MH2100 Calculus III

Midterm I (Solutions)

AY 2023/2024, Semester 1

Quantitative Research Society @NTU

September 2023

Overview

- **Coverage:** parametric curves and tangent lines; limits and continuity; partial and directional derivatives; tangent planes, the Chain Rule, implicit differentiation, and critical points.
- **Historical source:** MH2100 Midterm I, AY2023/2024 Semester 1. Every question below is transcribed from the official paper. Solutions and marking schemes are unofficial QRS@NTU material.
- **Question themes:** Q1 tangent lines of parametric curves and directional derivatives; Q2 limits, tangent planes of a level surface, continuity and differentiability at the origin; Q3 mixed partials, the Chain Rule, implicit differentiation, zero directional derivatives and critical points.
- **Current administration (AY2026/2027):** Tuesday 6 October 2026, 16:30–18:20 (110 minutes), venue TBA. One double-sided typed A4 sheet is permitted; attachments and sticky notes are not.

Instructions to candidates (historical, verbatim)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. ONE two-sided A4 piece of paper with handwritten notes is permitted.

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Question 1 (17 Marks)

QUESTION 1.

(i) [3 marks] Find parametric equations for the tangent line at the point when $t = 1$ to the curve with parametric equations

$$x = 3t^2, \qquad y = 5t, \qquad z = 7t^3.$$

(ii) [3 marks] The cartesian equation $y = ax + b$ describes the line tangent to the curve

$$\begin{cases} x(t) = \mathrm{e}^t, \\ y(t) = t^2 + 2t - 7 \end{cases}$$

at the point defined by $t = 0$. Find the values of a and b.

(iii) [3 marks] Find all points (P, Q) on the following curve where the tangent line is vertical.

$$x = t^3 - 9t + 18, \qquad y = \sin(t)$$

(iv) [3 marks] At the point $(0, \pi, 1)$, find the directional derivative of the function $f(x, y, z) = \sin(xyz)$ in the direction of the vector $\langle 1, 2, 2\rangle$.

(v) [5 marks] Find all unit direction vectors $\mathbf{u} = \langle a, b\rangle$ in which the directional derivative of the function $f(x, y) = y\mathrm{e}^{-xy}$ at the point $(0, 2)$ has the value 1.

Solution

Part (i), Method 1: Derivative of the position vector

Let

$$\mathbf{r}(t) = \langle 3t^2,\, 5t,\, 7t^3\rangle.$$

The point of tangency is

$$\mathbf{r}(1) = \langle 3(1)^2,\, 5(1),\, 7(1)^3\rangle = \langle 3, 5, 7\rangle.$$

Differentiate each component:

$$\frac{d}{dt}\big(3t^2\big) = 6t, \qquad \frac{d}{dt}(5t) = 5, \qquad \frac{d}{dt}\big(7t^3\big) = 21t^2,$$

so

$$\mathbf{r}'(t) = \langle 6t,\, 5,\, 21t^2\rangle, \qquad \mathbf{r}'(1) = \langle 6(1),\, 5,\, 21(1)^2\rangle = \langle 6, 5, 21\rangle \neq \mathbf{0}.$$

Since $\mathbf{r}'(1) \neq \mathbf{0}$, it is a direction vector of the tangent line at $t = 1$. The line through $\langle 3, 5, 7\rangle$ in this direction is

$$x = 3 + 6s, \qquad y = 5 + 5s, \qquad z = 7 + 21s, \qquad s \in \mathbb{R},$$

that is,

$$\boxed{\mathbf{l}(s) = \langle 3, 5, 7\rangle + s\langle 6, 5, 21\rangle, \quad s \in \mathbb{R}.}$$

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Part (i), Method 2: Eliminating the parameter (the curve as a graph over y)

From $y = 5t$ we get $t = y/5$. Substituting into the other two components,

$$x = 3\Big(\frac{y}{5}\Big)^2 = \frac{3y^2}{25}, \qquad z = 7\Big(\frac{y}{5}\Big)^3 = \frac{7y^3}{125},$$

so the curve is $y \mapsto \big(\frac{3y^2}{25},\, y,\, \frac{7y^3}{125}\big)$ with y as the parameter. The value $t = 1$ corresponds to $y = 5$, and there

$$x = \frac{3(25)}{25} = 3, \qquad z = \frac{7(125)}{125} = 7.$$

Differentiate with respect to y:

$$\frac{dx}{dy} = \frac{6y}{25}, \qquad \frac{dy}{dy} = 1, \qquad \frac{dz}{dy} = \frac{21y^2}{125}.$$

At $y = 5$,

$$\frac{dx}{dy} = \frac{30}{25} = \frac{6}{5}, \qquad \frac{dz}{dy} = \frac{21(25)}{125} = \frac{21}{5}.$$

A tangent vector is therefore

$$\Big\langle \frac{6}{5},\, 1,\, \frac{21}{5} \Big\rangle = \frac{1}{5}\langle 6, 5, 21\rangle,$$

and multiplying by 5 gives the direction $\langle 6, 5, 21\rangle$ through the point $\langle 3, 5, 7\rangle$:

$$\boxed{\mathbf{l}(s) = \langle 3, 5, 7\rangle + s\langle 6, 5, 21\rangle, \quad s \in \mathbb{R}.}$$

Part (ii), Method 1: Slope as the quotient of parameter derivatives

At $t = 0$,

$$x(0) = \mathrm{e}^0 = 1, \qquad y(0) = 0^2 + 2(0) - 7 = -7,$$

so the point of tangency is $(1, -7)$. Differentiate:

$$x'(t) = \mathrm{e}^t, \qquad y'(t) = 2t + 2.$$

Since $x'(0) = 1 \neq 0$, the curve is locally a graph $y = y(x)$ near $t = 0$, and the Chain Rule gives

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t + 2}{\mathrm{e}^t}.$$

At $t = 0$,

$$a = \frac{dy}{dx}\Big|_{t=0} = \frac{2(0) + 2}{\mathrm{e}^0} = \frac{2}{1} = 2.$$

The line $y = 2x + b$ passes through $(1, -7)$:

$$-7 = 2(1) + b \implies b = -9.$$

Hence

$$\boxed{a = 2, \qquad b = -9.}$$

Part (ii), Method 2: Eliminating the parameter

Since $x = \mathrm{e}^t > 0$, we have $t = \ln x$, and

$$y = (\ln x)^2 + 2\ln x - 7, \qquad x > 0.$$

The value $t = 0$ corresponds to $x = \mathrm{e}^0 = 1$, where $y = (\ln 1)^2 + 2\ln 1 - 7 = -7$. Differentiate:

$$\frac{dy}{dx} = 2\ln x \cdot \frac{1}{x} + \frac{2}{x} = \frac{2\ln x + 2}{x}.$$

