# MH2100 Calculus III

## Midterm II (Solutions)

AY 2022/2023, Semester 1

*Quantitative Research Society @NTU*

October 2022

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**Overview**

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- **Historical paper:** Midterm II, October 2022; **90 minutes** were allowed.
- **Current administration (AY2026/2027):** Tuesday 6 October 2026, 16:30–18:20, venue TBA. One double-sided typed A4 sheet is permitted; no attachments or sticky notes are permitted.
- **Original format:** three questions worth 50 marks in total. ONE two-sided A4 piece of paper with handwritten notes was permitted.
- **2026 scope note:** the current midterm scope ends at double integrals, including polar coordinates. Cylindrical and spherical coordinates, general Jacobian transformations, and line integrals are beyond that scope; these historical questions are retained and labelled accordingly.
- **Source:** MH2100 Midterm II, AY2022/2023 Semester 1. The instructions and questions below reproduce the official paper; solutions and marking schemes are unofficial QRS@NTU material.

## Instructions to candidates (verbatim)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates should write their answers in the separate answer book provided.
5. ONE two-sided A4 piece of paper with handwritten notes is permitted.

*For academic reference only. This document is provided for educational purposes and does not constitute an official question paper, answer key, or any formally authorised academic material. Its content is not guaranteed to be complete or accurate.*

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# Question 1 (16 Marks)

**QUESTION 1.**

(i) [5 marks] Find the *maximum* and *minimum* values of $xy$ satisfying the condition that $3x^2 + 2y^2 = 36$. Also give the points $(x, y)$ where the *maximum* and *minimum* values are achieved.

(ii) [3 marks] Evaluate the iterated integral.

$$\int_{-1}^{1} \int_{0}^{1} 2ye^{x} \, dx \, dy.$$

(iii) [4 marks] By reversing the order of integration, evaluate the integral

$$\int_{0}^{\sqrt{\pi/2}} \int_{y}^{\sqrt{\pi/2}} 2\cos(x^2) \, dx \, dy.$$

(iv) [4 marks] Find the area of the region $D$ (in the $xy$-plane) bounded by the cardioid expressed in polar coordinates as

$$r = 1 - \sin\theta; \qquad \text{and} \qquad 0 \leq \theta \leq 2\pi.$$

Hint: the formula $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$ may be useful.

## Solution

**Method 1: Lagrange multipliers, iterated integrals and the polar area formula**

(i) Let $f(x, y) = xy$ and $g(x, y) = 3x^2 + 2y^2$, and let $C = \{(x, y) : g(x, y) = 36\}$.

*Existence (Extreme Value Theorem).* The set $C$ is closed, being the preimage of the closed set $\{36\}$ under the continuous function $g$. It is bounded, because every point of $C$ satisfies

$$3x^2 \leq 36 \implies |x| \leq 2\sqrt{3}, \qquad 2y^2 \leq 36 \implies |y| \leq 3\sqrt{2}.$$

Hence $C$ is compact, $f$ is continuous on $C$, and by the Extreme Value Theorem $f$ attains an absolute maximum and an absolute minimum on $C$.

*Partial derivatives.*

$$f_x = y, \qquad f_y = x, \qquad g_x = 6x, \qquad g_y = 4y.$$

*Regularity.* $\nabla g = (6x, 4y) = (0, 0)$ only at $(0, 0)$, and $g(0, 0) = 0 \neq 36$, so $(0, 0) \notin C$. Therefore $\nabla g \neq 0$ at every point of $C$, and by the Lagrange multiplier theorem every extremum of $f$ on $C$ satisfies $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$:

$$y = 6\lambda x, \tag{1}$$
$$x = 4\lambda y, \tag{2}$$
$$3x^2 + 2y^2 = 36. \tag{3}$$

*Case 1:* $x = 0$. Then (1) gives $y = 6\lambda \cdot 0 = 0$, and (3) becomes $0 = 36$, which is impossible.

*Case 2:* $y = 0$. Then (2) gives $x = 4\lambda \cdot 0 = 0$, and (3) becomes $0 = 36$, which is impossible.

*Case 3:* $x \neq 0$ *and* $y \neq 0$. From (1) and (2),

$$\lambda = \frac{y}{6x} \qquad \text{and} \qquad \lambda = \frac{x}{4y}.$$

Equating the two expressions and cross-multiplying,

$$\frac{y}{6x} = \frac{x}{4y} \implies 4y^2 = 6x^2 \implies 2y^2 = 3x^2.$$

Substituting $2y^2 = 3x^2$ into (3),

$$3x^2 + 3x^2 = 36 \implies x^2 = 6 \implies x = \pm\sqrt{6},$$

and then

$$2y^2 = 3 \cdot 6 = 18 \implies y^2 = 9 \implies y = \pm 3.$$

This gives four candidate points, with $\lambda = \dfrac{y}{6x} = \pm\dfrac{3}{6\sqrt{6}} = \pm\dfrac{\sqrt{6}}{12}$. Evaluate $f$ at each:

$$f(\sqrt{6}, 3) = 3\sqrt{6}, \qquad f(-\sqrt{6}, -3) = 3\sqrt{6},$$
$$f(\sqrt{6}, -3) = -3\sqrt{6}, \qquad f(-\sqrt{6}, 3) = -3\sqrt{6}.$$

By the Extreme Value Theorem the absolute extrema exist, and by the Lagrange multiplier theorem they occur among these candidates. Comparing the values,

$$\boxed{\max xy = 3\sqrt{6} \text{ at } (\sqrt{6}, 3), (-\sqrt{6}, -3),}$$

$$\boxed{\min xy = -3\sqrt{6} \text{ at } (\sqrt{6}, -3), (-\sqrt{6}, 3).}$$

(ii) Integrate in $x$ first, with $y$ held fixed:

$$\int_0^1 2y\mathrm{e}^x \,\mathrm{d}x = 2y\left[\mathrm{e}^x\right]_0^1 = 2y(\mathrm{e}^1 - \mathrm{e}^0) = 2y(\mathrm{e} - 1).$$

Then integrate in $y$:

$$\begin{aligned}\int_{-1}^1 \int_0^1 2y\mathrm{e}^x \,\mathrm{d}x \,\mathrm{d}y &= \int_{-1}^1 2y(\mathrm{e} - 1) \,\mathrm{d}y \\ &= (\mathrm{e} - 1)\left[y^2\right]_{-1}^1 \\ &= (\mathrm{e} - 1)\left(1^2 - (-1)^2\right) \\ &= (\mathrm{e} - 1) \cdot 0 = \boxed{0}\end{aligned}$$

(iii) Write $c = \sqrt{\pi/2}$. The inner integral $\int \cos(x^2)\,\mathrm{d}x$ has no elementary antiderivative, so the order is reversed. The given limits describe the region as a Type II region:

$$D = \{(x, y) : 0 \leq y \leq c,\ y \leq x \leq c\},$$

which is the triangle with vertices $(0,0)$, $(c,0)$, $(c,c)$, bounded by $y=0$, $x=c$ and $y=x$. For a fixed $x \in [0,c]$, the vertical segment through $D$ runs from $y=0$ up to the line $y=x$, so the same region as a Type I region is

$$D = \{(x,y) : 0 \le x \le c,\ 0 \le y \le x\}.$$

The integrand $2\cos(x^2)$ is continuous on $D$, so by Fubini's theorem on Type I and Type II regions both iterated integrals equal $\iint_D 2\cos(x^2)\,\mathrm{d}A$:

$$\int_0^c \int_y^c 2\cos(x^2)\,\mathrm{d}x\,\mathrm{d}y = \int_0^c \int_0^x 2\cos(x^2)\,\mathrm{d}y\,\mathrm{d}x.$$

