

From (y),

$$\lambda = -\tfrac{1}{2}.$$

Substitute in (z):

$$2a + 2 = -4\left(-\tfrac{1}{2}\right) = 2 \implies a = 0.$$

Check (x) at $a = 0$:

$$-6 \cdot 0 \cdot \sin 0 = 0.$$

The (x)-equation alone also allows $3a^2 = k\pi$, $k \in \{1, 2, \ldots\}$, but (z) admits only $a = 0$. Since the second component of $\mathbf{r}'(a)$ is 1, $\mathbf{r}'(a) \neq \mathbf{0}$, so $\lambda \neq 0$ and the reading $\nabla f(1,2,3) = \mu\,\mathbf{r}'(a)$ (with $\mu = 1/\lambda = -2$) gives the same condition. Therefore

$$\boxed{a = 0.}$$

(v) Normalise the direction:

$$|\langle 1, -1\rangle| = \sqrt{1 + 1} = \sqrt{2}, \qquad \mathbf{u} = \tfrac{1}{\sqrt{2}}\langle 1, -1\rangle.$$

The formula $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ requires f to be differentiable at $(0,0)$, which is not known here, so use the definition

$$D_{\mathbf{u}} f(0,0) = \lim_{t\to 0} \frac{f(t\mathbf{u}) - f(0,0)}{t}.$$

For $t \neq 0$, $t\mathbf{u} = \left(\frac{t}{\sqrt{2}}, -\frac{t}{\sqrt{2}}\right) \neq (0,0)$, so the first formula applies with $x = \frac{t}{\sqrt{2}}$, $y = -\frac{t}{\sqrt{2}}$. The numerator:

$$\begin{aligned} x^3y^2 &= \frac{t^3}{2\sqrt{2}} \cdot \frac{t^2}{2} = \frac{t^5}{4\sqrt{2}}, \\ -3y^5 &= -3\left(-\frac{t^5}{4\sqrt{2}}\right) = \frac{3t^5}{4\sqrt{2}}, \\ x^3y^2 - 3y^5 &= \frac{4t^5}{4\sqrt{2}} = \frac{t^5}{\sqrt{2}}. \end{aligned}$$

The denominator:

$$\begin{aligned} x^4 + 2x^2y^2 + y^4 &= \frac{t^4}{4} + 2 \cdot \frac{t^2}{2} \cdot \frac{t^2}{2} + \frac{t^4}{4} \\ &= \frac{t^4}{4} + \frac{t^4}{2} + \frac{t^4}{4} = t^4. \end{aligned}$$

Hence, for $t \neq 0$,

$$f(t\mathbf{u}) = \frac{t^5/\sqrt{2}}{t^4} = \frac{t}{\sqrt{2}}, \qquad \frac{f(t\mathbf{u}) - f(0,0)}{t} = \frac{t/\sqrt{2} - 0}{t} = \frac{1}{\sqrt{2}}.$$

The difference quotient is constant, so

$$D_{\mathbf{u}} f(0,0) = \lim_{t \to 0} \frac{f(t\mathbf{u}) - f(0,0)}{t} = \boxed{\frac{1}{\sqrt{2}}}$$

The gradient formula would give a different number. Along the axes, $f(h,0) = 0$ and $f(0,h) = \dfrac{-3h^5}{h^4} = -3h$ for $h \neq 0$, so

$$f_x(0,0) = \lim_{h \to 0} \frac{0-0}{h} = 0, \qquad f_y(0,0) = \lim_{h \to 0} \frac{-3h - 0}{h} = -3,$$

$$\nabla f(0,0) \cdot \mathbf{u} = \frac{0 \cdot 1 + (-3)(-1)}{\sqrt{2}} = \frac{3}{\sqrt{2}} \neq \frac{1}{\sqrt{2}}.$$

Thus f is not differentiable at $(0,0)$, and the limit definition is the valid route.

Method 2: Cartesian slope dx/dy for the tangent line

This method applies to part (i).

(i) The point $(0,0)$ corresponds only to $t=1$, since $x(t)=(t-1)^2=0 \iff t=1$ and $y(1)=\sin 0=0$. The parameter derivatives are

$$\frac{dx}{dt}=2t-2, \qquad \frac{dy}{dt}=2t\cos(t^2-1).$$

At $t=1$,

$$\frac{dx}{dt}\Big|_{t=1}=0, \qquad \frac{dy}{dt}\Big|_{t=1}=2\cos 0=2\neq 0.$$

The slope $dy/dx=(dy/dt)/(dx/dt)$ has a zero denominator, so the tangent is not of the form $y=mx+c$. Since dy/dt is continuous and equals $2>0$ at $t=1$, y is strictly increasing in t near $t=1$, so near $(0,0)$ the curve is a graph $x=x(y)$ with

$$\frac{dx}{dy}=\frac{dx/dt}{dy/dt}=\frac{2t-2}{2t\cos(t^2-1)}, \qquad \frac{dx}{dy}\Big|_{t=1}=\frac{0}{2}=0.$$

The tangent line through $(0,0)$ with $dx/dy=0$ is

$$x-0=0\cdot(y-0), \qquad \text{i.e.} \qquad x=0,$$

the vertical line. Parametrising it by $y=2s$,

$$\boxed{\mathbf{l}(s)=\langle 0,0\rangle+s\langle 0,2\rangle,\ s\in\mathbb{R},} \iff \boxed{x=0,\ y=2s.}$$

Method 3: Cross-product parallelism test

This method applies to parts (ii) and (iv). For a nonzero vector $\mathbf{v}$, a vector $\mathbf{w}$ is a scalar multiple of $\mathbf{v}$ if and only if $\mathbf{w}\times\mathbf{v}=\mathbf{0}$.

