NANYANG TECHNOLOGICAL UNIVERSITY

MIDTERM II

MH2100 – Calculus III

October 2023 TIME ALLOWED: 90 minutes

INSTRUCTIONS TO CANDIDATES

1. **DO NOT TURN OVER PAPER UNTIL INSTRUCTED.**
2. This midterm paper contains **THREE (3)** questions.
3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
4. Candidates should write their answers in the separate answer book provided.
5. **ONE** two-sided A4 piece of paper with handwritten notes is permitted.

QUESTION 1. **(17 marks)**

(i) [5 marks] A wire has a shape of an ellipse described by $4x^2 + y^2 = 8$. The temperature $T(x, y)$ at every point (x, y) of the wire is described by $T(x, y) = e^{2xy}$. Use the method of Lagrange multipliers to find the point(s) on the wire where the wire has the highest and the lowest temperature.

(ii) [3 marks] Evaluate the double integral

$$\iint_R 3x^2 \sin(y)\,\mathrm{d}A,$$

where R is the rectangle $R = \{(x, y) \mid 0 \leq x \leq 2 \text{ and } 0 \leq y \leq \pi\}$.

(iii) [4 marks] By reversing the order of integration, evaluate the integral

$$\int_0^1 \int_x^1 (2 + 6x) \cos(y^2 + y^3)\,\mathrm{d}y\,\mathrm{d}x.$$

(iv) [5 marks] Let E be the solid lying below the plane $z = 2 - x$ and above the triangular region in the xy-plane with vertices $(0, 0, 0)$, $(4, 0, 0)$, and $(0, 4, 0)$. Evaluate the triple integral

$$\iiint_E x\,\mathrm{d}V.$$

QUESTION 2. **(17 marks)**

(i) [4 marks] Use cylindrical coordinates to evaluate the triple integral

$$\iiint_E 2z \, \mathrm{d}V,$$

where E is the solid above the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and below the ellipsoid $z = \sqrt{8 - 2x^2 - 2y^2}$.

(ii) [6 marks] The solid E is a spherical wedge such that

$$\iiint_E f(x, y, z) \mathrm{d}V = \int_0^1 \int_x^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} f(x, y, z) \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x.$$

The solid E can be described in spherical coordinates by $T_1 \leq \theta \leq T_2$, $P_1 \leq \phi \leq P_2$, and $R_1 \leq \rho \leq R_2$. Find the values of T_1, T_2, P_1, P_2, R_1, and R_2.

(iii) Consider the transformation $u = x + y$ and $v = y^2$.

(a) [2 marks] Calculate the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$.

(b) Using the transformation, one can evaluate the double integral

$$\iint_R 2y \cos\left((x + y)^3 - 3(x + y)\right) \, \mathrm{d}A,$$

where R is the region bounded by the curves $x = 0$, $y = 1$, and $x + y = 2$. One finds that

$$\iint_R 2y \cos\left((x + y)^3 - 3(x + y)\right) \, \mathrm{d}A = \int_a^b \int_c^d F(u, v) \mathrm{d}v \mathrm{d}u.$$

(I) [2 marks] Find the values of a, b, c, d. [Hint: they are not all constants].

(II) [1 mark] Find the integrand $F(u, v)$.

(III) [2 marks] It turns out that

$$\iint_R 2y \cos\left((x + y)^3 - 3(x + y)\right) \, \mathrm{d}A = C_1 \sin(C_2)$$

for some constants C_1 and C_2. Find the values of C_1 and C_2.

QUESTION 3. **(16 marks)**

(i) [4 marks] Let C be the part of the curve $y = x^2$ oriented from $(1, 1)$ to $(2, 4)$. Evaluate

$$\int_C xy \, \mathrm{d}x + (x + y) \, \mathrm{d}y.$$

(ii) [5 marks] Evaluate the line integral of $f(x, y, z) = x + y + z$ over the straight-line segment from $(3, 1, 2)$ to $(2, -3, -1)$.

(iii) Consider the vector field $\mathbf{F} = \langle 4z^2, \cos z, 8xz - y \sin z + 1 \rangle$.

(a) [4 marks] Show that $\mathbf{F}$ is conservative by finding a potential function f for $\mathbf{F}$.

(b) [3 marks] Evaluate

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where C is the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq \frac{\pi}{2}$.