At $x = 1$,

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{2(0)+2}{1} = 2.$$

The tangent line is

$$y - (-7) = 2(x-1) \iff y = 2x - 9.$$

Hence

$$\boxed{a = 2, \qquad b = -9.}$$

Part (ii), Method 3: Vector form of the tangent line

The tangent vector at $t = 0$ is

$$\langle x'(0),\, y'(0)\rangle = \langle \mathrm{e}^0,\, 2(0)+2\rangle = \langle 1, 2\rangle,$$

so the tangent line through $(1, -7)$ is

$$x = 1 + s, \qquad y = -7 + 2s, \qquad s \in \mathbb{R}.$$

From the first equation $s = x - 1$. Substituting into the second,

$$y = -7 + 2(x-1) = 2x - 9.$$

Comparing with $y = ax + b$,

$$\boxed{a = 2, \qquad b = -9.}$$

Part (iii), Method 1: Vertical-tangent condition on the tangent vector

The tangent vector is $\langle x'(t),\, y'(t)\rangle$. The tangent line is vertical exactly when its horizontal component is 0 and the vector itself is nonzero:

$$x'(t) = 0 \qquad \text{and} \qquad y'(t) \neq 0.$$

Differentiate:

$$x'(t) = 3t^2 - 9, \qquad y'(t) = \cos t.$$

Solve $x'(t) = 0$:

$$3t^2 - 9 = 0 \iff t^2 = 3 \iff t = \sqrt{3} \ \text{ or } \ t = -\sqrt{3}.$$

Check $y' \neq 0$ at these parameters. Since

$$\frac{\pi}{2} \approx 1.571 < \sqrt{3} \approx 1.732 < \pi \approx 3.142,$$

we have $\cos\sqrt{3} < 0$, so $\cos\sqrt{3} \neq 0$; and $\cos(-\sqrt{3}) = \cos\sqrt{3} \neq 0$ because cosine is even. Both parameters give vertical tangents.

For $t = \sqrt{3}$:

$$x = (\sqrt{3})^3 - 9\sqrt{3} + 18 = 3\sqrt{3} - 9\sqrt{3} + 18 = 18 - 6\sqrt{3}, \qquad y = \sin\sqrt{3}.$$

For $t = -\sqrt{3}$:

$$x = (-\sqrt{3})^3 - 9(-\sqrt{3}) + 18 = -3\sqrt{3} + 9\sqrt{3} + 18 = 18 + 6\sqrt{3}, \qquad y = \sin(-\sqrt{3}) = -\sin\sqrt{3}.$$

Hence

$$\boxed{(P, Q) = \big(18 - 6\sqrt{3}, \sin\sqrt{3}\big) \quad \text{or} \quad \big(18 + 6\sqrt{3}, -\sin\sqrt{3}\big).}$$

Part (iii), Method 2: Reciprocal slope dx/dy

First consider the parameters with $y'(t) = \cos t = 0$, that is $t = \frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$. These values are $\pm 1.571,\ \pm 4.712, \ldots$, none of which equals $\pm\sqrt{3} \approx \pm 1.732$, so $x'(t) = 3t^2 - 9 \neq 0$ there. At such t the tangent vector is $\langle x'(t), 0\rangle$ with $x'(t) \neq 0$: the tangent is horizontal, not vertical.

At every other parameter $\cos t \neq 0$, so the curve is locally a graph $x = x(y)$ and the Chain Rule gives

$$\frac{dx}{dy} = \frac{x'(t)}{y'(t)} = \frac{3t^2 - 9}{\cos t}.$$

The tangent is vertical exactly when x is momentarily not changing with y, i.e. $\dfrac{dx}{dy} = 0$:

$$\frac{3t^2 - 9}{\cos t} = 0 \iff 3t^2 - 9 = 0 \iff t = \pm\sqrt{3},$$

and these satisfy $\cos t \neq 0$ since $\frac{\pi}{2} < \sqrt{3} < \pi$. Substituting,

$$\begin{aligned} t = \sqrt{3}: &\quad x = 3\sqrt{3} - 9\sqrt{3} + 18 = 18 - 6\sqrt{3}, \quad y = \sin\sqrt{3}, \\ t = -\sqrt{3}: &\quad x = -3\sqrt{3} + 9\sqrt{3} + 18 = 18 + 6\sqrt{3}, \quad y = -\sin\sqrt{3}. \end{aligned}$$

Hence

$$\boxed{(P, Q) = \big(18 - 6\sqrt{3}, \sin\sqrt{3}\big) \quad \text{or} \quad \big(18 + 6\sqrt{3}, -\sin\sqrt{3}\big).}$$

Part (iv), Method 1: Gradient dotted with the unit direction

The function $f(x,y,z)=\sin(xyz)$ is a composition of sin with a polynomial, so its first partial derivatives exist and are continuous; hence f is differentiable and $D_{\mathbf{u}}f=\nabla f\cdot\mathbf{u}$. By the Chain Rule,

$$f_x=\cos(xyz)\cdot yz,\qquad f_y=\cos(xyz)\cdot xz,\qquad f_z=\cos(xyz)\cdot xy.$$

At $(0,\pi,1)$ we have $xyz=0$ and $\cos 0=1$, so

$$f_x=\pi\cdot 1=\pi,\qquad f_y=0\cdot 1=0,\qquad f_z=0\cdot\pi=0,\qquad \nabla f(0,\pi,1)=\langle\pi,0,0\rangle.$$

Normalise the direction:

$$|\langle 1,2,2\rangle|=\sqrt{1+4+4}=3,\qquad \mathbf{u}=\frac{\langle 1,2,2\rangle}{3}=\Big\langle\frac{1}{3},\frac{2}{3},\frac{2}{3}\Big\rangle.$$

Then

$$\nabla f(0,\pi,1)\cdot\mathbf{u}=\pi\cdot\frac{1}{3}+0\cdot\frac{2}{3}+0\cdot\frac{2}{3}=\frac{\pi}{3},$$

that is,

$$\boxed{D_{\mathbf{u}}f(0,\pi,1)=\langle\pi,0,0\rangle\cdot\frac{\langle 1,2,2\rangle}{3}=\frac{\pi}{3}.}$$

Part (iv), Method 2: Restriction to the line through the point

By definition, $D_{\mathbf{u}}f(P)=g'(0)$ where $g(h)=f(P+h\mathbf{u})$. With $P=(0,\pi,1)$ and $\mathbf{u}=\langle\frac{1}{3},\frac{2}{3},\frac{2}{3}\rangle$,

$$P+h\mathbf{u}=\Big(\frac{h}{3},\ \pi+\frac{2h}{3},\ 1+\frac{2h}{3}\Big),\qquad g(h)=\sin\big(\varphi(h)\big),\qquad \varphi(h)=\frac{h}{3}\Big(\pi+\frac{2h}{3}\Big)\Big(1+\frac{2h}{3}\Big).$$

Then $\varphi(0)=0$, and by the product rule

$$\varphi'(h)=\frac{1}{3}\Big(\pi+\frac{2h}{3}\Big)\Big(1+\frac{2h}{3}\Big)+\frac{h}{3}\cdot\frac{2}{3}\Big(1+\frac{2h}{3}\Big)+\frac{h}{3}\Big(\pi+\frac{2h}{3}\Big)\cdot\frac{2}{3},$$

so

$$\varphi'(0)=\frac{1}{3}\cdot\pi\cdot 1+0+0=\frac{\pi}{3}.$$

By the one-variable Chain Rule,

$$g'(0)=\cos\big(\varphi(0)\big)\,\varphi'(0)=\cos 0\cdot\frac{\pi}{3}=\frac{\pi}{3}.$$