The inner integral in $y$ treats $2\cos(x^2)$ as a constant:

$$\int_0^x 2\cos(x^2)\,\mathrm{d}y = 2\cos(x^2)\big[y\big]_0^x = 2x\cos(x^2).$$

For the outer integral substitute $u = x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$; when $x=0$, $u=0$, and when $x = c = \sqrt{\pi/2}$, $u = \pi/2$:

$$\begin{aligned}\int_0^{\sqrt{\pi/2}} 2x\cos(x^2)\,\mathrm{d}x &= \int_0^{\pi/2} \cos u\,\mathrm{d}u \\ &= \big[\sin u\big]_0^{\pi/2} \\ &= \sin\frac{\pi}{2} - \sin 0 = \boxed{1}\end{aligned}$$

(iv) Since $\sin\theta \le 1$, $r = 1-\sin\theta \ge 0$ for every $\theta$, and the cardioid is traced once as $\theta$ runs over $[0,2\pi]$. Hence the polar area formula applies:

$$A = \frac{1}{2}\int_0^{2\pi} r^2\,\mathrm{d}\theta = \frac{1}{2}\int_0^{2\pi} (1-\sin\theta)^2\,\mathrm{d}\theta.$$

Expand the square:

$$(1-\sin\theta)^2 = 1 - 2\sin\theta + \sin^2\theta.$$

Integrate term by term over $[0,2\pi]$:

$$\int_0^{2\pi} 1\,\mathrm{d}\theta = 2\pi,$$

$$\int_0^{2\pi} 2\sin\theta\,\mathrm{d}\theta = \big[-2\cos\theta\big]_0^{2\pi} = -2\cos 2\pi + 2\cos 0 = -2+2 = 0,$$

and, using the hint $\sin^2\theta = \frac{1-\cos 2\theta}{2}$,

$$\begin{aligned}\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta &= \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta \\ &= \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi} \\ &= \left(\pi - \frac{\sin 4\pi}{4}\right) - \left(0 - \frac{\sin 0}{4}\right) = \pi.\end{aligned}$$

Therefore $A = \frac{1}{2}(2\pi - 0 + \pi)$, that is,

$$\boxed{A = \frac{1}{2}(2\pi+\pi) = \frac{3\pi}{2}.}$$

**Method 2: Completing the square; Fubini in the other order; integration by parts; polar double integral**

(i) From the constraint,

$$2y^2 = 36 - 3x^2 \implies y^2 = \frac{36 - 3x^2}{2}.$$

Let $a = x^2$. Since $y^2 \geq 0$, we need $36 - 3a \geq 0$, so $0 \leq a \leq 12$. Then

$$(xy)^2 = x^2y^2 = a \cdot \frac{36 - 3a}{2} = 18a - \frac{3}{2}a^2.$$

Complete the square in $a$:

$$\begin{aligned} 18a - \frac{3}{2}a^2 &= -\frac{3}{2}\left(a^2 - 12a\right) \\ &= -\frac{3}{2}\left[(a-6)^2 - 36\right] \\ &= 54 - \frac{3}{2}(a-6)^2. \end{aligned}$$

Since $\frac{3}{2}(a-6)^2 \geq 0$, every point of the ellipse satisfies

$$(xy)^2 \leq 54 \implies |xy| \leq \sqrt{54} = 3\sqrt{6} \implies -3\sqrt{6} \leq xy \leq 3\sqrt{6},$$

with $(xy)^2 = 54$ if and only if $a = 6$, which lies in $[0, 12]$. At $a = 6$,

$$x^2 = 6, \qquad y^2 = \frac{36 - 18}{2} = 9,$$

so $x = \pm\sqrt{6}$ and $y = \pm 3$. Then $xy = 3\sqrt{6}$ when $x, y$ have the same sign and $xy = -3\sqrt{6}$ when they have opposite signs. Both bounds are attained, so

$$\boxed{\max xy = 3\sqrt{6} \text{ at } (\sqrt{6}, 3), (-\sqrt{6}, -3),}$$

$$\boxed{\min xy = -3\sqrt{6} \text{ at } (\sqrt{6}, -3), (-\sqrt{6}, 3).}$$

(ii) The integrand $2y\mathrm{e}^x$ is continuous on the rectangle $R = [0,1] \times [-1,1]$, so by Fubini's theorem on a rectangle the order of integration may be swapped:

$$\int_{-1}^{1}\int_{0}^{1} 2y\mathrm{e}^x \,\mathrm{d}x\,\mathrm{d}y = \int_{0}^{1}\int_{-1}^{1} 2y\mathrm{e}^x \,\mathrm{d}y\,\mathrm{d}x.$$

Inner integral in $y$, with $x$ held fixed:

$$\int_{-1}^{1} 2y\mathrm{e}^x \,\mathrm{d}y = \mathrm{e}^x\left[y^2\right]_{-1}^{1} = \mathrm{e}^x\left(1 - 1\right) = 0.$$

Outer integral:

$$\int_{0}^{1} 0 \,\mathrm{d}x = \boxed{0}$$

(iii) *This method should be used only if the course allows this interpretation* (it evaluates the integral without reversing the order).

Write $c = \sqrt{\pi/2}$ and define

$$F(y) = \int_y^c 2\cos(x^2)\,\mathrm{d}x, \qquad 0 \le y \le c.$$

By the Fundamental Theorem of Calculus ($2\cos(x^2)$ is continuous), $F$ is differentiable with

$$F'(y) = -2\cos(y^2), \qquad \text{and} \qquad F(c) = \int_c^c 2\cos(x^2)\,\mathrm{d}x = 0.$$

The given integral is $\int_0^c F(y)\,\mathrm{d}y$. Integrate by parts with $p = F(y)$, $\mathrm{d}q = \mathrm{d}y$, so $\mathrm{d}p = F'(y)\,\mathrm{d}y$ and $q = y$:

$$\begin{aligned}\int_0^c F(y)\,\mathrm{d}y &= \big[yF(y)\big]_0^c - \int_0^c yF'(y)\,\mathrm{d}y \\ &= \big(c\cdot F(c) - 0\cdot F(0)\big) - \int_0^c y\big(-2\cos(y^2)\big)\,\mathrm{d}y \\ &= 0 + \int_0^c 2y\cos(y^2)\,\mathrm{d}y.\end{aligned}$$

Substitute $w = y^2$, $\mathrm{d}w = 2y\,\mathrm{d}y$; when $y = 0$, $w = 0$, and when $y = c$, $w = \pi/2$:

$$\begin{aligned}\int_0^c 2y\cos(y^2)\,\mathrm{d}y &= \int_0^{\pi/2} \cos w\,\mathrm{d}w \\ &= \big[\sin w\big]_0^{\pi/2} = 1 - 0 = \boxed{1}\end{aligned}$$

(iv) In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the area element is $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$. Since $1 - \sin\theta \ge 0$ for every $\theta$, the region is

$$D = \{(r,\theta) : 0 \le \theta \le 2\pi,\ 0 \le r \le 1 - \sin\theta\}.$$

Hence

$$A = \iint_D 1\,\mathrm{d}A = \int_0^{2\pi}\int_0^{1-\sin\theta} r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Inner integral:

$$\int_0^{1-\sin\theta} r\,\mathrm{d}r = \left[\frac{r^2}{2}\right]_0^{1-\sin\theta} = \frac{(1-\sin\theta)^2}{2} = \frac{1 - 2\sin\theta + \sin^2\theta}{2}.$$

For the outer integral, use the following evaluations on $[0, 2\pi]$. First,

$$\int_0^{2\pi} \sin\theta\,\mathrm{d}\theta = \big[-\cos\theta\big]_0^{2\pi} = -1 + 1 = 0.$$

Second, evaluate $\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta$ by periodicity instead of the half-angle formula. The function $\sin^2\theta$ has period $2\pi$, so its integral over any interval of length $2\pi$ is the same. Substituting $\theta = \varphi + \frac{\pi}{2}$, with $\sin\left(\varphi + \frac{\pi}{2}\right) = \cos\varphi$,

$$\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta = \int_{-\pi/2}^{3\pi/2} \cos^2\varphi\,\mathrm{d}\varphi = \int_0^{2\pi} \cos^2\varphi\,\mathrm{d}\varphi.$$