(ii) With $\mathbf{r}(a)=\langle 2a,\ 3a^2+a,\ -3a^2-a\rangle$ and $\mathbf{r}'(a)=\langle 2,\ 6a+1,\ -6a-1\rangle\neq\mathbf{0}$, the point $P=(0,-12,12)$ lies on the tangent line at $t=a$ if and only if $P-\mathbf{r}(a)$ is a scalar multiple of $\mathbf{r}'(a)$. Compute

$$P-\mathbf{r}(a)=\langle 0-2a,\ -12-3a^2-a,\ 12+3a^2+a\rangle=\langle -2a,\ -B,\ B\rangle, \qquad B:=3a^2+a+12,$$

and write $\mathbf{r}'(a)=\langle 2,\ C,\ -C\rangle$ with $C:=6a+1$. Expanding the determinant along the first row,

$$\begin{aligned}(P-\mathbf{r}(a))\times\mathbf{r}'(a)&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-2a&-B&B\\2&C&-C\end{vmatrix}\\&=\mathbf{i}\big((-B)(-C)-BC\big)-\mathbf{j}\big((-2a)(-C)-2B\big)+\mathbf{k}\big((-2a)C-(-B)2\big)\\&=\langle 0,\ 2B-2aC,\ 2B-2aC\rangle.\end{aligned}$$

This is $\mathbf{0}$ if and only if $B=aC$:

$$\begin{aligned}&3a^2+a+12=a(6a+1)=6a^2+a\\&\iff 3a^2=12\\&\iff a^2=4.\end{aligned}$$

Therefore

$$\boxed{a=-2 \text{ or } a=2.}$$

(iv) Differentiating componentwise,

$$\mathbf{r}'(a) = \langle X,\ 1,\ Z\rangle, \qquad X := -6a\sin(3a^2), \quad Z := 2a+2.$$

The partial derivatives $f_x = 2x-2$, $f_y = 2y-2z$, $f_z = -2y$ at $(1,2,3)$ give

$$\nabla f(1,2,3) = \langle 2-2,\ 4-6,\ -4\rangle = \langle 0,-2,-4\rangle \neq \mathbf{0}.$$

So $\mathbf{r}'(a)$ is a scalar multiple of $\nabla f(1,2,3)$ if and only if $\mathbf{r}'(a)\times\nabla f(1,2,3) = \mathbf{0}$. Expanding the determinant,

$$\begin{aligned}\mathbf{r}'(a)\times\nabla f(1,2,3) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ X & 1 & Z\\ 0 & -2 & -4\end{vmatrix}\\ &= \mathbf{i}\big(1\cdot(-4) - Z(-2)\big) - \mathbf{j}\big(X(-4) - Z\cdot 0\big) + \mathbf{k}\big(X(-2) - 1\cdot 0\big)\\ &= \langle -4+2Z,\ 4X,\ -2X\rangle.\end{aligned}$$

The first component vanishes if and only if

$$-4+2(2a+2) = 4a = 0 \iff a = 0,$$

and at $a=0$ the other two components are $4X = -24\cdot 0\cdot\sin 0 = 0$ and $-2X = 0$. Therefore

$$\boxed{a = 0.}$$

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Method 4: Squeeze bound near the origin

This method applies to part (iii).

(iii) Let $r = \sqrt{x^2+y^2}$, so $|x| \le r$ and $|y| \le r$. For the denominator, since $x^2+y^2 \ge 0$,

$$\begin{aligned} x^2 - 3x + y^2 + 7y + 1 &\ge 1 - 3|x| - 7|y| \\ &\ge 1 - 10r \ge \tfrac{1}{2} \qquad \text{whenever } r \le \tfrac{1}{20}. \end{aligned}$$

For the numerator,

$$|3xy - y| \le 3|x||y| + |y| \le 3r^2 + r.$$

Hence, for $0 < r \le \frac{1}{20}$,

$$0 \le \left| \frac{3xy - y}{x^2 - 3x + y^2 + 7y + 1} \right| \le \frac{3r^2 + r}{1/2} = 6r^2 + 2r.$$

Since $6r^2 + 2r \to 0$ as $(x,y) \to (0,0)$, the Squeeze Theorem gives

$$\lim_{(x,y)\to(0,0)} \frac{3xy - y}{x^2 - 3x + y^2 + 7y + 1} = \boxed{0}$$

Method 5: Degree-one homogeneity

This method applies to part (v).

(v) For $(x,y) \neq (0,0)$ the numerator $x^3y^2 - 3y^5$ is homogeneous of degree 5 and the denominator $x^4 + 2x^2y^2 + y^4 = (x^2+y^2)^2$ is homogeneous of degree 4. Hence, for every real $t \neq 0$,

$$f(tx, ty) = \frac{t^5\,(x^3y^2 - 3y^5)}{t^4\,(x^4 + 2x^2y^2 + y^4)} = t\,f(x,y),$$

and this also holds at $(0,0)$ because $f(0,0) = 0$. With $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, -1\rangle$,

$$\frac{f(t\mathbf{u}) - f(0,0)}{t} = \frac{t\,f(\mathbf{u})}{t} = f(\mathbf{u}) \qquad (t \neq 0),$$

so $D_{\mathbf{u}} f(0,0) = f(\mathbf{u})$. Evaluate at $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$:

$$\begin{aligned} x^3y^2 - 3y^5 &= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} - 3\left(-\frac{1}{4\sqrt{2}}\right) = \frac{1}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} = \frac{1}{\sqrt{2}}, \\ (x^2+y^2)^2 &= \left(\tfrac{1}{2} + \tfrac{1}{2}\right)^2 = 1. \end{aligned}$$

Therefore

$$D_{\mathbf{u}} f(0,0) = f(\mathbf{u}) = \boxed{\frac{1}{\sqrt{2}}}$$

UNOFFICIAL QRS marking scheme (17 marks)

(i) **(3 marks)** 1: identifies $t = 1$; 1: finds the tangent vector; 1: gives the line.

(ii) **(3 marks)** 1: writes the tangent line; 1: solves $a^2 = 4$; 1: gives both values and checks the remaining component.

(iii) **(3 marks)** 1: identifies a nonzero denominator; 1: justifies substitution; 1: obtains 0.