This agrees with $\nabla f(0,\pi,1)\cdot\mathbf{u}$ computed from $\nabla f(0,\pi,1)=\langle\pi,0,0\rangle$ (Method 1):

$$\boxed{D_{\mathbf{u}}f(0,\pi,1)=\langle\pi,0,0\rangle\cdot\frac{\langle 1,2,2\rangle}{3}=\frac{\pi}{3}.}$$

Part (v), Method 1: Gradient and substitution into the unit-circle equation

Compute the partial derivatives of $f(x,y) = ye^{-xy}$. Holding y fixed,

$$f_x = y \cdot e^{-xy} \cdot (-y) = -y^2 e^{-xy}.$$

Holding x fixed and using the product rule,

$$f_y = 1 \cdot e^{-xy} + y \cdot e^{-xy} \cdot (-x) = (1 - xy)e^{-xy}.$$

Both are continuous, so f is differentiable and $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$. At $(0,2)$, $e^{-xy} = e^0 = 1$:

$$f_x(0,2) = -(2)^2 = -4, \qquad f_y(0,2) = 1 - 0 = 1, \qquad \nabla f(0,2) = \langle -4, 1 \rangle.$$

The conditions on $\mathbf{u} = \langle a, b \rangle$ are

$$-4a + b = 1, \qquad a^2 + b^2 = 1.$$

From the first, $b = 1 + 4a$. Substitute into the second:

$$a^2 + (1+4a)^2 = 1 \iff a^2 + 1 + 8a + 16a^2 = 1 \iff 17a^2 + 8a = 0 \iff a(17a + 8) = 0.$$

Case $a = 0$: $b = 1 + 0 = 1$. Check: $0^2 + 1^2 = 1$ and $-4(0) + 1 = 1$.

Case $a = -\frac{8}{17}$: $b = 1 - \frac{32}{17} = -\frac{15}{17}$. Check:

$$\left(-\frac{8}{17}\right)^2 + \left(-\frac{15}{17}\right)^2 = \frac{64 + 225}{289} = 1, \qquad -4\left(-\frac{8}{17}\right) - \frac{15}{17} = \frac{32 - 15}{17} = 1.$$

Hence

$$\boxed{\mathbf{u} = \langle 0, 1 \rangle \quad \text{or} \quad \mathbf{u} = \left\langle -\frac{8}{17}, -\frac{15}{17} \right\rangle.}$$

Part (v), Method 2: Orthogonal decomposition along the gradient

Let $\mathbf{g} = \nabla f(0,2) = \langle -4, 1 \rangle$ (computed in Method 1) and $\mathbf{n} = \langle 1, 4 \rangle$. Then

$$\mathbf{g} \cdot \mathbf{n} = -4 + 4 = 0, \qquad |\mathbf{g}|^2 = 16 + 1 = 17, \qquad |\mathbf{n}|^2 = 1 + 16 = 17,$$

so $\mathbf{g}, \mathbf{n}$ form an orthogonal basis of $\mathbb{R}^2$, and every $\mathbf{u}$ can be written $\mathbf{u} = \alpha\mathbf{g} + \beta\mathbf{n}$. Using orthogonality,

$$\mathbf{g} \cdot \mathbf{u} = \alpha |\mathbf{g}|^2 = 17\alpha, \qquad |\mathbf{u}|^2 = \alpha^2 |\mathbf{g}|^2 + \beta^2 |\mathbf{n}|^2 = 17\alpha^2 + 17\beta^2.$$

The condition $D_{\mathbf{u}} f(0,2) = \mathbf{g} \cdot \mathbf{u} = 1$ gives $\alpha = \frac{1}{17}$. The condition $|\mathbf{u}| = 1$ then gives

$$17 \cdot \frac{1}{289} + 17\beta^2 = 1 \iff 17\beta^2 = \frac{16}{17} \iff \beta^2 = \frac{16}{289} \iff \beta = \pm\frac{4}{17}.$$

Therefore

$$\mathbf{u} = \frac{1}{17}\langle -4, 1 \rangle \pm \frac{4}{17}\langle 1, 4 \rangle.$$

With $+$: $\mathbf{u} = \frac{1}{17}\langle -4 + 4,\ 1 + 16 \rangle = \langle 0, 1 \rangle$. With $-$: $\mathbf{u} = \frac{1}{17}\langle -4 - 4,\ 1 - 16 \rangle = \langle -\frac{8}{17}, -\frac{15}{17} \rangle$. Hence

$$\boxed{\mathbf{u} = \langle 0, 1 \rangle \quad \text{or} \quad \mathbf{u} = \left\langle -\frac{8}{17}, -\frac{15}{17} \right\rangle.}$$

UNOFFICIAL QRS marking scheme (17 marks)

(i) **(3 marks)** 1: point at $t = 1$; 1: tangent vector; 1: correct line.

(ii) **(3 marks)** 1: point; 1: slope by dy/dx; 1: both a and b.

(iii) **(3 marks)** 1: vertical-tangent condition; 1: parameters and nonzero y' check; 1: both points.

(iv) **(3 marks)** 1: gradient; 1: normalised direction; 1: dot product.

(v) **(5 marks)** 1: partial derivatives; 1: gradient at the point; 1: directional-derivative equation; 1: unit-vector equation; 1: both vectors.

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Question 2 (16 Marks)

QUESTION 2.

(i) [4 marks] Evaluate each of the following limits or show that the limit does not exist. Justify your answers.

(a)
$$\lim_{(x,y)\to(1,0)} \frac{\sin(xy)}{x^3y + (x-y)^4}$$

(b)
$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6 + y^2}$$

(ii) [5 marks] Find all points (P, Q, R) of the level surface given by
$$x^3 + 3x^2y + 6yz - 3z^2 = 7$$
where the tangent plane has an equation of the form $Ax + Ay = B$ for some constants A and B.

(iii) [7 marks] Define
$$f(x,y) = \begin{cases} \dfrac{y^3}{x^2+y^2}; & \text{if } x^2+y^2 \neq 0, \\ 0; & \text{if } x^2+y^2 = 0. \end{cases}$$

(a) Is f continuous at $(0,0)$? Justify your answer.

(b) Find $f_x(0,0)$ and $f_y(0,0)$.

(c) Is f differentiable at $(0,0)$? Justify your answer.