Adding $\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta$ to both sides,

$$2\int_0^{2\pi} \sin^2\theta\,\mathrm{d}\theta = \int_0^{2\pi} \left(\sin^2\theta + \cos^2\theta\right)\mathrm{d}\theta = \int_0^{2\pi} 1\,\mathrm{d}\theta = 2\pi,$$

so $\int_0^{2\pi} \sin^2\theta \,\mathrm{d}\theta = \pi$. Therefore

$$A = \frac{1}{2}\left(\int_0^{2\pi} 1 \,\mathrm{d}\theta - 2\int_0^{2\pi} \sin\theta \,\mathrm{d}\theta + \int_0^{2\pi} \sin^2\theta \,\mathrm{d}\theta\right) = \frac{1}{2}(2\pi - 0 + \pi),$$

that is,

$$\boxed{A = \frac{1}{2}(2\pi + \pi) = \frac{3\pi}{2}.}$$

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**Method 3: Parametrising the constraint; odd symmetry**

(i) Divide the constraint by 36:

$$\frac{x^2}{12} + \frac{y^2}{18} = 1.$$

This ellipse is parametrised once by

$$x = 2\sqrt{3}\cos t, \qquad y = 3\sqrt{2}\sin t, \qquad 0 \le t \le 2\pi,$$

since $3(2\sqrt{3}\cos t)^2 + 2(3\sqrt{2}\sin t)^2 = 36\cos^2 t + 36\sin^2 t = 36$. Along the ellipse,

$$h(t) = xy = (2\sqrt{3})(3\sqrt{2})\cos t \sin t = 6\sqrt{6}\sin t\cos t = 3\sqrt{6}\sin 2t.$$

The function $h$ is continuous on the closed interval $[0, 2\pi]$. Its derivative is

$$h'(t) = 6\sqrt{6}\cos 2t,$$

which vanishes when $2t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}\right\}$, that is, $t \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$. Evaluate $h$ and the point at each critical number and at the endpoints:

$$\begin{aligned}
t = \tfrac{\pi}{4}: &\quad (x,y) = \left(2\sqrt{3}\cdot\tfrac{\sqrt{2}}{2},\, 3\sqrt{2}\cdot\tfrac{\sqrt{2}}{2}\right) = (\sqrt{6}, 3), &&\quad h = 3\sqrt{6}\sin\tfrac{\pi}{2} = 3\sqrt{6},\\
t = \tfrac{3\pi}{4}: &\quad (x,y) = \left(-2\sqrt{3}\cdot\tfrac{\sqrt{2}}{2},\, 3\sqrt{2}\cdot\tfrac{\sqrt{2}}{2}\right) = (-\sqrt{6}, 3), &&\quad h = 3\sqrt{6}\sin\tfrac{3\pi}{2} = -3\sqrt{6},\\
t = \tfrac{5\pi}{4}: &\quad (x,y) = \left(-2\sqrt{3}\cdot\tfrac{\sqrt{2}}{2},\, -3\sqrt{2}\cdot\tfrac{\sqrt{2}}{2}\right) = (-\sqrt{6}, -3), &&\quad h = 3\sqrt{6}\sin\tfrac{5\pi}{2} = 3\sqrt{6},\\
t = \tfrac{7\pi}{4}: &\quad (x,y) = \left(2\sqrt{3}\cdot\tfrac{\sqrt{2}}{2},\, -3\sqrt{2}\cdot\tfrac{\sqrt{2}}{2}\right) = (\sqrt{6}, -3), &&\quad h = 3\sqrt{6}\sin\tfrac{7\pi}{2} = -3\sqrt{6},\\
t = 0,\ 2\pi: &\quad (x,y) = (2\sqrt{3}, 0), &&\quad h = 0.
\end{aligned}$$

Comparing these values (closed interval method),

$$\boxed{\max xy = 3\sqrt{6} \text{ at } (\sqrt{6}, 3), (-\sqrt{6}, -3),}$$

$$\boxed{\min xy = -3\sqrt{6} \text{ at } (\sqrt{6}, -3), (-\sqrt{6}, 3).}$$

(ii) The integrand $2ye^x$ changes sign under $y \mapsto -y$, and the $y$-interval $[-1, 1]$ is symmetric about 0. Let $I = \int_{-1}^{1}\int_{0}^{1} 2ye^x \,\mathrm{d}x\,\mathrm{d}y$. In the outer integral substitute $y = -w$, $\mathrm{d}y = -\,\mathrm{d}w$; when $y = -1$, $w = 1$, and when $y = 1$, $w = -1$:

$$\begin{aligned}
I &= \int_{1}^{-1}\int_{0}^{1} 2(-w)\mathrm{e}^x \,\mathrm{d}x\,(-\,\mathrm{d}w)\\
&= \int_{1}^{-1}\int_{0}^{1} 2w\mathrm{e}^x \,\mathrm{d}x\,\mathrm{d}w\\
&= -\int_{-1}^{1}\int_{0}^{1} 2w\mathrm{e}^x \,\mathrm{d}x\,\mathrm{d}w = -I.
\end{aligned}$$

Hence $2I = 0$, so

$$I = \boxed{0}$$

**Method 4: AM–GM**

(i) For real numbers $p, q \geq 0$, the AM–GM inequality states $p + q \geq 2\sqrt{pq}$, with equality if and only if $p = q$. Take $p = 3x^2 \geq 0$ and $q = 2y^2 \geq 0$:

$$36 = 3x^2 + 2y^2 \geq 2\sqrt{3x^2 \cdot 2y^2} = 2\sqrt{6x^2y^2} = 2\sqrt{6}\,|xy|.$$

Hence

$$|xy| \leq \frac{36}{2\sqrt{6}} = \frac{18}{\sqrt{6}} = 3\sqrt{6} \implies -3\sqrt{6} \leq xy \leq 3\sqrt{6}.$$

Equality holds if and only if $3x^2 = 2y^2$. Together with $3x^2 + 2y^2 = 36$ this gives

$$3x^2 = 2y^2 = 18 \iff x^2 = 6,\ y^2 = 9 \iff x = \pm\sqrt{6},\ y = \pm 3.$$

The value $xy = 3\sqrt{6}$ needs $x, y$ of the same sign, and $xy = -3\sqrt{6}$ needs opposite signs. Therefore

$$\boxed{\max xy = 3\sqrt{6} \text{ at } (\sqrt{6}, 3), (-\sqrt{6}, -3),}$$

$$\boxed{\min xy = -3\sqrt{6} \text{ at } (\sqrt{6}, -3), (-\sqrt{6}, 3).}$$

**Marking scheme (16 marks)**

(i) **(5 marks)**

- 2 marks: reduces the constraint to a correct one-variable expression for $(xy)^2$.
- 1 mark: maximises that expression at $x^2 = 6$, $y^2 = 9$.
- 2 marks: gives both extrema and all four attaining points.

(ii) **(3 marks)**

- 1 mark: evaluates the inner integral correctly.
- 1 mark: recognises the odd outer factor or completes the outer integral.
- 1 mark: obtains 0.

(iii) **(4 marks)**

- 2 marks: reverses the order with correct bounds.
- 1 mark: reduces to $\int 2x\cos(x^2)\,\mathrm{d}x$.
- 1 mark: obtains 1.

(iv) **(4 marks)**

- 1 mark: states the polar area formula.
- 2 marks: expands and integrates the square correctly.
- 1 mark: obtains $3\pi/2$.

# Question 2 (17 Marks)

***(beyond 2026 midterm scope)***

**QUESTION 2.**

(i) *(beyond 2026 midterm scope)* [4 marks] Use cylindrical coordinates to evaluate

$$\iiint_E \sqrt{x^2+y^2}\,\mathrm{d}V,$$

where $E$ is the solid bounded by the cylinder $x^2+y^2=1$ and the planes $z=1$ and $z=2$.