(iv) **(4 marks)** 1: differentiates $\mathbf{r}$; 1: computes the gradient; 1: imposes proportionality; 1: verifies $a = 0$.

(v) **(4 marks)** 1: normalizes the direction; 1: substitutes the path; 1: expands both expressions; 1: evaluates the limit.

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Question 2 (16 Marks)

QUESTION 2.

(i) [4 marks] Evaluate each of the following limits or show that the limit does not exist. Justify your answers.

(a)
$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^3y+(x-y)^4}$$

(b)
$$\lim_{(x,y)\to(1,0)} \frac{7x^2-7}{x^2+y^2-1}$$

(ii) [4 marks]

(a) Find k such that the function
$$f(x,y)=\begin{cases}\dfrac{x^4+3x^2-xy^3+3y^2}{x^2+y^2} & \text{if } x^2+y^2>0;\\ k & \text{if } x^2+y^2=0\end{cases}$$
is continuous. (No justification is needed).

(b) For the k found in Part (a), prove that
$$\lim_{(x,y)\to(0,0)} \frac{x^4+3x^2-xy^3+3y^2}{x^2+y^2}=k.$$

(iii) [8 marks] Define
$$f(x,y)=\begin{cases}\dfrac{x^3-xy^2}{x^2+y^2}; & \text{if } x^2+y^2\neq 0,\\ 0; & \text{if } x^2+y^2=0.\end{cases}$$

(a) Is f continuous at $(0,0)$? Justify your answer.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

(c) Is f differentiable at $(0,0)$? Justify your answer.

Solution

Method 1: Two-path test, Squeeze Theorem and the limit definitions

(i) (a) Let $g(x,y)=\dfrac{x^3y}{x^3y+(x-y)^4}$, defined wherever the denominator is nonzero.

Along the path $y=0$, $x\neq 0$: the denominator is $x^3\cdot 0+(x-0)^4=x^4\neq 0$, and
$$g(x,0)=\frac{x^3\cdot 0}{x^4}=0 \quad\Longrightarrow\quad \lim_{x\to 0} g(x,0)=0.$$

Along the path $y=x$, $x\neq 0$: the numerator is $x^3\cdot x=x^4$ and the denominator is $x^4+(x-x)^4=x^4\neq 0$, so
$$g(x,x)=\frac{x^4}{x^4}=1 \quad\Longrightarrow\quad \lim_{x\to 0} g(x,x)=1.$$

The two paths give the different limits $0\neq 1$, so by the two-path test

$$\boxed{\text{the limit does not exist.}}$$

(b) Let $h(x,y) = \dfrac{7x^2-7}{x^2+y^2-1}$, defined wherever $x^2+y^2 \neq 1$.

Along the path $x=1$, $y \neq 0$: the numerator is $7\cdot 1 - 7 = 0$ and the denominator is $1+y^2-1 = y^2 \neq 0$, so

$$h(1,y) = \frac{0}{y^2} = 0 \quad \Longrightarrow \quad \lim_{y\to 0} h(1,y) = 0.$$

Along the path $y=0$, $x>0$, $x\neq 1$: the denominator is $x^2-1 \neq 0$, so

$$h(x,0) = \frac{7(x^2-1)}{x^2-1} = 7 \quad \Longrightarrow \quad \lim_{x\to 1} h(x,0) = 7.$$

The two paths give the different limits $0 \neq 7$, so by the two-path test

$$\boxed{\text{the limit does not exist.}}$$

(ii) (a) For $(x,y) \neq (0,0)$, f is a quotient of polynomials with denominator $x^2+y^2 > 0$, so it is continuous there. At $(0,0)$, continuity requires $k = f(0,0) = \lim_{(x,y)\to(0,0)} f(x,y)$. If this limit exists it equals the limit along the path $y=0$, $x\neq 0$:

$$f(x,0) = \frac{x^4+3x^2}{x^2} = x^2+3 \;\to\; 3 \qquad (x\to 0).$$

So the only possible value is

$$\boxed{k=3}$$

and part (b) proves that the limit is indeed 3.

(b) Let $r = \sqrt{x^2+y^2}$, so $|x| \le r$ and $|y| \le r$. Split off $3(x^2+y^2)$ from the numerator:

$$x^4+3x^2-xy^3+3y^2 = 3(x^2+y^2) + x^4 - xy^3,$$

hence, for $(x,y) \neq (0,0)$,

$$\frac{x^4+3x^2-xy^3+3y^2}{x^2+y^2} = 3 + \frac{x^4-xy^3}{x^2+y^2}.$$

Bound the remainder with the triangle inequality and $|x|, |y| \le r$:

$$0 \le \left|\frac{x^4-xy^3}{x^2+y^2}\right| \le \frac{|x|^4+|x||y|^3}{r^2} \le \frac{r^4+r\cdot r^3}{r^2} = 2r^2.$$

Since $2r^2 \to 0$ as $(x,y)\to(0,0)$, the Squeeze Theorem gives $\dfrac{x^4-xy^3}{x^2+y^2} \to 0$, and therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^4+3x^2-xy^3+3y^2}{x^2+y^2} = 3+0 = \boxed{3=k}$$

(iii) (a) For $(x,y) \neq (0,0)$ factor out x:

$$f(x,y) = x\cdot\frac{x^2-y^2}{x^2+y^2}.$$

Since $|x^2-y^2| \le x^2+y^2$, the fraction has absolute value at most 1, so

$$0 \le |f(x,y)| \le |x| \le \sqrt{x^2+y^2}.$$

As $(x,y)\to(0,0)$ the right-hand side tends to 0, so by the Squeeze Theorem $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. Therefore

$$\boxed{f \text{ is continuous at } (0,0).}$$

(b) By definition,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}, \qquad f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}.$$