Solution

Part (i)(a), Method 1: Continuity of the quotient

Let
$$N(x,y) = \sin(xy), \qquad D(x,y) = x^3y + (x-y)^4.$$
D is a polynomial and N is sin composed with a polynomial, so both are continuous on $\mathbb{R}^2$. At $(1,0)$,
$$N(1,0) = \sin(1\cdot 0) = \sin 0 = 0, \qquad D(1,0) = 1^3\cdot 0 + (1-0)^4 = 1.$$
Since $D(1,0) = 1 \neq 0$, the quotient law for limits applies and N/D is continuous at $(1,0)$. Hence
$$\lim_{(x,y)\to(1,0)} \frac{\sin(xy)}{x^3y + (x-y)^4} = \frac{N(1,0)}{D(1,0)} = \frac{0}{1} = \boxed{0.}$$

Part (i)(a), Method 2: Squeeze with explicit bounds

Restrict to points with $|x-1| < \frac{1}{10}$ and $|y| < \frac{1}{10}$. Then $0.9 < x < 1.1$, so
$$|x^3y| < (1.1)^3(0.1) = 0.1331,$$

and $x - y > 0.9 - 0.1 = 0.8$, so

$$(x-y)^4 > (0.8)^4 = 0.4096.$$

Therefore

$$x^3y + (x-y)^4 \;\geq\; (x-y)^4 - |x^3y| \;>\; 0.4096 - 0.1331 = 0.2765 > \frac{1}{4}.$$

Using $|\sin\theta| \leq |\theta|$,

$$|\sin(xy)| \leq |x||y| < 1.1|y|.$$

Combining the two bounds,

$$0 \leq \left|\frac{\sin(xy)}{x^3y + (x-y)^4}\right| \leq \frac{1.1|y|}{1/4} = 4.4|y|.$$

As $(x,y) \to (1,0)$, $4.4|y| \to 0$, so by the Squeeze Theorem

$$\lim_{(x,y)\to(1,0)} \frac{\sin(xy)}{x^3y + (x-y)^4} = \boxed{0.}$$

Part (i)(b), Method 1: Two-path test

Let $f(x,y) = \dfrac{x^3y}{x^6+y^2}$ for $(x,y) \neq (0,0)$.

Path 1, the x-axis $y = 0$ with $x \neq 0$:

$$f(x,0) = \frac{x^3 \cdot 0}{x^6 + 0} = 0 \;\longrightarrow\; 0 \quad (x \to 0).$$

Path 2, the cubic $y = x^3$ with $x \neq 0$:

$$f(x,x^3) = \frac{x^3 \cdot x^3}{x^6 + (x^3)^2} = \frac{x^6}{2x^6} = \frac{1}{2} \;\longrightarrow\; \frac{1}{2} \quad (x \to 0).$$

Both paths approach $(0,0)$ and give different limits $0 \neq \frac{1}{2}$, so

$$\boxed{\text{the limit does not exist.}}$$

Part (i)(b), Method 2: The family of cubic curves y = m x cubed

First test all straight lines. On $x = 0$, $y \neq 0$: $f(0,y) = 0$. On $y = mx$ with $m \neq 0$, $x \neq 0$:

$$f(x,mx) = \frac{x^3 \cdot mx}{x^6 + m^2x^2} = \frac{mx^2}{x^4 + m^2} \;\longrightarrow\; \frac{0}{m^2} = 0.$$

Every line gives 0, which does not decide the question, because the powers x^6 and y^2 balance only when y is of order x^3. On $y = mx^3$ with $x \neq 0$:

$$f(x,mx^3) = \frac{x^3 \cdot mx^3}{x^6 + m^2x^6} = \frac{m}{1+m^2},$$

which is constant along each curve. For $m = 0$ the value is 0; for $m = 1$ it is $\frac{1}{2}$; for $m = -1$ it is $-\frac{1}{2}$. The limiting value depends on the curve of approach, so

$$\boxed{\text{the limit does not exist.}}$$

Unofficial solutions by QRS@NTU

Part (i)(b), Method 3: Substitution u = x cubed, then polar coordinates

The map $(x, y) \mapsto (u, y) = (x^3, y)$ is a continuous bijection of $\mathbb{R}^2$ with continuous inverse $(u, y) \mapsto (u^{1/3}, y)$, and $(x, y) \to (0, 0)$ if and only if $(u, y) \to (0, 0)$. Hence the limit of f exists if and only if

$$\lim_{(u,y)\to(0,0)} \frac{uy}{u^2 + y^2}$$

exists. Write $u = \rho \cos\theta$, $y = \rho \sin\theta$ with $\rho > 0$:

$$\frac{uy}{u^2 + y^2} = \frac{\rho^2 \cos\theta \sin\theta}{\rho^2} = \cos\theta \sin\theta = \frac{1}{2} \sin 2\theta.$$

This does not depend on ρ. On every circle $\rho = \text{const}$ it takes the value 0 at $\theta = 0$ and $\frac{1}{2}$ at $\theta = \frac{\pi}{4}$, so no single limit exists. (Back in (x, y), $\theta = 0$ is the path $y = 0$ and $\theta = \frac{\pi}{4}$ is the path $y = u = x^3$.) Hence

$$\boxed{\text{the limit does not exist.}}$$

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Part (ii), Method 1: Gradient parallel to the normal (1, 1, 0)

Let

$$F(x, y, z) = x^3 + 3x^2y + 6yz - 3z^2,$$

so the surface is $F = 7$. The partial derivatives are

$$F_x = 3x^2 + 6xy, \qquad F_y = 3x^2 + 6z, \qquad F_z = 6y - 6z.$$

F is a polynomial, hence C^1; at a point P of the surface with $\nabla F(P) \neq \mathbf{0}$ the tangent plane is $\nabla F(P) \cdot (\mathbf{X} - P) = 0$. A plane $Ax + Ay = B$ with $A \neq 0$ has normal $A\langle 1, 1, 0\rangle$, so we need $\nabla F(P) = \lambda\langle 1, 1, 0\rangle$ with $\lambda \neq 0$:

$$F_z = 0, \qquad F_x = F_y, \qquad F_x \neq 0.$$

From $F_z = 0$:

$$6y - 6z = 0 \implies z = y.$$

From $F_x = F_y$ and $z = y$:

$$3x^2 + 6xy = 3x^2 + 6z \implies 6xy = 6y \implies 6y(x - 1) = 0 \implies y = 0 \text{ or } x = 1.$$

Branch $y = 0$. Then $z = y = 0$, and the level equation becomes

$$x^3 + 3x^2(0) + 6(0)(0) - 3(0)^2 = 7 \implies x^3 = 7 \implies x = \sqrt[3]{7}.$$

At $(\sqrt[3]{7}, 0, 0)$: $F_x = 3 \cdot 7^{2/3} + 0 = 3 \cdot 7^{2/3}$, $F_y = 3 \cdot 7^{2/3} + 0 = 3 \cdot 7^{2/3}$, $F_z = 0$, so $\nabla F = 3 \cdot 7^{2/3}\langle 1, 1, 0\rangle \neq \mathbf{0}$. The tangent plane is

$$3 \cdot 7^{2/3}\big[(x - \sqrt[3]{7}) + (y - 0)\big] = 0 \iff x + y = \sqrt[3]{7}.$$

Branch $x = 1$. With $z = y$, the level equation becomes

$$1 + 3y + 6y^2 - 3y^2 = 7 \iff 3y^2 + 3y - 6 = 0 \iff y^2 + y - 2 = 0 \iff (y - 1)(y + 2) = 0,$$

so $y = 1$ or $y = -2$.

- $(1, 1, 1)$: $F_x = 3 + 6 = 9$, $F_y = 3 + 6 = 9$, $F_z = 6 - 6 = 0$, so $\nabla F = \langle 9, 9, 0\rangle \neq \mathbf{0}$. Tangent plane: $9(x - 1) + 9(y - 1) = 0 \iff x + y = 2$.
- $(1, -2, -2)$: $F_x = 3 - 12 = -9$, $F_y = 3 - 12 = -9$, $F_z = -12 + 12 = 0$, so $\nabla F = \langle -9, -9, 0\rangle \neq \mathbf{0}$. Tangent plane: $-9(x - 1) - 9(y + 2) = 0 \iff x + y = -1$.