(ii) *(beyond 2026 midterm scope)* Consider the transformation $u=x+4y$ and $v=y-x$.

(a) [3 marks] Calculate the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$.

(b) [5 marks] Use the transformation to evaluate the iterated integral

$$\int_0^1 \int_y^{5-4y} (x+4y)\mathrm{e}^{y-x}\,\mathrm{d}x\,\mathrm{d}y.$$

Hint: you can use $\displaystyle\int x\mathrm{e}^{ax}\,\mathrm{d}x = \frac{\mathrm{e}^{ax}(ax-1)}{a^2}+C.$

(iii) *(beyond 2026 midterm scope)* [5 marks] Use spherical coordinates to find the volume of the region inside the sphere $x^2+y^2+z^2=4$ and outside the cone $z^2=x^2+y^2$. That is, the volume of the region $E$ given by

$$E=\{(x,y,z)\mid x^2+y^2+z^2\le 4 \text{ and } z^2\le x^2+y^2\}.$$

## Solution

**Method 1: Cylindrical coordinates, the given transformation, spherical coordinates**

(i) Use cylindrical coordinates $x=r\cos\theta$, $y=r\sin\theta$, $z=z$ with $r\ge 0$. The Jacobian determinant, expanded along the third row, is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\cdot\left(r\cos^2\theta + r\sin^2\theta\right) = r,$$

so $\mathrm{d}V = r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$. The cylinder $x^2+y^2=1$ is $r=1$, so

$$E=\{(r,\theta,z): 0\le r\le 1,\ 0\le\theta\le 2\pi,\ 1\le z\le 2\},$$

and the integrand is $\sqrt{x^2+y^2}=\sqrt{r^2}=r$ (since $r\ge 0$). Hence

$$\iiint_E \sqrt{x^2+y^2}\,\mathrm{d}V = \int_0^{2\pi}\int_0^1\int_1^2 r\cdot r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

Innermost integral in $z$:

$$\int_1^2 r^2\,\mathrm{d}z = r^2\big[z\big]_1^2 = r^2(2-1) = r^2.$$

Middle integral in $r$:

$$\int_0^1 r^2 \,\mathrm{d}r = \left[\frac{r^3}{3}\right]_0^1 = \frac{1}{3}.$$

Outer integral in $\theta$:

$$\int_0^{2\pi} \frac{1}{3}\,\mathrm{d}\theta = \frac{1}{3}\big[\theta\big]_0^{2\pi} = \frac{2\pi}{3}.$$

Therefore

$$\iiint_E \sqrt{x^2+y^2}\,\mathrm{d}V = \boxed{2\pi/3}$$

(ii) (a) With $u = x + 4y$ and $v = y - x$, the partial derivatives are

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = 4, \qquad \frac{\partial v}{\partial x} = -1, \qquad \frac{\partial v}{\partial y} = 1.$$

Hence

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ -1 & 1 \end{vmatrix} = 1\cdot 1 - 4\cdot(-1) = 1 + 4 = \boxed{5}$$

(b) *Inverse transformation.* From $u = x + 4y$ and $v = y - x$,

$$u - 4v = x + 4y - 4y + 4x = 5x \implies x = \frac{u-4v}{5},$$
$$u + v = x + 4y + y - x = 5y \implies y = \frac{u+v}{5}.$$

*Jacobian.* With $x_u = \frac{1}{5}$, $x_v = -\frac{4}{5}$, $y_u = \frac{1}{5}$, $y_v = \frac{1}{5}$,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{5} & -\frac{4}{5} \\ \frac{1}{5} & \frac{1}{5} \end{vmatrix} = \frac{1}{5}\cdot\frac{1}{5} - \left(-\frac{4}{5}\right)\cdot\frac{1}{5} = \frac{1}{25} + \frac{4}{25} = \frac{1}{5},$$

so $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{5}$, the reciprocal of the value 5 found in (a).

*Region.* The limits describe

$$D = \{(x,y) : 0 \le y \le 1,\ y \le x \le 5 - 4y\}.$$

Its boundary consists of the lines $x = y$, $x + 4y = 5$ and $y = 0$; at $y = 1$ the two $x$-limits meet at the single point $(1,1)$. So $D$ is the triangle with vertices $(0,0)$, $(5,0)$, $(1,1)$. Map each boundary line:

$$x = y \implies v = y - x = 0,$$
$$x + 4y = 5 \implies u = 5,$$
$$y = 0 \implies u = x,\ v = -x \implies v = -u.$$

The vertices map as

$$(0,0) \mapsto (0,0), \qquad (5,0) \mapsto (5,-5), \qquad (1,1) \mapsto (1+4,\ 1-1) = (5,0).$$

Hence the image is the triangle with vertices $(0,0)$, $(5,-5)$, $(5,0)$:

$$S = \{(u,v) : 0 \le u \le 5,\ -u \le v \le 0\}.$$

(Check with the interior point $(2, \frac{1}{2}) \in D$: it maps to $(u, v) = (4, -\frac{3}{2})$, and $-4 \le -\frac{3}{2} \le 0$.)

*Integrand.* $(x + 4y)\mathrm{e}^{y-x} = u\mathrm{e}^{v}$.

*Change of variables.* The map $(u, v) \mapsto (x, y)$ is linear, so it has continuous first partial derivatives, and its Jacobian $\frac{1}{5}$ is nonzero, so it is one-to-one from $S$ onto $D$. The integrand is continuous on $D$. By the change of variables theorem,

$$\int_0^1 \int_y^{5-4y} (x + 4y)\mathrm{e}^{y-x}\,\mathrm{d}x\,\mathrm{d}y = \iint_S u\mathrm{e}^{v} \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}A = \frac{1}{5}\int_0^5 \int_{-u}^0 u\mathrm{e}^{v}\,\mathrm{d}v\,\mathrm{d}u.$$

Inner integral in $v$:

$$\int_{-u}^0 u\mathrm{e}^{v}\,\mathrm{d}v = u\big[\mathrm{e}^{v}\big]_{-u}^0 = u\left(1 - \mathrm{e}^{-u}\right) = u - u\mathrm{e}^{-u}.$$

Outer integral: first,

$$\int_0^5 u\,\mathrm{d}u = \left[\frac{u^2}{2}\right]_0^5 = \frac{25}{2}.$$

Second, the hint with $a = -1$ gives $\int u\mathrm{e}^{-u}\,\mathrm{d}u = \dfrac{\mathrm{e}^{-u}(-u-1)}{(-1)^2} + C = -(u+1)\mathrm{e}^{-u} + C$, so

$$\int_0^5 u\mathrm{e}^{-u}\,\mathrm{d}u = \Big[-(u+1)\mathrm{e}^{-u}\Big]_0^5 = -6\mathrm{e}^{-5} - (-1) = 1 - 6\mathrm{e}^{-5}.$$

Therefore

$$\begin{aligned}\int_0^1 \int_y^{5-4y} (x + 4y)\mathrm{e}^{y-x}\,\mathrm{d}x\,\mathrm{d}y &= \frac{1}{5}\left(\frac{25}{2} - \left(1 - 6\mathrm{e}^{-5}\right)\right) \\ &= \frac{1}{5}\left(\frac{23}{2} + 6\mathrm{e}^{-5}\right) \\ &= \boxed{\frac{23}{10} + \frac{6}{5\mathrm{e}^5}.}\end{aligned}$$

(iii) Use spherical coordinates $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, with $\rho \ge 0$, $0 \le \phi \le \pi$ measured from the $+z$-axis, $0 \le \theta \le 2\pi$. Expanding the Jacobian determinant along the third row,

$$\begin{aligned}\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\ &= \cos\phi\big(\rho^2\sin\phi\cos\phi\cos^2\theta + \rho^2\sin\phi\cos\phi\sin^2\theta\big) \\ &\quad + \rho\sin\phi\big(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\big) \\ &= \rho^2\sin\phi\cos^2\phi + \rho^2\sin^3\phi = \rho^2\sin\phi,\end{aligned}$$

which is $\ge 0$ for $0 \le \phi \le \pi$; so $\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$.