For $h \neq 0$,

$$f(h,0) = \frac{h^3 - h\cdot 0}{h^2} = h, \qquad \frac{f(h,0) - f(0,0)}{h} = \frac{h-0}{h} = 1,$$

$$f(0,h) = \frac{0 - 0\cdot h^2}{h^2} = 0, \qquad \frac{f(0,h) - f(0,0)}{h} = \frac{0-0}{h} = 0.$$

Both quotients are constant, so

$$\boxed{f_x(0,0) = 1, \qquad f_y(0,0) = 0.}$$

(c) If f is differentiable at $(0,0)$, its linearisation must be

$$L(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y = x,$$

and the definition of differentiability requires

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - L(x,y)}{\sqrt{x^2+y^2}} = 0.$$

Test this along the path $y = x$, $x = t \neq 0$:

$$f(t,t) = \frac{t^3 - t\cdot t^2}{t^2 + t^2} = \frac{0}{2t^2} = 0,$$

so

$$\frac{f(t,t) - f(0,0) - L(t,t)}{\sqrt{t^2+t^2}} = \frac{0 - 0 - t}{\sqrt{2}\,|t|} = -\frac{\operatorname{sgn} t}{\sqrt{2}}.$$

For every $t > 0$ this equals $-\frac{1}{\sqrt{2}}$, so its limit as $t \to 0^+$ along this path is $-\frac{1}{\sqrt{2}} \neq 0$. Hence the two-variable limit is not 0, and

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

Method 2: One-parameter families of paths

This method applies to part (i). If the limit L existed, every path in the family would give the same value L; a family whose path limits depend on the parameter therefore shows that the limit does not exist.

(i) (a) Use the lines $y = mx$, $x \neq 0$. The numerator is $x^3(mx) = mx^4$ and the denominator is

$$mx^4 + (x - mx)^4 = x^4\big(m + (1-m)^4\big).$$

The bracket is never 0: for $m \geq 0$ it vanishes only if $m = 0$ and $(1-m)^4 = 0$, which is impossible; for $m < 0$, $1 - m > 1$ gives $(1-m)^4 > 1 - m > -m$, so $m + (1-m)^4 > 0$. Hence

$$g(x, mx) = \frac{mx^4}{x^4\big(m + (1-m)^4\big)} = \frac{m}{m + (1-m)^4} \qquad (x \neq 0),$$

a constant along each line. For $m = 0$, $m = 1$, $m = 2$:

$$\frac{0}{0+1} = 0, \qquad \frac{1}{1+0} = 1, \qquad \frac{2}{2+1} = \frac{2}{3}.$$

The path limits depend on m, so

$$\boxed{\text{the limit does not exist.}}$$

(b) Use the curves $x = \sqrt{1 + my^2}$ with $m \geq 0$, $y \neq 0$. Each curve passes through $(1, 0)$ since $x \to 1$ as $y \to 0$. On it,

$$7x^2 - 7 = 7(1 + my^2) - 7 = 7my^2, \qquad x^2 + y^2 - 1 = my^2 + y^2 = (m+1)y^2 \neq 0,$$

so

$$h\big(\sqrt{1+my^2}, y\big) = \frac{7my^2}{(m+1)y^2} = \frac{7m}{m+1} \qquad (y \neq 0).$$

For $m = 0$ (the line $x = 1$) the path limit is 0; for $m = 1$ it is $\frac{7}{2}$. The path limits depend on m, so

$$\boxed{\text{the limit does not exist.}}$$

Method 3: Polar coordinates

This method applies to parts (ii)(b), (iii)(a) and (iii)(c). Write $x = r\cos\theta$, $y = r\sin\theta$ with $r = \sqrt{x^2+y^2} > 0$. If $|F(x,y)-L| \le C\,r^p$ with C and $p > 0$ independent of θ, then $F(x,y) \to L$ as $(x,y) \to (0,0)$.

(ii) (b) Write the numerator as $3(x^2+y^2) + x^4 - xy^3$ and substitute:

$$x^4 - xy^3 = r^4\cos^4\theta - r\cos\theta \cdot r^3\sin^3\theta = r^4\left(\cos^4\theta - \cos\theta\sin^3\theta\right),$$
$$x^2 + y^2 = r^2.$$

Hence, for $r > 0$,

$$\frac{x^4+3x^2-xy^3+3y^2}{x^2+y^2} = \frac{3r^2 + r^4\left(\cos^4\theta - \cos\theta\sin^3\theta\right)}{r^2}$$
$$= 3 + r^2\left(\cos^4\theta - \cos\theta\sin^3\theta\right).$$

Since $|\cos\theta| \le 1$ and $|\sin\theta| \le 1$,

$$\left|r^2\left(\cos^4\theta - \cos\theta\sin^3\theta\right)\right| \le r^2(1+1) = 2r^2 \to 0 \quad (r \to 0^+),$$

uniformly in θ. Therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^4+3x^2-xy^3+3y^2}{x^2+y^2} = \boxed{3 = k}$$

(iii) (a) Substitute:

$$x^3 - xy^2 = r^3\cos^3\theta - r^3\cos\theta\sin^2\theta$$
$$= r^3\cos\theta\left(\cos^2\theta - \sin^2\theta\right) = r^3\cos\theta\cos 2\theta,$$
$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos\theta\cos 2\theta}{r^2} = r\cos\theta\cos 2\theta.$$

Hence

$$|f(x,y) - 0| = r\,|\cos\theta|\,|\cos 2\theta| \le r \to 0 \quad (r \to 0^+),$$

uniformly in θ, so $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$ and

$$\boxed{f \text{ is continuous at } (0,0).}$$

(c) The partial derivatives come from the axis quotients: for $h \neq 0$, $\dfrac{f(h,0)-0}{h} = \dfrac{h^3/h^2}{h} = 1$ and $\dfrac{f(0,h)-0}{h} = \dfrac{0}{h} = 0$, so $f_x(0,0) = 1$, $f_y(0,0) = 0$ and the candidate linearisation is $L(x,y) = x$. The differentiability quotient in polar form is

$$E(r,\theta) := \frac{f(x,y) - f(0,0) - L(x,y)}{\sqrt{x^2+y^2}} = \frac{r\cos\theta\cos 2\theta - 0 - r\cos\theta}{r}$$
$$= \cos\theta\,(\cos 2\theta - 1) = \cos\theta\left(-2\sin^2\theta\right) = -2\cos\theta\sin^2\theta.$$