The two branches meet only where $x = 1$, $y = z = 0$, and there $F = 1 \neq 7$, so no point is counted twice or missed. Hence

$$\boxed{(P, Q, R) = \big(\sqrt[3]{7}, 0, 0\big),\ (1, 1, 1),\ \text{or } (1, -2, -2).}$$

Part (ii), Method 2: Tangent plane containing two prescribed directions

A plane $Ax + Ay = B$ with $A \neq 0$ is exactly a plane containing the directions

$$\mathbf{e}_3 = \langle 0, 0, 1\rangle \qquad \text{and} \qquad \mathbf{v} = \langle 1, -1, 0\rangle.$$

Indeed $\langle 1, 1, 0\rangle \cdot \mathbf{e}_3 = 0$ and $\langle 1, 1, 0\rangle \cdot \mathbf{v} = 1 - 1 = 0$; conversely, a plane containing both directions has normal parallel to

$$\mathbf{e}_3 \times \mathbf{v} = \big\langle 0 \cdot 0 - 1 \cdot (-1),\ 1 \cdot 1 - 0 \cdot 0,\ 0 \cdot (-1) - 0 \cdot 1\big\rangle = \langle 1, 1, 0\rangle.$$

The tangent plane of $F = 7$ at P (with $\nabla F(P) \neq \mathbf{0}$) consists of the directions $\mathbf{w}$ with $D_{\mathbf{w}}F(P) = \nabla F(P) \cdot \mathbf{w} = 0$. So the required points are those on the surface with $\nabla F \neq \mathbf{0}$ and

$$\nabla F \cdot \mathbf{e}_3 = F_z = 6y - 6z = 0, \qquad \nabla F \cdot \mathbf{v} = F_x - F_y = (3x^2 + 6xy) - (3x^2 + 6z) = 6xy - 6z = 0.$$

The first gives $z = y$. Restricting F to the plane $z = y$,

$$F(x, y, y) = x^3 + 3x^2y + 6y^2 - 3y^2 = x^3 + 3x^2y + 3y^2,$$

and the second condition becomes $6xy - 6y = 6y(x - 1) = 0$.

- $y = 0$: $F(x, 0, 0) = x^3 = 7$, so $x = \sqrt[3]{7}$; $\nabla F(\sqrt[3]{7}, 0, 0) = \langle 3 \cdot 7^{2/3}, 3 \cdot 7^{2/3}, 0\rangle \neq \mathbf{0}$.
- $x = 1$: $F(1, y, y) = 1 + 3y + 3y^2 = 7$, so $y^2 + y - 2 = (y - 1)(y + 2) = 0$ and $y = 1$ or $y = -2$; $\nabla F(1, 1, 1) = \langle 9, 9, 0\rangle \neq \mathbf{0}$ and $\nabla F(1, -2, -2) = \langle -9, -9, 0\rangle \neq \mathbf{0}$.

The corresponding planes are $x + y = \sqrt[3]{7}$, $x + y = 2$, $x + y = -1$, all of the form $Ax + Ay = B$. Hence

$$\boxed{(P, Q, R) = \big(\sqrt[3]{7}, 0, 0\big),\ (1, 1, 1),\ \text{or } (1, -2, -2).}$$

Part (iii)(a), Method 1: Squeeze with the bound by the absolute value of y

For $(x,y) \neq (0,0)$ we have $y^2 \leq x^2+y^2$, so $\dfrac{y^2}{x^2+y^2} \leq 1$ and

$$|f(x,y) - f(0,0)| = \frac{|y|^3}{x^2+y^2} = |y| \cdot \frac{y^2}{x^2+y^2} \leq |y|.$$

As $(x,y) \to (0,0)$, $|y| \to 0$. By the Squeeze Theorem $f(x,y) \to 0 = f(0,0)$, so

$$\boxed{f \text{ is continuous at } (0,0).}$$

Part (iii)(a), Method 2: Polar coordinates

Write $x = r\cos\theta$, $y = r\sin\theta$ with $r = \sqrt{x^2+y^2} > 0$. Then

$$f(x,y) = \frac{r^3\sin^3\theta}{r^2} = r\sin^3\theta, \qquad |f(x,y) - f(0,0)| = r|\sin\theta|^3 \leq r.$$

The bound r does not depend on θ. Given $\varepsilon > 0$, take $\delta = \varepsilon$: if $0 < \sqrt{x^2+y^2} < \delta$ then $|f(x,y) - f(0,0)| \leq r < \varepsilon$. Hence $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$, and

$$\boxed{f \text{ is continuous at } (0,0).}$$

Part (iii)(b), Method 1: Limit definition of the partial derivatives

For $h \neq 0$,

$$f(h,0) = \frac{0^3}{h^2+0} = 0, \qquad f(0,h) = \frac{h^3}{0+h^2} = h.$$

Therefore

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$
$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1,$$

so

$$\boxed{f_x(0,0) = 0, \qquad f_y(0,0) = 1.}$$

Part (iii)(b), Method 2: Restriction to the coordinate axes

On the x-axis, $f(x,0) = 0$ for $x \neq 0$ and $f(0,0) = 0$, so $\varphi(x) := f(x,0) = 0$ for every x, and $f_x(0,0) = \varphi'(0) = 0$. On the y-axis, $f(0,y) = \dfrac{y^3}{y^2} = y$ for $y \neq 0$ and $f(0,0) = 0$, so $\psi(y) := f(0,y) = y$ for every y, and $f_y(0,0) = \psi'(0) = 1$. Hence

$$\boxed{f_x(0,0) = 0, \qquad f_y(0,0) = 1.}$$

Part (iii)(c), Method 1: Linearisation remainder along two paths

If f were differentiable at $(0,0)$, its linearisation would have to be

$$L(x,y) = f(0,0) + f_x(0,0)\,x + f_y(0,0)\,y = 0 + 0\cdot x + 1\cdot y = y,$$

and the remainder would have to satisfy

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = 0.$$

For $(x,y) \neq (0,0)$,

$$f(x,y) - L(x,y) = \frac{y^3}{x^2+y^2} - y = \frac{y^3 - y(x^2+y^2)}{x^2+y^2} = -\frac{x^2 y}{x^2+y^2},$$

so the quotient is

$$E(x,y) := \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = -\frac{x^2 y}{(x^2+y^2)^{3/2}}.$$

Path 1, the x-axis $y = 0$, $x \neq 0$: $E(x,0) = 0 \to 0$.