*Region.* $x^2 + y^2 + z^2 = \rho^2 \le 4$ gives $0 \le \rho \le 2$. For $\rho > 0$,

$$z^2 \le x^2 + y^2 \iff \rho^2\cos^2\phi \le \rho^2\sin^2\phi \iff |\cos\phi| \le \sin\phi \iff \frac{\pi}{4} \le \phi \le \frac{3\pi}{4},$$

using $\sin\phi \ge 0$ on $[0, \pi]$. There is no restriction on $\theta$. Hence

$$E = \left\{(\rho, \phi, \theta) : 0 \le \rho \le 2,\ \frac{\pi}{4} \le \phi \le \frac{3\pi}{4},\ 0 \le \theta \le 2\pi\right\}.$$

*Volume.*

$$V = \int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_0^2 \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta.$$

Innermost integral in $\rho$:

$$\int_0^2 \rho^2 \sin\phi \, \mathrm{d}\rho = \sin\phi \left[\frac{\rho^3}{3}\right]_0^2 = \frac{8}{3}\sin\phi.$$

Middle integral in $\phi$:

$$\int_{\pi/4}^{3\pi/4} \frac{8}{3}\sin\phi \, \mathrm{d}\phi = \frac{8}{3}\big[-\cos\phi\big]_{\pi/4}^{3\pi/4} = \frac{8}{3}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = \frac{8\sqrt{2}}{3}.$$

Outer integral in $\theta$:

$$\int_0^{2\pi} \frac{8\sqrt{2}}{3} \, \mathrm{d}\theta = 2\pi \cdot \frac{8\sqrt{2}}{3}.$$

Therefore

$$\boxed{V = \frac{16\sqrt{2}\pi}{3}.}$$

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**Method 2: Slicing with a polar double integral; inverse Jacobian; direct integration by parts; complement in the ball**

(i) Slice $E$ by the planes $z = \text{const}$. For each $z \in [1, 2]$ the cross-section is the disk $D = \{x^2 + y^2 \le 1\}$, and the integrand does not depend on $z$. By Fubini's theorem with $z$ outermost,

$$\iiint_E \sqrt{x^2+y^2}\,\mathrm{d}V = \int_1^2 \left( \iint_D \sqrt{x^2+y^2}\,\mathrm{d}A \right) \mathrm{d}z.$$

Evaluate the double integral in polar coordinates, $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, $D = \{0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$:

$$\iint_D \sqrt{x^2+y^2}\,\mathrm{d}A = \int_0^{2\pi}\int_0^1 r \cdot r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^1 \mathrm{d}\theta = \int_0^{2\pi} \frac{1}{3}\,\mathrm{d}\theta = \frac{2\pi}{3}.$$

Then

$$\int_1^2 \frac{2\pi}{3}\,\mathrm{d}z = \frac{2\pi}{3}\big[z\big]_1^2 = \frac{2\pi}{3}(2-1) = \boxed{2\pi/3}$$

(ii) (a) Solve for $x, y$: $u - 4v = 5x$ and $u + v = 5y$, so

$$x = \frac{u-4v}{5}, \qquad y = \frac{u+v}{5}.$$

The partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{1}{5}, \qquad \frac{\partial x}{\partial v} = -\frac{4}{5}, \qquad \frac{\partial y}{\partial u} = \frac{1}{5}, \qquad \frac{\partial y}{\partial v} = \frac{1}{5},$$

so

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5}\cdot\frac{1}{5} - \left(-\frac{4}{5}\right)\cdot\frac{1}{5} = \frac{1}{25} + \frac{4}{25} = \frac{1}{5}.$$

The maps $(x, y) \mapsto (u, v)$ and $(u, v) \mapsto (x, y)$ are inverse to each other, so by the Chain Rule the product of their Jacobian matrices is the identity matrix, and taking determinants,

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1 \implies \frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{1/5} = \boxed{5}$$

(b) *This method should be used only if the course allows this interpretation* (it integrates directly in $x$ and $y$ without the transformation).

*Inner integral in $x$.* With $y$ fixed, write $\mathrm{e}^{y-x} = \mathrm{e}^y\mathrm{e}^{-x}$ and integrate by parts with $p = x + 4y$, $\mathrm{d}q = \mathrm{e}^{-x}\,\mathrm{d}x$, so $\mathrm{d}p = \mathrm{d}x$ and $q = -\mathrm{e}^{-x}$:

$$\begin{aligned}\int (x+4y)\mathrm{e}^{-x}\,\mathrm{d}x &= -(x+4y)\mathrm{e}^{-x} + \int \mathrm{e}^{-x}\,\mathrm{d}x \\ &= -(x+4y)\mathrm{e}^{-x} - \mathrm{e}^{-x} + C = -(x+4y+1)\mathrm{e}^{-x} + C.\end{aligned}$$

Multiplying by $\mathrm{e}^y$, an antiderivative of $(x+4y)\mathrm{e}^{y-x}$ in $x$ is $-(x+4y+1)\mathrm{e}^{y-x}$. At $x = 5-4y$: $x+4y+1 = 6$ and $y - x = 5y - 5$. At $x = y$: $x+4y+1 = 5y+1$ and $y - x = 0$. Hence

$$\int_y^{5-4y} (x+4y)\mathrm{e}^{y-x}\,\mathrm{d}x = \Big[-(x+4y+1)\mathrm{e}^{y-x}\Big]_{x=y}^{x=5-4y} = -6\mathrm{e}^{5y-5} + (5y+1).$$

*Outer integral in $y$.* First,

$$\int_0^1 (5y+1)\,\mathrm{d}y = \left[\frac{5y^2}{2}+y\right]_0^1 = \frac{5}{2}+1 = \frac{7}{2}.$$

Second, substitute $s = 5y-5$, $\mathrm{d}s = 5\,\mathrm{d}y$; when $y=0$, $s=-5$, and when $y=1$, $s=0$:

$$\int_0^1 6\mathrm{e}^{5y-5}\,\mathrm{d}y = \frac{6}{5}\int_{-5}^0 \mathrm{e}^s\,\mathrm{d}s = \frac{6}{5}\left[\mathrm{e}^s\right]_{-5}^0 = \frac{6}{5}\left(1-\mathrm{e}^{-5}\right).$$

Therefore

$$\begin{aligned}\int_0^1\int_y^{5-4y}(x+4y)\mathrm{e}^{y-x}\,\mathrm{d}x\,\mathrm{d}y &= \frac{7}{2}-\frac{6}{5}\left(1-\mathrm{e}^{-5}\right)\\ &= \frac{35}{10}-\frac{12}{10}+\frac{6}{5}\mathrm{e}^{-5}\\ &= \boxed{\frac{23}{10}+\frac{6}{5\mathrm{e}^5}.}\end{aligned}$$

(iii) The region $E$ is the ball $B=\{x^2+y^2+z^2\le 4\}$ with the two solid cones $\{z^2>x^2+y^2,\ z>0\}$ and $\{z^2>x^2+y^2,\ z<0\}$ (inside the ball) removed. In the spherical coordinates of Method 1, with $\mathrm{d}V=\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$, these pieces are $0\le\phi\le\pi$ (ball), $0\le\phi<\pi/4$ (upper cone) and $3\pi/4<\phi\le\pi$ (lower cone), each with $0\le\rho\le 2$, $0\le\theta\le 2\pi$. The $\rho$ and $\theta$ integrals contribute $\int_0^2\rho^2\,\mathrm{d}\rho=\frac{8}{3}$ and $\int_0^{2\pi}\mathrm{d}\theta=2\pi$ in each case.