E does not depend on r. Along the ray $\theta = \frac{\pi}{4}$,

$$E\left(r, \tfrac{\pi}{4}\right) = -2\cdot\frac{1}{\sqrt{2}}\cdot\frac{1}{2} = -\frac{1}{\sqrt{2}} \quad \text{for every } r > 0,$$

so $E \not\to 0$ as $(x,y) \to (0,0)$. Therefore

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

Method 4: The ε–δ definition of the limit

This method applies to parts (ii)(b) and (iii)(a). By definition, $\lim_{(x,y)\to(0,0)} F(x,y) = L$ means: for every $\varepsilon > 0$ there is $\delta > 0$ such that $0 < \sqrt{x^2+y^2} < \delta$ implies $|F(x,y) - L| < \varepsilon$.

(ii) (b) Let $r = \sqrt{x^2+y^2} > 0$. Since $x^4 + 3x^2 - xy^3 + 3y^2 = 3(x^2+y^2) + x^4 - xy^3$,

$$\left|\frac{x^4+3x^2-xy^3+3y^2}{x^2+y^2} - 3\right| = \frac{|x^4 - xy^3|}{r^2} \le \frac{|x|^4 + |x||y|^3}{r^2} \le \frac{r^4+r^4}{r^2} = 2r^2,$$

using $|x|, |y| \le r$. Given $\varepsilon > 0$, choose $\delta = \sqrt{\varepsilon/2}$. Then $0 < r < \delta$ implies

$$\left|\frac{x^4+3x^2-xy^3+3y^2}{x^2+y^2} - 3\right| \le 2r^2 < 2\delta^2 = 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^4+3x^2-xy^3+3y^2}{x^2+y^2} = \boxed{3 = k}$$

(iii) (a) For $(x,y) \neq (0,0)$, $f(x,y) = x \cdot \dfrac{x^2-y^2}{x^2+y^2}$ and $|x^2 - y^2| \le x^2 + y^2$, so

$$|f(x,y) - 0| \le |x| \le \sqrt{x^2+y^2}.$$

Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $0 < \sqrt{x^2+y^2} < \delta$ implies

$$|f(x,y) - f(0,0)| \le \sqrt{x^2+y^2} < \delta = \varepsilon.$$

So $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, and

$$\boxed{f \text{ is continuous at } (0,0).}$$

Method 5: Directional derivative compared with the gradient

This method applies to part (iii)(c). Theorem: if f is differentiable at $(0,0)$, then for every unit vector $\mathbf{u}$ the directional derivative exists and $D_{\mathbf{u}} f(0,0) = \nabla f(0,0) \cdot \mathbf{u}$.

(iii) (c) From the axis quotients, for $h \neq 0$,

$$\frac{f(h,0) - f(0,0)}{h} = \frac{h^3/h^2}{h} = 1, \qquad \frac{f(0,h) - f(0,0)}{h} = \frac{0/h^2}{h} = 0,$$

so $\nabla f(0,0) = \langle 1, 0\rangle$. Take $\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1,1\rangle$. For $t \neq 0$, $t\mathbf{u} = \left(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$ and

$$f(t\mathbf{u}) = \frac{\left(\frac{t}{\sqrt{2}}\right)^3 - \frac{t}{\sqrt{2}}\left(\frac{t}{\sqrt{2}}\right)^2}{\left(\frac{t}{\sqrt{2}}\right)^2 + \left(\frac{t}{\sqrt{2}}\right)^2} = \frac{0}{t^2} = 0,$$

so

$$D_{\mathbf{u}} f(0,0) = \lim_{t\to 0} \frac{f(t\mathbf{u}) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0.$$

On the other hand,

$$\nabla f(0,0) \cdot \mathbf{u} = \frac{1\cdot 1 + 0 \cdot 1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \neq 0.$$

The conclusion of the theorem fails, so its hypothesis fails:

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

UNOFFICIAL QRS marking scheme (16 marks)

(i) **(4 marks)**

(a) **(2 marks)** 1: $y = x$; 1: $y = 0$ and conclusion.

(b) **(2 marks)** 1: two valid paths; 1: nonexistence conclusion.

(ii) **(4 marks)**

(a) **(1 mark)** $k = 3$.

(b) **(3 marks)** 1: separates 3; 1: bounds the remainder; 1: concludes the limit.

(iii) **(8 marks)**

(a) **(2 marks)** 1: valid bound; 1: continuity.

(b) **(3 marks)** 1: x-axis quotient; 1: y-axis quotient; 1: stated partials.

(c) **(3 marks)** 1: derivative L; 1: path calculation; 1: failure of the remainder condition.

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Question 3 (17 Marks)

QUESTION 3.

(i) [3 marks] Find an equation for the tangent plane to the surface given by the equation $z = \ln(8x^2 + 9y^2 + 1)$ at the point $(0, 0, 0)$.

(ii) [3 marks] Use the Chain Rule to find $\dfrac{\partial u}{\partial y}$ at $(x, y, z) = (3, 2, 0)$ for the function

$$u(p, q, r) = \mathrm{e}^{pq} \cos(r),$$

where

$$p = \frac{1}{x}, \qquad q = x^2 \ln y, \qquad r = z.$$

(iii) [3 marks] Assume that z is uniquely determined by the equation

$$9\sin(x + y) + 4\sin(x + z) + 3\sin(y + z) = 0.$$

Find the value of $\dfrac{\partial z}{\partial x}$ at the point $(2\pi, -\pi, 4\pi)$.

(iv) [3 marks] In which directions is the directional derivative of

$$f(x, y) = \sin(xy) - x^2 + xy$$

at $(\pi, 0)$ equal to zero? Write the unit vector for each direction.