Path 2, the diagonal $x = y = t$, $t \neq 0$: $f(t,t) = \dfrac{t^3}{2t^2} = \dfrac{t}{2}$ and $L(t,t) = t$, so

$$E(t,t) = \frac{t/2 - t}{\sqrt{t^2+t^2}} = \frac{-t/2}{\sqrt{2}\,|t|} = -\frac{t}{2\sqrt{2}|t|} = \begin{cases} -\dfrac{1}{2\sqrt{2}}, & t > 0, \\ \dfrac{1}{2\sqrt{2}}, & t < 0. \end{cases}$$

Along $t \to 0^+$ the limit is $-\frac{1}{2\sqrt{2}} \neq 0$, while along the x-axis it is 0. So $\lim E$ does not exist, in particular it is not 0, and

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

Part (iii)(c), Method 2: Directional derivatives versus the gradient formula

If f is differentiable at $(0,0)$, then for every unit vector $\mathbf{u} = \langle a, b\rangle$ the directional derivative exists and

$$D_{\mathbf{u}} f(0,0) = \nabla f(0,0)\cdot \mathbf{u} = 0\cdot a + 1\cdot b = b.$$

Compute $D_{\mathbf{u}} f(0,0)$ from the definition. For $h \neq 0$, using $a^2 + b^2 = 1$,

$$f(ha, hb) = \frac{h^3 b^3}{h^2(a^2+b^2)} = hb^3, \qquad \frac{f(ha,hb) - f(0,0)}{h} = \frac{hb^3}{h} = b^3,$$

so $D_{\mathbf{u}} f(0,0) = b^3$ for every unit $\mathbf{u}$. Take $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$:

$$D_{\mathbf{u}} f(0,0) = \left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{2\sqrt{2}}, \qquad \nabla f(0,0)\cdot \mathbf{u} = \frac{1}{\sqrt{2}} = \frac{2}{2\sqrt{2}} \neq \frac{1}{2\sqrt{2}}.$$

The gradient formula fails, so the differentiability assumption is false:

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

Part (iii)(c), Method 3: Polar form of the remainder

With $L(x,y) = y$ as in Method 1, the remainder quotient is $E(x,y) = -\dfrac{x^2 y}{(x^2+y^2)^{3/2}}$. In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$,

$$E = -\frac{r^2\cos^2\theta \cdot r\sin\theta}{r^3} = -\cos^2\theta\sin\theta,$$

which does not depend on r. At $\theta = 0$ it equals 0; at $\theta = \frac{\pi}{4}$ it equals $-\frac{1}{2}\cdot\frac{1}{\sqrt{2}} = -\frac{1}{2\sqrt{2}}$. So on every circle around the origin, however small, E takes both values 0 and $-\frac{1}{2\sqrt{2}}$, and E cannot tend to 0. Hence

$$\boxed{f \text{ is not differentiable at } (0,0).}$$

UNOFFICIAL QRS marking scheme (16 marks)

(i) **(4 marks)**

(a) **(2 marks)** 1: nonzero limiting denominator; 1: limit 0.

(b) **(2 marks)** 1: two valid paths; 1: nonexistence conclusion.

(ii) **(5 marks)** 1: gradient; 1: plane-normal conditions; 1: branch $y = 0$; 1: branch $x = 1$; 1: all three points.

(iii) **(7 marks)**

(a) **(2 marks)** 1: valid bound; 1: continuity conclusion.

(b) **(2 marks)** 1: f_x; 1: f_y.

(c) **(3 marks)** 1: candidate derivative; 1: diagonal-path remainder; 1: non-differentiability conclusion.

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Question 3 (17 Marks)

QUESTION 3.

(i) [3 marks] Find the second order partial derivative w_{xy} where

$$w = y\sin(3x^2 - 5y).$$

(ii) [3 marks] Use the Chain Rule to find $\dfrac{\partial u}{\partial y}$ at $(x, y, z) = (-1, 0, 1)$ for the function

$$u(p, q, r) = \frac{p^2}{q + r},$$

where

$$p = x + 2y, \qquad q = y + 2z, \qquad r = x + y + z.$$

(iii) [3 marks] Assume that z is uniquely determined in terms of x and y by the equation

$$z^3 - xy + yz + y^3 = 2.$$

Find the value of $\dfrac{\partial z}{\partial y}$ at the point $(1, 1, 1)$.

(iv) [3 marks] In which directions is the directional derivative of

$$f(x, y) = \sin(xy) - x^2 + xy$$

at $(\pi, 0)$ equal to zero? Write the unit vector for each direction.

(v) [5 marks] Let

$$f(x, y) = x^3 - y^2 + xy - \frac{x}{2}.$$

(a) Find all critical points of $f(x, y)$.

(b) For each critical point found in Part (a), determine whether or not it is a local max/min or a saddle point.

Solution

Part (i), Method 1: Differentiate in x, then in y

Let $v = 3x^2 - 5y$, so $v_x = 6x$ and $v_y = -5$. Differentiate $w = y\sin v$ with respect to x (with y constant), using the Chain Rule:

$$w_x = y\cos v \cdot v_x = 6xy\cos(3x^2 - 5y).$$

Differentiate $w_x = 6xy\cos v$ with respect to y (with x constant), using the product rule and the Chain Rule:

$$w_{xy} = \frac{\partial}{\partial y}\big(6xy\big)\cos v + 6xy \cdot (-\sin v) \cdot v_y = 6x\cos v + 6xy(-\sin v)(-5) = 6x\cos v + 30xy\sin v.$$

Hence

$$\boxed{w_{xy} = 6x\cos(3x^2 - 5y) + 30xy\sin(3x^2 - 5y).}$$

Part (i), Method 2: Differentiate in y first and apply Clairaut's Theorem

The function $w = y\sin(3x^2 - 5y)$ is built from polynomials and sin by products and compositions, so all its second partial derivatives exist and are continuous on $\mathbb{R}^2$. By Clairaut's Theorem, $w_{xy} = w_{yx}$ everywhere. With $v = 3x^2 - 5y$, the product rule gives

$$w_y = 1 \cdot \sin v + y\cos v \cdot v_y = \sin v - 5y\cos v.$$

Differentiate with respect to x, using $v_x = 6x$:

$$w_{yx} = \cos v \cdot 6x - 5y \cdot (-\sin v) \cdot 6x = 6x\cos v + 30xy\sin v.$$

By Clairaut, this equals w_{xy}:

$$\boxed{w_{xy} = 6x\cos(3x^2 - 5y) + 30xy\sin(3x^2 - 5y).}$$

Part (ii), Method 1: Chain Rule through p, q, r

The inner functions p, q, r are linear in (x, y, z), and $u = p^2/(q + r)$ has continuous partial derivatives wherever $q + r \neq 0$. At $(x, y, z) = (-1, 0, 1)$:

$$p = -1 + 2(0) = -1, \qquad q = 0 + 2(1) = 2, \qquad r = -1 + 0 + 1 = 0, \qquad q + r = 2 \neq 0,$$

so the Chain Rule applies:

$$\frac{\partial u}{\partial y} = u_p\, p_y + u_q\, q_y + u_r\, r_y.$$

Inner derivatives:

$$p_y = 2, \qquad q_y = 1, \qquad r_y = 1.$$

Outer derivatives:

$$u_p = \frac{2p}{q + r}, \qquad u_q = -\frac{p^2}{(q + r)^2}, \qquad u_r = -\frac{p^2}{(q + r)^2}.$$

At $(p, q, r) = (-1, 2, 0)$:

$$u_p = \frac{2(-1)}{2} = -1, \qquad u_q = -\frac{(-1)^2}{2^2} = -\frac{1}{4}, \qquad u_r = -\frac{1}{4}.$$

Therefore

$$\frac{\partial u}{\partial y} = (-1)(2) + \left(-\frac{1}{4}\right)(1) + \left(-\frac{1}{4}\right)(1) = -2 - \frac{1}{4} - \frac{1}{4} = \boxed{-\frac{5}{2}}$$

Part (ii), Method 2: Direct substitution and the quotient rule

Note: the question asks for the Chain Rule, so this method should be used only to verify the value found in Method 1.