Ball:

$$V_B = 2\pi\cdot\frac{8}{3}\left[-\cos\phi\right]_0^{\pi} = \frac{16\pi}{3}\big(-(-1)+1\big) = \frac{32\pi}{3}.$$

Upper cone:

$$V_+ = 2\pi\cdot\frac{8}{3}\left[-\cos\phi\right]_0^{\pi/4} = \frac{16\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right).$$

Lower cone:

$$V_- = 2\pi\cdot\frac{8}{3}\left[-\cos\phi\right]_{3\pi/4}^{\pi} = \frac{16\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right).$$

The cone surfaces $\phi=\pi/4$, $\phi=3\pi/4$ have zero volume, so

$$V = V_B - V_+ - V_- = \frac{32\pi}{3}-\frac{32\pi}{3}\left(1-\frac{\sqrt{2}}{2}\right) = \frac{32\pi}{3}\cdot\frac{\sqrt{2}}{2},$$

that is,

$$\boxed{V=\frac{16\sqrt{2}\pi}{3}.}$$

**Method 3: Fubini in the other order in the $uv$-plane; cylindrical coordinates**

(ii) (b) Use the inverse transformation, the Jacobian $|\partial(x,y)/\partial(u,v)| = \frac{1}{5}$ and the image triangle $S$ with vertices $(0,0)$, $(5,-5)$, $(5,0)$ exactly as in Method 1. Describe $S$ with $v$ outermost: for each $v \in [-5,0]$, the horizontal segment through $S$ runs from the line $v=-u$ (that is, $u=-v$) to the line $u=5$, so

$$S = \{(u,v) : -5 \le v \le 0,\ -v \le u \le 5\}.$$

By Fubini's theorem (the integrand $ue^v$ is continuous on $S$),

$$\int_0^1 \int_y^{5-4y} (x+4y)e^{y-x}\,dx\,dy = \frac{1}{5}\int_{-5}^0 \int_{-v}^5 ue^v\,du\,dv.$$

Inner integral in $u$:

$$\int_{-v}^5 ue^v\,du = e^v\left[\frac{u^2}{2}\right]_{-v}^5 = \frac{e^v\,(25-v^2)}{2}.$$

So the integral equals $\dfrac{1}{10}\displaystyle\int_{-5}^0 \left(25-v^2\right)e^v\,dv$. First,

$$\int_{-5}^0 25e^v\,dv = 25\left[e^v\right]_{-5}^0 = 25\left(1-e^{-5}\right).$$

Second, integrate by parts twice. With $p=v^2$, $dq = e^v\,dv$:

$$\int v^2e^v\,dv = v^2e^v - \int 2ve^v\,dv,$$

and the hint with $a=1$ gives $\int ve^v\,dv = e^v(v-1)+C$, so

$$\int v^2e^v\,dv = v^2e^v - 2e^v(v-1) + C = e^v\left(v^2-2v+2\right)+C.$$

Hence

$$\int_{-5}^0 v^2e^v\,dv = \left[e^v\left(v^2-2v+2\right)\right]_{-5}^0 = 2 - e^{-5}(25+10+2) = 2-37e^{-5}.$$

Therefore

$$\begin{aligned}\int_0^1 \int_y^{5-4y} (x+4y)e^{y-x}\,dx\,dy &= \frac{1}{10}\left(25-25e^{-5}-2+37e^{-5}\right)\\ &= \frac{1}{10}\left(23+12e^{-5}\right)\\ &= \boxed{\frac{23}{10}+\frac{6}{5e^5}.}\end{aligned}$$

(iii) *This method should be used only if the course allows this interpretation* (the question asks for spherical coordinates; this route uses cylindrical coordinates).

Use cylindrical coordinates, $dV = r\,dz\,dr\,d\theta$ (Jacobian computed in part (i) of Method 1). The conditions become

$$r^2+z^2 \le 4 \iff |z| \le \sqrt{4-r^2}, \qquad z^2 \le r^2 \iff |z| \le r.$$

So for fixed $r \in [0,2]$, $|z| \le \min\left(r, \sqrt{4-r^2}\right)$. Since

$$r \le \sqrt{4-r^2} \iff r^2 \le 4-r^2 \iff r^2 \le 2 \iff r \le \sqrt{2},$$

the region splits at $r = \sqrt{2}$:

$$V = \int_0^{2\pi}\int_0^{\sqrt{2}}\int_{-r}^{r} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta + \int_0^{2\pi}\int_{\sqrt{2}}^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta.$$

*First piece.* The $z$-integral is $r\big[z\big]_{-r}^{r} = 2r^2$, and

$$\int_0^{\sqrt{2}} 2r^2\,\mathrm{d}r = \left[\frac{2r^3}{3}\right]_0^{\sqrt{2}} = \frac{2\cdot 2\sqrt{2}}{3} = \frac{4\sqrt{2}}{3},$$

so the first piece is $2\pi\cdot\dfrac{4\sqrt{2}}{3} = \dfrac{8\sqrt{2}\pi}{3}$.

*Second piece.* The $z$-integral is $r\big[z\big]_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} = 2r\sqrt{4-r^2}$. Substitute $w = 4-r^2$, $\mathrm{d}w = -2r\,\mathrm{d}r$; when $r=\sqrt{2}$, $w=2$, and when $r=2$, $w=0$:

$$\int_{\sqrt{2}}^{2} 2r\sqrt{4-r^2}\,\mathrm{d}r = \int_2^0 w^{1/2}(-\,\mathrm{d}w) = \int_0^2 w^{1/2}\,\mathrm{d}w = \left[\frac{2}{3}w^{3/2}\right]_0^2 = \frac{2}{3}\cdot 2\sqrt{2} = \frac{4\sqrt{2}}{3},$$

so the second piece is $2\pi\cdot\dfrac{4\sqrt{2}}{3} = \dfrac{8\sqrt{2}\pi}{3}$. Adding,

$$V = \frac{8\sqrt{2}\pi}{3} + \frac{8\sqrt{2}\pi}{3},$$

that is,

$$\boxed{V = \frac{16\sqrt{2}\pi}{3}.}$$

**Marking scheme (17 marks)**

(i) **(4 marks)** 1 mark: correct cylindrical bounds; 1 mark: correct integrand and Jacobian; 1 mark: correct integration; 1 mark: obtains $2\pi/3$.

(ii) **(8 marks)**

(a) **(3 marks)** 2 marks: forms the determinant; 1 mark: obtains 5.

(b) **(5 marks)** 1 mark: inverts the transformation and finds $|\partial(x,y)/\partial(u,v)|$; 2 marks: maps the region correctly; 1 mark: transforms the integral; 1 mark: obtains $23/10 + 6/(5\mathrm{e}^5)$.

(iii) **(5 marks)** 2 marks: correct spherical bounds; 2 marks: correct volume integral; 1 mark: obtains $16\sqrt{2}\pi/3$.

# Question 3 (17 Marks)

***(beyond 2026 midterm scope)***

**QUESTION 3.**

(i) *(beyond 2026 midterm scope)* [4 marks] Let $C$ be the curve parametrised by $x = 3t$ and $y = t^3$ where $0 \le t \le 1$. Evaluate $\displaystyle\int_C (x^3 + y)\,\mathrm{d}s$.

(ii) *(beyond 2026 midterm scope)* [4 marks] Find the work done by the vector field $\mathbf{F} = \langle x^3 - y^3, xy^2 \rangle$ along the curve parametrised by $x = t^2$ and $y = t^3$ from $t = -1$ to $t = 0$.

(iii) *(beyond 2026 midterm scope)* [4 marks] Determine if the vector field $\mathbf{F} = \langle 3x^2y^2 + \mathrm{e}^x, 2x^3y + \sin(y) \rangle$ is conservative. If so, find a potential function $f$ for $\mathbf{F}$.