(v) [5 marks] Let

$$f(x, y) = 2y^3 + x^2 - 6xy - 8x + 4.$$

(a) Find all critical points of $f(x, y)$.

(b) For each critical point found in Part (a), determine whether or not it is a local max/min or a saddle point.

Solution

Method 1: Tangent-plane formula, Chain Rule, Implicit Function Theorem and Second Derivative Test

(i) Let $g(x, y) = \ln(8x^2 + 9y^2 + 1)$. The argument satisfies $8x^2 + 9y^2 + 1 \geq 1 > 0$, so g is a composition of C^1 functions and hence differentiable; its tangent plane at $(0, 0, g(0, 0))$ is

$$z = g(0, 0) + g_x(0, 0)(x - 0) + g_y(0, 0)(y - 0).$$

The point lies on the surface: $g(0, 0) = \ln 1 = 0$. By the Chain Rule ($\frac{d}{dw} \ln w = \frac{1}{w}$),

$$g_x = \frac{1}{8x^2 + 9y^2 + 1} \cdot 16x = \frac{16x}{8x^2 + 9y^2 + 1}, \qquad g_y = \frac{1}{8x^2 + 9y^2 + 1} \cdot 18y = \frac{18y}{8x^2 + 9y^2 + 1}.$$

At $(0, 0)$,

$$g_x(0, 0) = \frac{0}{1} = 0, \qquad g_y(0, 0) = \frac{0}{1} = 0.$$

Substituting,

$$z = 0 + 0 \cdot x + 0 \cdot y \qquad \Longrightarrow \qquad \boxed{z = 0}$$

(ii) The functions $u(p,q,r) = \mathrm{e}^{pq}\cos r$ and $p = \frac{1}{x}$, $q = x^2 \ln y$, $r = z$ are C^1 near $(x,y,z) = (3,2,0)$ (there $x \neq 0$ and $y > 0$), so the Chain Rule applies:

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}.$$

The outer partial derivatives are

$$\frac{\partial u}{\partial p} = q\,\mathrm{e}^{pq}\cos r, \qquad \frac{\partial u}{\partial q} = p\,\mathrm{e}^{pq}\cos r, \qquad \frac{\partial u}{\partial r} = -\mathrm{e}^{pq}\sin r,$$

and the inner ones are

$$\frac{\partial p}{\partial y} = 0, \qquad \frac{\partial q}{\partial y} = \frac{x^2}{y}, \qquad \frac{\partial r}{\partial y} = 0.$$

Hence

$$\frac{\partial u}{\partial y} = q\,\mathrm{e}^{pq}\cos r \cdot 0 + p\,\mathrm{e}^{pq}\cos r \cdot \frac{x^2}{y} - \mathrm{e}^{pq}\sin r \cdot 0 = p\,\mathrm{e}^{pq}\cos r \cdot \frac{x^2}{y}.$$

At $(x,y,z) = (3,2,0)$:

$$p = \frac{1}{3}, \qquad q = 9\ln 2, \qquad r = 0, \qquad pq = 3\ln 2, \qquad \mathrm{e}^{pq} = \mathrm{e}^{\ln 8} = 8, \qquad \frac{x^2}{y} = \frac{9}{2}.$$

Therefore

$$\boxed{u_y = \frac{1}{3}\mathrm{e}^{3\ln 2}\cdot\frac{9}{2} = \frac{1}{3}\cdot 8\cdot\frac{9}{2} = 12.}$$

(iii) Let $F(x,y,z) = 9\sin(x+y) + 4\sin(x+z) + 3\sin(y+z)$. At $P = (2\pi, -\pi, 4\pi)$,

$$x + y = \pi, \qquad x + z = 6\pi, \qquad y + z = 3\pi,$$

so $F(P) = 9\sin\pi + 4\sin 6\pi + 3\sin 3\pi = 0$ and P lies on the surface. The partial derivatives are

$$\begin{aligned} F_x &= 9\cos(x+y) + 4\cos(x+z), \\ F_y &= 9\cos(x+y) + 3\cos(y+z), \\ F_z &= 4\cos(x+z) + 3\cos(y+z). \end{aligned}$$

With $\cos\pi = -1$, $\cos 6\pi = 1$, $\cos 3\pi = -1$,

$$\begin{aligned} F_x(P) &= 9(-1) + 4(1) = -5, \\ F_z(P) &= 4(1) + 3(-1) = 1 \neq 0. \end{aligned}$$

F is C^1, $F(P) = 0$ and $F_z(P) \neq 0$, so the Implicit Function Theorem gives $z = z(x,y)$ differentiable near $(2\pi, -\pi)$ with

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.$$

Therefore

$$\boxed{z_x = -F_x/F_z = 5}$$

(iv) The function $f(x,y) = \sin(xy) - x^2 + xy$ is C^1 on $\mathbb{R}^2$, so it is differentiable and $D_{\mathbf{u}} f(\pi, 0) = \nabla f(\pi, 0) \cdot \mathbf{u}$ for every unit vector $\mathbf{u}$. The partial derivatives (Chain Rule on $\sin(xy)$) are

$$f_x = y\cos(xy) - 2x + y, \qquad f_y = x\cos(xy) + x.$$

At $(\pi, 0)$, $\cos(\pi \cdot 0) = 1$, so

$$f_x(\pi,0) = 0 - 2\pi + 0 = -2\pi, \qquad f_y(\pi,0) = \pi + \pi = 2\pi, \qquad \nabla f(\pi,0) = 2\pi\langle -1, 1\rangle.$$

For a unit vector $\mathbf{u} = \langle u_1, u_2\rangle$,

$$D_{\mathbf{u}} f(\pi,0) = -2\pi u_1 + 2\pi u_2 = 2\pi(u_2 - u_1) = 0 \iff u_1 = u_2.$$