Substitute the inner functions:

$$q + r = (y + 2z) + (x + y + z) = x + 2y + 3z, \qquad u = \frac{(x + 2y)^2}{x + 2y + 3z}.$$

Let $N = (x+2y)^2$ and $D = x+2y+3z$. Then

$$N_y = 2(x+2y)\cdot 2 = 4(x+2y), \qquad D_y = 2,$$

and the quotient rule gives

$$\frac{\partial u}{\partial y} = \frac{N_y D - N D_y}{D^2} = \frac{4(x+2y)(x+2y+3z) - 2(x+2y)^2}{(x+2y+3z)^2}.$$

At $(-1,0,1)$: $x+2y = -1$ and $x+2y+3z = 2$, so

$$\frac{\partial u}{\partial y} = \frac{4(-1)(2) - 2(-1)^2}{2^2} = \frac{-8-2}{4} = \boxed{-\frac{5}{2}}$$

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Part (iii), Method 1: Implicit differentiation with respect to y

The point lies on the surface: $1^3 - 1 \cdot 1 + 1 \cdot 1 + 1^3 = 2$. Treat $z = z(x, y)$ and differentiate the equation with respect to y, holding x fixed. Term by term,

$$\frac{\partial}{\partial y}\left(z^3\right) = 3z^2 z_y, \qquad \frac{\partial}{\partial y}(-xy) = -x, \qquad \frac{\partial}{\partial y}(yz) = z + yz_y,$$
$$\frac{\partial}{\partial y}\left(y^3\right) = 3y^2, \qquad \frac{\partial}{\partial y}(2) = 0.$$

Hence

$$3z^2 z_y - x + z + yz_y + 3y^2 = 0 \iff (3z^2 + y)\, z_y = x - z - 3y^2 \iff z_y = \frac{x - z - 3y^2}{3z^2 + y}.$$

At $(1, 1, 1)$:

$$z_y = \frac{1 - 1 - 3}{3 + 1} = -\frac{3}{4},$$

that is,

$$\boxed{\frac{\partial z}{\partial y} = -\frac{3}{4}.}$$

Part (iii), Method 2: Implicit Function Theorem formula

Let $F(x, y, z) = z^3 - xy + yz + y^3 - 2$, a polynomial, hence C^1, with $F(1, 1, 1) = 1 - 1 + 1 + 1 - 2 = 0$. Its partial derivatives are

$$F_y = -x + z + 3y^2, \qquad F_z = 3z^2 + y.$$

At $(1, 1, 1)$:

$$F_y = -1 + 1 + 3 = 3, \qquad F_z = 3 + 1 = 4 \neq 0.$$

Since $F_z(1, 1, 1) \neq 0$, the Implicit Function Theorem gives z as a C^1 function of (x, y) near $(1, 1)$ with

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3}{4},$$

so

$$\boxed{\frac{\partial z}{\partial y} = -\frac{3}{4}.}$$

Part (iii), Method 3: Tangent plane solved for z

With F as in Method 2, also $F_x = -y$, so $F_x(1, 1, 1) = -1$ and $\nabla F(1, 1, 1) = \langle -1, 3, 4 \rangle$. The tangent plane to $F = 0$ at $(1, 1, 1)$ is

$$-1(x - 1) + 3(y - 1) + 4(z - 1) = 0.$$

Solve for z:

$$z = 1 + \frac{1}{4}(x - 1) - \frac{3}{4}(y - 1).$$

This plane is the graph of the linearisation of $z(x, y)$ at $(1, 1)$, whose coefficient of $(y - 1)$ is $\dfrac{\partial z}{\partial y}(1, 1)$. Hence

$$\boxed{\frac{\partial z}{\partial y} = -\frac{3}{4}.}$$

Part (iv), Method 1: Gradient and the orthogonality equation

Differentiate $f(x,y) = \sin(xy) - x^2 + xy$ term by term:

$$f_x = y\cos(xy) - 2x + y, \qquad f_y = x\cos(xy) + x.$$

At $(\pi, 0)$ we have $xy = 0$ and $\cos 0 = 1$, so

$$f_x(\pi,0) = 0 - 2\pi + 0 = -2\pi, \qquad f_y(\pi,0) = \pi + \pi = 2\pi, \qquad \nabla f(\pi,0) = 2\pi\langle -1, 1\rangle \neq \mathbf{0}.$$

Both partials are continuous, so $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$. For $\mathbf{u} = \langle a, b\rangle$,

$$D_{\mathbf{u}}f(\pi,0) = 2\pi(-a+b) = 0 \iff b = a.$$

With $a^2 + b^2 = 1$:

$$2a^2 = 1 \iff a = \pm\frac{1}{\sqrt{2}}, \qquad b = a.$$

Hence

$$\boxed{\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle \quad \text{or} \quad \mathbf{u} = \frac{1}{\sqrt{2}}\langle -1, -1\rangle.}$$

Part (iv), Method 2: Rotating the gradient through a right angle

Since $D_{\mathbf{u}}f = |\nabla f|\,|\mathbf{u}|\cos\phi$ with ϕ the angle between $\mathbf{u}$ and ∇f, and $|\nabla f(\pi,0)| = 2\sqrt{2}\,\pi \neq 0$, we have $D_{\mathbf{u}}f(\pi,0) = 0$ exactly when $\cos\phi = 0$, i.e. $\mathbf{u} \perp \nabla f(\pi,0)$. In $\mathbb{R}^2$ the vectors perpendicular to a nonzero $\langle p, q\rangle$ are the multiples of $\langle -q, p\rangle$, because

$$\langle p, q\rangle \cdot \langle -q, p\rangle = -pq + qp = 0.$$

With $\langle p, q\rangle = \langle -2\pi, 2\pi\rangle$ this gives $\langle -2\pi, -2\pi\rangle$ and its negative $\langle 2\pi, 2\pi\rangle$, each of length $2\sqrt{2}\,\pi$. Dividing by the length,

$$\boxed{\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle \quad \text{or} \quad \mathbf{u} = \frac{1}{\sqrt{2}}\langle -1, -1\rangle.}$$

Part (iv), Method 3: Tangent direction of the level curve

The level curve through $(\pi, 0)$ is $f(x,y) = f(\pi,0) = \sin 0 - \pi^2 + 0 = -\pi^2$. Since $f_y(\pi,0) = 2\pi \neq 0$ (Method 1), the Implicit Function Theorem gives this curve near $(\pi,0)$ as a graph $y = y(x)$ with

$$\frac{dy}{dx} = -\frac{f_x(\pi,0)}{f_y(\pi,0)} = -\frac{-2\pi}{2\pi} = 1.$$

A tangent vector to the level curve is $\langle 1, \frac{dy}{dx}\rangle = \langle 1, 1\rangle$. Along the level curve f is constant, so its rate of change in the two tangent directions $\pm\frac{1}{\sqrt{2}}\langle 1, 1\rangle$ is 0. Since $\nabla f(\pi,0) \neq \mathbf{0}$, the vectors with $\nabla f \cdot \mathbf{u} = 0$ form a single line through the origin, which meets the unit circle in exactly two points; so there are no other directions. Hence

$$\boxed{\mathbf{u} = \frac{1}{\sqrt{2}}\langle 1, 1\rangle \quad \text{or} \quad \mathbf{u} = \frac{1}{\sqrt{2}}\langle -1, -1\rangle.}$$