(iv) *(beyond 2026 midterm scope)* [5 marks] Let $C$ be part of the curve (in the $xy$-plane) described by polar coordinates as $r = \sin(\theta)$ where $0 \le \theta \le \pi$. Find the length of $C$.

## Solution

**Method 1: Parametric line integrals, the cross-partial test, polar arc length**

(i) With $x(t) = 3t$ and $y(t) = t^3$,

$$x'(t) = 3, \qquad y'(t) = 3t^2,$$

so the arc-length element is

$$\mathrm{d}s = \sqrt{x'(t)^2 + y'(t)^2}\,\mathrm{d}t = \sqrt{9 + 9t^4}\,\mathrm{d}t = 3\sqrt{1 + t^4}\,\mathrm{d}t.$$

Along $C$ the integrand is

$$x^3 + y = (3t)^3 + t^3 = 27t^3 + t^3 = 28t^3.$$

Hence

$$\int_C (x^3 + y)\,\mathrm{d}s = \int_0^1 28t^3 \cdot 3\sqrt{1 + t^4}\,\mathrm{d}t = 84 \int_0^1 t^3\sqrt{1 + t^4}\,\mathrm{d}t.$$

Substitute $u = 1 + t^4$, $\mathrm{d}u = 4t^3\,\mathrm{d}t$, so $t^3\,\mathrm{d}t = \frac{1}{4}\,\mathrm{d}u$; when $t = 0$, $u = 1$, and when $t = 1$, $u = 2$:

$$\begin{aligned} 84 \int_0^1 t^3\sqrt{1 + t^4}\,\mathrm{d}t &= 84 \cdot \frac{1}{4} \int_1^2 u^{1/2}\,\mathrm{d}u = 21 \left[\frac{2}{3}u^{3/2}\right]_1^2 \\ &= 14\left(2^{3/2} - 1\right) = \boxed{14(2\sqrt{2} - 1)} \end{aligned}$$

(ii) Let $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $-1 \le t \le 0$. Then

$$\mathbf{r}'(t) = \langle 2t, 3t^2 \rangle.$$

On the curve, $x^3 = t^6$, $y^3 = t^9$ and $xy^2 = t^2 \cdot t^6 = t^8$, so

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^6 - t^9,\ t^8 \rangle.$$

The dot product is

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = (t^6 - t^9)(2t) + t^8(3t^2) = 2t^7 - 2t^{10} + 3t^{10} = 2t^7 + t^{10}.$$

Hence the work is

$$\begin{aligned} W = \int_{-1}^{0} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,\mathrm{d}t &= \int_{-1}^{0} (2t^7 + t^{10})\,\mathrm{d}t = \left[\frac{t^8}{4} + \frac{t^{11}}{11}\right]_{-1}^{0} \\ &= 0 - \left(\frac{(-1)^8}{4} + \frac{(-1)^{11}}{11}\right) = -\left(\frac{1}{4} - \frac{1}{11}\right) \\ &= -\frac{11-4}{44} = \boxed{-7/44} \end{aligned}$$

(iii) Write $\mathbf{F} = \langle P, Q\rangle$ with $P = 3x^2y^2 + \mathrm{e}^x$ and $Q = 2x^3y + \sin y$. The partial derivatives are

$$P_y = \frac{\partial}{\partial y}\left(3x^2y^2 + \mathrm{e}^x\right) = 6x^2y, \qquad Q_x = \frac{\partial}{\partial x}\left(2x^3y + \sin y\right) = 6x^2y.$$

$P$ and $Q$ have continuous first partial derivatives on $\mathbb{R}^2$, which is open and simply connected, and $P_y = Q_x$ everywhere. By the cross-partial test for conservative fields, $\mathbf{F}$ is conservative.

To find $f$ with $\nabla f = \mathbf{F}$, integrate $f_x = P$ in $x$:

$$f(x,y) = \int \left(3x^2y^2 + \mathrm{e}^x\right)\,\mathrm{d}x = x^3y^2 + \mathrm{e}^x + h(y).$$

Differentiate in $y$ and match with $Q$:

$$f_y = 2x^3y + h'(y) = 2x^3y + \sin y \implies h'(y) = \sin y \implies h(y) = -\cos y + K.$$

Taking $K = 0$ and checking, $f_x = 3x^2y^2 + \mathrm{e}^x = P$ and $f_y = 2x^3y + \sin y = Q$. Hence $\mathbf{F}$ is conservative with potential

$$\boxed{f(x,y) = x^3y^2 + \mathrm{e}^x - \cos y.}$$

(iv) For a polar curve $r = r(\theta)$, $\alpha \le \theta \le \beta$, traced once, the arc length is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$

Here

$$r = \sin\theta, \qquad \frac{\mathrm{d}r}{\mathrm{d}\theta} = \cos\theta,$$

so

$$r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \sin^2\theta + \cos^2\theta = 1.$$

The curve is traced once on $[0,\pi]$: for $0 < \theta < \pi$ we have $r = \sin\theta > 0$, so different values of $\theta$ give points in different directions from the origin, and only $\theta = 0$ and $\theta = \pi$ give the origin. Hence

$$L = \int_0^{\pi} \sqrt{1}\,\mathrm{d}\theta = \left[\theta\right]_0^{\pi} = \pi,$$

that is,

$$\boxed{L = \pi}$$

**Method 2: Reparametrisation by $x$; potential for one component; integrating $Q$ first; Cartesian parametric arc length**

(i) The scalar line integral $\int_C h\,\mathrm{d}s$ does not depend on the parametrisation of $C$. Since $x = 3t$, $t = x/3$ and $y = t^3 = x^3/27$, so $C$ is the graph

$$y = \varphi(x) = \frac{x^3}{27}, \qquad 0 \le x \le 3.$$

Then $\varphi'(x) = \dfrac{x^2}{9}$ and

$$\mathrm{d}s = \sqrt{1 + \varphi'(x)^2}\,\mathrm{d}x = \sqrt{1 + \frac{x^4}{81}}\,\mathrm{d}x.$$

Along $C$ the integrand is

$$x^3 + y = x^3 + \frac{x^3}{27} = \frac{28x^3}{27}.$$

Substitute $w = 1 + \dfrac{x^4}{81}$, $\mathrm{d}w = \dfrac{4x^3}{81}\,\mathrm{d}x$, so $x^3\,\mathrm{d}x = \dfrac{81}{4}\,\mathrm{d}w$; when $x = 0$, $w = 1$, and when $x = 3$, $w = 1 + \frac{81}{81} = 2$:

$$\begin{aligned}\int_C (x^3 + y)\,\mathrm{d}s &= \int_0^3 \frac{28}{27}x^3\sqrt{1 + \frac{x^4}{81}}\,\mathrm{d}x = \frac{28}{27}\cdot\frac{81}{4}\int_1^2 w^{1/2}\,\mathrm{d}w \\ &= 21\left[\frac{2}{3}w^{3/2}\right]_1^2 = 14\left(2\sqrt{2} - 1\right) = \boxed{14(2\sqrt{2} - 1)}\end{aligned}$$

(ii) Split $\mathbf{F} = \mathbf{G} + \mathbf{H}$ with

$$\mathbf{G} = \langle x^3, 0\rangle, \qquad \mathbf{H} = \langle -y^3,\ xy^2\rangle.$$

*The field* $\mathbf{G}$. $\mathbf{G} = \nabla\psi$ with $\psi(x, y) = \dfrac{x^4}{4}$, since $\psi_x = x^3$ and $\psi_y = 0$. The curve starts at $\mathbf{r}(-1) = (1, -1)$ and ends at $\mathbf{r}(0) = (0, 0)$. By the Fundamental Theorem for line integrals,

$$\int_C \mathbf{G}\cdot\,\mathrm{d}\mathbf{r} = \psi(0, 0) - \psi(1, -1) = 0 - \frac{1}{4} = -\frac{1}{4}.$$