With $u_1^2 + u_2^2 = 1$:

$$2u_1^2 = 1 \implies u_1 = \pm\frac{1}{\sqrt{2}}, \qquad u_2 = u_1.$$

Therefore

$$\boxed{\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle \quad \text{or} \quad \frac{1}{\sqrt{2}}\langle -1, -1\rangle.}$$

(v) (a) The first partial derivatives of $f(x,y) = 2y^3 + x^2 - 6xy - 8x + 4$ are

$$f_x = 2x - 6y - 8, \qquad f_y = 6y^2 - 6x.$$

Critical points solve $f_x = 0$ and $f_y = 0$. From $f_y = 0$,

$$6y^2 = 6x \implies x = y^2.$$

Substitute in $f_x = 0$:

$$\begin{aligned} 2y^2 - 6y - 8 &= 0 \\ y^2 - 3y - 4 &= 0 \\ (y-4)(y+1) &= 0. \end{aligned}$$

For $y = 4$: $x = 16$. For $y = -1$: $x = 1$. Check in f_x: $2(16) - 24 - 8 = 0$ and $2(1) + 6 - 8 = 0$. So

$$\boxed{(16,4) \text{ and } (1,-1)}$$

(b) The second partial derivatives are

$$f_{xx} = \frac{\partial}{\partial x}(2x - 6y - 8) = 2, \qquad f_{xy} = \frac{\partial}{\partial y}(2x - 6y - 8) = -6,$$

$$f_{yx} = \frac{\partial}{\partial x}(6y^2 - 6x) = -6, \qquad f_{yy} = \frac{\partial}{\partial y}(6y^2 - 6x) = 12y.$$

They are continuous (f is a polynomial), so Clairaut's Theorem holds ($f_{xy} = f_{yx} = -6$) and the Second Derivative Test applies with

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 2(12y) - (-6)^2 = 24y - 36.$$

At each critical point:

$$\begin{aligned} (16,4): &\ D = 24(4) - 36 = 60 > 0, \quad f_{xx} = 2 > 0 \implies \text{local minimum}, \\ (1,-1): &\ D = 24(-1) - 36 = -60 < 0 \implies \text{saddle point}. \end{aligned}$$

Therefore

$$\boxed{(16,4) \text{ is a local minimum; } (1,-1) \text{ is a saddle point.}}$$

Method 2: Normal vector of the level surface

This method applies to part (i).

(i) Write the surface as the level set $G(x, y, z) = 0$ of

$$G(x, y, z) = \ln(8x^2 + 9y^2 + 1) - z.$$

The point lies on it: $G(0, 0, 0) = \ln 1 - 0 = 0$. The partial derivatives are

$$G_x = \frac{16x}{8x^2 + 9y^2 + 1}, \qquad G_y = \frac{18y}{8x^2 + 9y^2 + 1}, \qquad G_z = -1,$$

so

$$\nabla G(0, 0, 0) = \langle 0, 0, -1 \rangle \neq \mathbf{0}.$$

The tangent plane is $\nabla G(0, 0, 0) \cdot \langle x - 0,\, y - 0,\, z - 0 \rangle = 0$:

$$0 \cdot x + 0 \cdot y + (-1) \cdot z = 0 \qquad \Longrightarrow \qquad \boxed{z = 0}$$

Method 3: Direct substitution and term-by-term implicit differentiation

This method applies to parts (ii) and (iii). For part (ii) this method should be used only if the course allows this interpretation, since the question asks for the Chain Rule.

(ii) For $x \neq 0$ and $y > 0$, substitute p, q, r into u:

$$pq = \frac{1}{x} \cdot x^2 \ln y = x \ln y, \qquad \mathrm{e}^{pq} = \mathrm{e}^{x \ln y} = y^x, \qquad u = y^x \cos z.$$

Differentiate in y with x and z held fixed (power rule):

$$\frac{\partial u}{\partial y} = x\, y^{x-1} \cos z.$$

At $(3, 2, 0)$:

$$\frac{\partial u}{\partial y} = 3 \cdot 2^2 \cdot \cos 0 = 3 \cdot 4 \cdot 1 = 12.$$

In the variables p, q, r this is the same expression, since

$$x\, y^{x-1} \cos z = \frac{1}{x} \cdot y^x \cdot \frac{x^2}{y} \cdot \cos z = p\, \mathrm{e}^{pq} \cos r \cdot \frac{x^2}{y},$$

which at $(3, 2, 0)$ reads

$$\boxed{u_y = \frac{1}{3} \mathrm{e}^{3 \ln 2} \cdot \frac{9}{2} = \frac{1}{3} \cdot 8 \cdot \frac{9}{2} = 12.}$$

(iii) At $P = (2\pi, -\pi, 4\pi)$, $x + y = \pi$, $x + z = 6\pi$, $y + z = 3\pi$, and the coefficient of z_x below is $4 \cos 6\pi + 3 \cos 3\pi = 1 \neq 0$, so by the Implicit Function Theorem $z = z(x, y)$ is differentiable near $(2\pi, -\pi)$. Differentiate each term of the equation with respect to x, holding y fixed and using the Chain Rule on each sine:

$$\frac{\partial}{\partial x}\big[9 \sin(x + y)\big] = 9 \cos(x + y) \cdot 1,$$
$$\frac{\partial}{\partial x}\big[4 \sin(x + z)\big] = 4 \cos(x + z) \cdot (1 + z_x),$$

$$\frac{\partial}{\partial x}\big[3\sin(y+z)\big] = 3\cos(y+z)\cdot z_x.$$

The derivative of the right-hand side 0 is 0, so

$$9\cos(x+y) + 4\cos(x+z)(1+z_x) + 3\cos(y+z)\,z_x = 0.$$

Collecting the terms with z_x, and writing F for the left-hand side of the given equation,

$$\underbrace{9\cos(x+y) + 4\cos(x+z)}_{F_x} + \underbrace{\big(4\cos(x+z) + 3\cos(y+z)\big)}_{F_z}\,z_x = 0.$$

Substitute $\cos\pi = -1$, $\cos 6\pi = 1$, $\cos 3\pi = -1$:

$$\begin{aligned} 9(-1) + 4(1)(1+z_x) + 3(-1)z_x &= 0 \\ -9 + 4 + 4z_x - 3z_x &= 0 \\ z_x - 5 &= 0. \end{aligned}$$

Here $F_x(P) = -9 + 4 = -5$ and $F_z(P) = 4 - 3 = 1$. Therefore

$$\boxed{z_x = -F_x/F_z = 5}$$

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Method 4: Angle parametrisation of the unit vector

This method applies to part (iv).