Part (v), Method 1: First-order conditions and the Second Derivative Test

(a) The function $f(x,y) = x^3 - y^2 + xy - \frac{x}{2}$ is a polynomial, so it is differentiable everywhere and its critical points are the solutions of $f_x = f_y = 0$:

$$f_x = 3x^2 + y - \frac{1}{2}, \qquad f_y = -2y + x.$$

From $f_y = 0$:

$$x = 2y \iff y = \frac{x}{2}.$$

Substitute into $f_x = 0$ and multiply by 2:

$$3x^2 + \frac{x}{2} - \frac{1}{2} = 0 \iff 6x^2 + x - 1 = 0 \iff (3x-1)(2x+1) = 0 \iff x = \frac{1}{3} \text{ or } x = -\frac{1}{2}.$$

Then $y = \frac{x}{2}$ gives $y = \frac{1}{6}$ and $y = -\frac{1}{4}$ respectively. Hence

$$\boxed{\left(\frac{1}{3}, \frac{1}{6}\right) \quad \text{and} \quad \left(-\frac{1}{2}, -\frac{1}{4}\right).}$$

(b) The second partial derivatives are

$$f_{xx} = 6x, \qquad f_{xy} = 1, \qquad f_{yy} = -2,$$

all continuous, so the Second Derivative Test applies with

$$D = f_{xx} f_{yy} - f_{xy}^2 = 6x(-2) - 1^2 = -12x - 1.$$

At $\left(\frac{1}{3}, \frac{1}{6}\right)$:

$$D = -12 \cdot \frac{1}{3} - 1 = -5 < 0 \implies \text{saddle point.}$$

At $\left(-\frac{1}{2}, -\frac{1}{4}\right)$:

$$D = -12\left(-\frac{1}{2}\right) - 1 = 5 > 0, \qquad f_{xx} = 6\left(-\frac{1}{2}\right) = -3 < 0 \implies \text{local maximum.}$$

Hence

$$\boxed{\left(\tfrac{1}{3}, \tfrac{1}{6}\right) \text{ saddle; } \left(-\tfrac{1}{2}, -\tfrac{1}{4}\right) \text{ local maximum.}}$$

Unofficial QRS@NTU Solutions

Part (v), Method 2: Completing the square in y

Complete the square in the terms containing y:

$$-y^2+xy=-(y^2-xy)=-\Big[\Big(y-\frac{x}{2}\Big)^2-\frac{x^2}{4}\Big]=-\Big(y-\frac{x}{2}\Big)^2+\frac{x^2}{4}.$$

Hence

$$f(x,y)=g(x)-\Big(y-\frac{x}{2}\Big)^2,\qquad g(x)=x^3+\frac{x^2}{4}-\frac{x}{2}.$$

(a) Differentiate the new form:

$$f_y=-2\Big(y-\frac{x}{2}\Big),\qquad f_x=g'(x)-2\Big(y-\frac{x}{2}\Big)\Big(-\frac{1}{2}\Big)=g'(x)+\Big(y-\frac{x}{2}\Big).$$

$f_y=0$ forces $y=\frac{x}{2}$, and then $f_x=g'(x)$, so the critical points are $\big(x_0,\frac{x_0}{2}\big)$ with $g'(x_0)=0$:

$$g'(x)=3x^2+\frac{x}{2}-\frac{1}{2}=0 \iff 6x^2+x-1=0$$
$$\iff (3x-1)(2x+1)=0 \iff x_0=\frac{1}{3} \text{ or } x_0=-\frac{1}{2}.$$

Hence

$$\boxed{\Big(\frac{1}{3},\frac{1}{6}\Big) \quad \text{and} \quad \Big(-\frac{1}{2},-\frac{1}{4}\Big).}$$

(b) Differentiate again: $g''(x)=6x+\frac{1}{2}$.

At $x_0=-\frac{1}{2}$: $g''\big(-\frac{1}{2}\big)=-3+\frac{1}{2}=-\frac{5}{2}<0$, so by the one-variable second derivative test g has a strict local maximum at $-\frac{1}{2}$: there is $\delta>0$ with $g(x)\le g\big(-\frac{1}{2}\big)$ for $|x+\frac{1}{2}|<\delta$. For such x and every y,

$$f(x,y)=g(x)-\Big(y-\frac{x}{2}\Big)^2\le g\Big(-\frac{1}{2}\Big)-0=f\Big(-\frac{1}{2},-\frac{1}{4}\Big)=-\frac{1}{8}+\frac{1}{16}+\frac{1}{4}=\frac{3}{16},$$

so $\big(-\frac{1}{2},-\frac{1}{4}\big)$ is a local maximum.

At $x_0=\frac{1}{3}$: $g''\big(\frac{1}{3}\big)=2+\frac{1}{2}=\frac{5}{2}>0$, so by the one-variable second derivative test g has a strict local minimum at $\frac{1}{3}$: there is $\delta'>0$ with $g(x)>g\big(\frac{1}{3}\big)$ for $0<|x-\frac{1}{3}|<\delta'$. Here $g\big(\frac{1}{3}\big)=\frac{1}{27}+\frac{1}{36}-\frac{1}{6}=-\frac{11}{108}=f\big(\frac{1}{3},\frac{1}{6}\big)$. Along the curve $y=\frac{x}{2}$,

$$f\Big(x,\frac{x}{2}\Big)=g(x)-0>g\Big(\frac{1}{3}\Big)=f\Big(\frac{1}{3},\frac{1}{6}\Big)\qquad \text{for } 0<\Big|x-\frac{1}{3}\Big|<\delta',$$

while along the line $x=\frac{1}{3}$,

$$f\Big(\frac{1}{3},y\Big)=g\Big(\frac{1}{3}\Big)-\Big(y-\frac{1}{6}\Big)^2<f\Big(\frac{1}{3},\frac{1}{6}\Big)\qquad \text{for } y\ne\frac{1}{6}.$$

Every neighbourhood of $\big(\frac{1}{3},\frac{1}{6}\big)$ contains points with larger and with smaller values of f, so it is a saddle point. Hence

$$\boxed{\big(\tfrac{1}{3},\tfrac{1}{6}\big) \text{ saddle; } \big(-\tfrac{1}{2},-\tfrac{1}{4}\big) \text{ local maximum.}}$$

UNOFFICIAL QRS marking scheme (17 marks)

(i) **(3 marks)** 1: first partial; 1: product and Chain Rules; 1: correct w_{xy}.

(ii) **(3 marks)** 1: Chain Rule; 1: intermediate values and derivatives; 1: value $-5/2$.

(iii) **(3 marks)** 1: implicit differentiation; 1: isolates z_y; 1: evaluates at the point.

(iv) **(3 marks)** 1: gradient; 1: orthogonality equation; 1: both unit vectors.

(v) **(5 marks)**

(a) **(2 marks)** 1: solves the first-derivative equations; 1: both critical points.

(b) **(3 marks)** 1: Hessian discriminant; 1: saddle classification; 1: maximum classification.

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