*The field* $\mathbf{H}$. With $\mathbf{r}(t) = \langle t^2, t^3\rangle$, $\mathbf{r}'(t) = \langle 2t, 3t^2\rangle$ and $\mathbf{H}(\mathbf{r}(t)) = \langle -t^9,\ t^8\rangle$,

$$\mathbf{H}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = (-t^9)(2t) + t^8(3t^2) = -2t^{10} + 3t^{10} = t^{10},$$

so

$$\int_C \mathbf{H}\cdot\,\mathrm{d}\mathbf{r} = \int_{-1}^0 t^{10}\,\mathrm{d}t = \left[\frac{t^{11}}{11}\right]_{-1}^0 = 0 - \left(-\frac{1}{11}\right) = \frac{1}{11}.$$

Adding the two pieces,

$$W = -\frac{1}{4} + \frac{1}{11} = \frac{-11 + 4}{44} = \boxed{-7/44}$$

(iii) The cross-partial test is the same as in Method 1: $P_y = 6x^2y = Q_x$ on the simply connected domain $\mathbb{R}^2$, so $\mathbf{F}$ is conservative. Now integrate $f_y = Q$ in $y$ first:

$$f(x, y) = \int \left(2x^3y + \sin y\right)\,\mathrm{d}y = x^3y^2 - \cos y + g(x).$$

Differentiate in $x$ and match with $P$:

$$f_x = 3x^2y^2 + g'(x) = 3x^2y^2 + \mathrm{e}^x \implies g'(x) = \mathrm{e}^x \implies g(x) = \mathrm{e}^x + K.$$

Taking $K = 0$,

$$\boxed{f(x,y) = x^3y^2 + \mathrm{e}^x - \cos y.}$$

(iv) In Cartesian form, $x = r\cos\theta$ and $y = r\sin\theta$ with $r = \sin\theta$ give

$$x(\theta) = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta, \qquad y(\theta) = \sin^2\theta = \frac{1-\cos 2\theta}{2}, \qquad 0 \le \theta \le \pi.$$

Differentiate:

$$x'(\theta) = \frac{1}{2}\cdot 2\cos 2\theta = \cos 2\theta, \qquad y'(\theta) = \frac{1}{2}\cdot 2\sin 2\theta = \sin 2\theta.$$

The speed is

$$\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1.$$

The point $(x,y) = \frac{1}{2}(\sin 2\theta,\ 1-\cos 2\theta)$ is determined by the angle $2\theta$, which runs over $[0, 2\pi)$ exactly once for $\theta \in [0,\pi)$, so no arc is counted twice. Hence

$$L = \int_0^{\pi} 1\,\mathrm{d}\theta = \pi,$$

that is,

$$\boxed{L = \pi}$$

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**Method 3: Elimination of the parameter; potential by integration along a path; circle circumference**

(ii) Write the work as $W = \int_C P\,\mathrm{d}x + Q\,\mathrm{d}y$ with $P = x^3 - y^3$, $Q = xy^2$, and use $x$ as the parameter. For $-1 \le t \le 0$ we have $t \le 0$ and $x = t^2$, so $t = -\sqrt{x}$ and

$$y = t^3 = -x^{3/2}, \qquad \mathrm{d}y = -\frac{3}{2}x^{1/2}\,\mathrm{d}x.$$

As $t$ runs from $-1$ to $0$, $x = t^2$ runs from 1 to 0. On the curve,

$$P = x^3 - \left(-x^{3/2}\right)^3 = x^3 + x^{9/2}, \qquad Q = x\left(-x^{3/2}\right)^2 = x \cdot x^3 = x^4.$$

Hence

$$P\,\mathrm{d}x + Q\,\mathrm{d}y = \left(x^3 + x^{9/2}\right)\,\mathrm{d}x + x^4\left(-\frac{3}{2}x^{1/2}\right)\,\mathrm{d}x = \left(x^3 - \frac{1}{2}x^{9/2}\right)\,\mathrm{d}x,$$

and

$$\begin{aligned} W &= \int_1^0 \left(x^3 - \frac{1}{2}x^{9/2}\right)\,\mathrm{d}x = -\int_0^1 \left(x^3 - \frac{1}{2}x^{9/2}\right)\,\mathrm{d}x \\ &= -\left[\frac{x^4}{4} - \frac{1}{2}\cdot\frac{2}{11}x^{11/2}\right]_0^1 = -\left(\frac{1}{4} - \frac{1}{11}\right) = \boxed{-7/44} \end{aligned}$$

(iii) Define $f(x, y)$ as the line integral of $\mathbf{F}$ from $(0, 0)$ along the segment to $(x, 0)$ and then along the segment to $(x, y)$.

*First segment.* $\mathbf{r}_1(s) = \langle s, 0\rangle$, $s$ from 0 to $x$, $\mathbf{r}_1'(s) = \langle 1, 0\rangle$:

$$\int_0^x P(s, 0)\,\mathrm{d}s = \int_0^x \left(3s^2 \cdot 0 + \mathrm{e}^s\right)\,\mathrm{d}s = \left[\mathrm{e}^s\right]_0^x = \mathrm{e}^x - 1.$$

*Second segment.* $\mathbf{r}_2(s) = \langle x, s\rangle$, $s$ from 0 to $y$, $\mathbf{r}_2'(s) = \langle 0, 1\rangle$:

$$\int_0^y Q(x, s)\,\mathrm{d}s = \int_0^y \left(2x^3 s + \sin s\right)\,\mathrm{d}s = \left[x^3 s^2 - \cos s\right]_0^y = x^3 y^2 - \cos y + 1.$$

Adding,

$$f(x, y) = \mathrm{e}^x - 1 + x^3 y^2 - \cos y + 1 = x^3 y^2 + \mathrm{e}^x - \cos y.$$

Verification of $\nabla f = \mathbf{F}$:

$$f_x = 3x^2 y^2 + \mathrm{e}^x = P, \qquad f_y = 2x^3 y + \sin y = Q.$$

So $\mathbf{F}$ has a potential on $\mathbb{R}^2$ and is therefore conservative, with

$$\boxed{f(x, y) = x^3 y^2 + \mathrm{e}^x - \cos y.}$$

(iv) Multiply $r = \sin\theta$ by $r$: $r^2 = r\sin\theta$, that is, $x^2 + y^2 = y$ (this adds only the origin, which is already on the curve at $\theta = 0$). Complete the square in $y$:

$$x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \iff x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2.$$

So $C$ lies on the circle with centre $\left(0, \frac{1}{2}\right)$ and radius $\frac{1}{2}$. Using $x = \frac{1}{2}\sin 2\theta$ and $y = \frac{1-\cos 2\theta}{2}$,

$$\left(x,\ y - \frac{1}{2}\right) = \frac{1}{2}\left(\sin 2\theta,\ -\cos 2\theta\right).$$

As $\theta$ runs over $[0, \pi)$, the angle $2\theta$ runs over $[0, 2\pi)$ exactly once, so each point of the circle is reached exactly once (and $\theta = \pi$ returns to the start). Hence $L$ is the circumference:

$$L = 2\pi \cdot \frac{1}{2} = \pi,$$

that is,

$$\boxed{L = \pi}$$

**Marking scheme (17 marks)**

(i) **(4 marks)** 1 mark: obtains $\mathrm{d}s$; 1 mark: substitutes the integrand; 1 mark: uses a valid substitution; 1 mark: obtains $14(2\sqrt{2} - 1)$.

(ii) **(4 marks)** 1 mark: finds $\mathbf{r}'$; 2 marks: forms the work integrand; 1 mark: obtains $-7/44$.

(iii) **(4 marks)** 1 mark: checks $P_y = Q_x$; 2 marks: constructs a potential; 1 mark: gives a valid final potential.

(iv) **(5 marks)** 2 marks: states the polar arc-length formula; 2 marks: simplifies its integrand; 1 mark: obtains $\pi$.

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