(iv) As f is C^1, $D_{\mathbf{u}}f(\pi,0) = \nabla f(\pi,0)\cdot\mathbf{u}$. With $f_x = y\cos(xy) - 2x + y$ and $f_y = x\cos(xy) + x$,

$$\nabla f(\pi,0) = \langle 0 - 2\pi + 0,\ \pi + \pi\rangle = \langle -2\pi,\ 2\pi\rangle.$$

Every unit vector is $\mathbf{u} = \langle\cos\theta, \sin\theta\rangle$ for one $\theta \in [0, 2\pi)$. Then

$$D_{\mathbf{u}}f(\pi,0) = -2\pi\cos\theta + 2\pi\sin\theta = 2\pi(\sin\theta - \cos\theta).$$

Since $\sin\left(\theta - \frac{\pi}{4}\right) = \sin\theta\cos\frac{\pi}{4} - \cos\theta\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$,

$$D_{\mathbf{u}}f(\pi,0) = 2\sqrt{2}\,\pi\,\sin\left(\theta - \frac{\pi}{4}\right).$$

This is zero if and only if $\theta - \frac{\pi}{4} = k\pi$, $k \in \mathbb{Z}$; in $[0, 2\pi)$,

$$\theta = \frac{\pi}{4} \qquad \text{or} \qquad \theta = \frac{5\pi}{4}.$$

The corresponding vectors are

$$\left\langle\cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{4}\right\rangle = \left\langle\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right\rangle, \qquad \left\langle\cos\tfrac{5\pi}{4}, \sin\tfrac{5\pi}{4}\right\rangle = \left\langle-\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right\rangle.$$

Therefore

$$\boxed{\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle \quad \text{or} \quad \frac{1}{\sqrt{2}}\langle -1, -1\rangle.}$$

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Method 5: Completing the square in x

This method applies to part (v).

(v) (a) Collect the terms containing x and complete the square:

$$\begin{aligned} x^2 - 6xy - 8x &= x^2 - 2x(3y+4) \\ &= \big(x - (3y+4)\big)^2 - (3y+4)^2 \\ &= (x - 3y - 4)^2 - 9y^2 - 24y - 16. \end{aligned}$$

Hence

$$f(x,y) = (x - 3y - 4)^2 + h(y), \qquad h(y) := 2y^3 - 9y^2 - 24y - 12.$$

Let $s = x - 3y - 4$. Differentiating $f = s^2 + h(y)$,

$$f_x = 2s, \qquad f_y = 2s \cdot (-3) + h'(y) = -6s + h'(y), \qquad h'(y) = 6y^2 - 18y - 24.$$

$f_x = 0$ gives $s = 0$, and then $f_y = 0$ gives

$$h'(y) = 6(y^2 - 3y - 4) = 6(y-4)(y+1) = 0 \implies y = 4 \text{ or } y = -1.$$

With $s = 0$, $x = 3y + 4$: $y = 4$ gives $x = 16$, and $y = -1$ gives $x = 1$. So

$$\boxed{(16,4) \text{ and } (1,-1)}$$

(b) The values of h at the two roots are

$$\begin{aligned} h(4) &= 2(64) - 9(16) - 24(4) - 12 = 128 - 144 - 96 - 12 = -124 = f(16,4), \\ h(-1) &= 2(-1) - 9(1) - 24(-1) - 12 = -2 - 9 + 24 - 12 = 1 = f(1,-1). \end{aligned}$$

The sign of $h'(y) = 6(y-4)(y+1)$ is positive for $y < -1$, negative for $-1 < y < 4$, and positive for $y > 4$. So h decreases then increases through $y = 4$ (local minimum of h) and increases then decreases through $y = -1$ (local maximum of h).

At $(16,4)$: there is $\delta > 0$ with $h(y) \geq h(4)$ for $|y - 4| < \delta$. For every (x,y) with $|y-4| < \delta$,

$$f(x,y) = s^2 + h(y) \geq 0 + h(4) = f(16,4),$$

so $(16,4)$ is a local minimum.

At $(1,-1)$: along the line $s = 0$, i.e. $x = 3y + 4$, through $(1,-1)$,

$$f(3y+4, y) = h(y) < h(-1) = f(1,-1) \qquad \text{for } y \neq -1 \text{ near } -1.$$

Along the line $y = -1$, where $s = x - 1$,

$$f(x,-1) = (x-1)^2 + h(-1) = (x-1)^2 + 1 > 1 = f(1,-1) \qquad \text{for } x \neq 1.$$

Every disc around $(1,-1)$ contains points with $f > f(1,-1)$ and points with $f < f(1,-1)$, so $(1,-1)$ is a saddle point. Therefore

$$\boxed{(16,4) \text{ is a local minimum; } (1,-1) \text{ is a saddle point.}}$$

UNOFFICIAL QRS marking scheme (17 marks)

(i) **(3 marks)** 1: both partials; 1: evaluates slopes; 1: plane $z = 0$.

(ii) **(3 marks)** 1: Chain Rule; 1: intermediate derivatives; 1: obtains 12.

(iii) **(3 marks)** 1: F_x, F_z; 1: evaluates cosines; 1: obtains 5.

(iv) **(3 marks)** 1: gradient; 1: orthogonality condition; 1: both unit vectors.

(v) **(5 marks)**

(a) **(2 marks)** 1: solves first-derivative equations; 1: both points.

(b) **(3 marks)** 1: Hessian and discriminant; 1: minimum classification; 1: saddle classification.

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