MH2100 Calculus III

Midterm II (Solutions)

AY 2023/2024, Semester 1

Quantitative Research Society @NTU

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Overview

- **Coverage:** Lagrange multipliers; double integrals and reversing order; polar coordinates; and volume by a height double integral.
- **Historical source:** MH2100 Midterm II, AY2023/2024 Semester 1. Every question below is transcribed from the official paper. Solutions and marking schemes are unofficial QRS@NTU material.
- **Current administration (AY2026/2027):** Tuesday 6 October 2026, 16:30–18:20 (110 minutes), venue TBA. One double-sided typed A4 sheet is permitted; attachments and sticky notes are not.
- **2026 scope note:** explicit triple integrals, cylindrical and spherical coordinates, general Jacobian transformations, and line integrals are beyond the current midterm scope. They are retained and labelled for historical coverage.

Instructions to candidates (historical, verbatim)

1. DO NOT TURN OVER PAPER UNTIL INSTRUCTED.
2. This midterm paper contains THREE (3) questions.
3. Answer ALL questions. The marks for each question are indicated at the beginning of each question.
4. Candidates should write their answers in the separate answer book provided.
5. ONE two-sided A4 piece of paper with handwritten notes is permitted.

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Unofficial QRS@NTU Solutions

Question 1 (17 Marks)

QUESTION 1.

(i) [5 marks] A wire has a shape of an ellipse described by $4x^2 + y^2 = 8$. The temperature $T(x, y)$ at every point (x, y) of the wire is described by $T(x, y) = \mathrm{e}^{2xy}$. Use the method of Lagrange multipliers to find the point(s) on the wire where the wire has the highest and the lowest temperature.

(ii) [3 marks] Evaluate the double integral

$$\iint_R 3x^2 \sin(y)\,\mathrm{d}A,$$

where R is the rectangle $R = \{(x, y) \mid 0 \le x \le 2 \text{ and } 0 \le y \le \pi\}$.

(iii) [4 marks] By reversing the order of integration, evaluate the integral

$$\int_0^1 \int_x^1 (2 + 6x)\cos(y^2 + y^3)\,\mathrm{d}y\,\mathrm{d}x.$$

(iv) *(beyond 2026 midterm scope)* [5 marks] Let E be the solid lying below the plane $z = 2 - x$ and above the triangular region in the xy-plane with vertices $(0, 0, 0)$, $(4, 0, 0)$, and $(0, 4, 0)$. Evaluate the triple integral

$$\iiint_E x\,\mathrm{d}V.$$

Solution

Method 1: Lagrange multipliers and iterated integrals

(i) Let

$$T(x, y) = \mathrm{e}^{2xy}, \qquad g(x, y) = 4x^2 + y^2,$$

so the wire is the level curve $g(x, y) = 8$.

Existence (EVT). The ellipse $4x^2 + y^2 = 8$ is closed and bounded, and T is continuous on $\mathbb{R}^2$. By the Extreme Value Theorem, T attains a maximum and a minimum on the wire.

Lagrange condition. The partial derivatives are

$$T_x = 2y\,\mathrm{e}^{2xy}, \qquad T_y = 2x\,\mathrm{e}^{2xy}, \qquad g_x = 8x, \qquad g_y = 2y.$$

Hence

$$\nabla g = \langle 8x, 2y\rangle = \mathbf{0} \iff (x, y) = (0, 0),$$

and $(0, 0)$ is not on the wire because $g(0, 0) = 0 \neq 8$. Therefore $\nabla g \neq \mathbf{0}$ at every point of the wire, and every extreme point satisfies $\nabla T = \lambda \nabla g$ for some λ:

$$2y\,\mathrm{e}^{2xy} = 8\lambda x, \tag{1}$$
$$2x\,\mathrm{e}^{2xy} = 2\lambda y, \tag{2}$$
$$4x^2 + y^2 = 8. \tag{3}$$

Case $x = 0$. Then (3) gives $y^2 = 8$, so $y = \pm 2\sqrt{2} \neq 0$. Equation (1) becomes

$$2y\,\mathrm{e}^0 = 0 \implies y = 0,$$

a contradiction. So $x \neq 0$.

Case $y = 0$. Then (3) gives $x^2 = 2$, so $x = \pm\sqrt{2} \neq 0$. Equation (2) becomes

$$2x\,\mathrm{e}^0 = 0 \implies x = 0,$$

a contradiction. So $y \neq 0$.

Case $x \neq 0$ and $y \neq 0$. If $\lambda = 0$, then (1) gives $2y\,\mathrm{e}^{2xy} = 0$, so $y = 0$, which is excluded. So $\lambda \neq 0$, and dividing (1) by (2) (both sides of (2) are nonzero) gives

$$\frac{2y\,\mathrm{e}^{2xy}}{2x\,\mathrm{e}^{2xy}} = \frac{8\lambda x}{2\lambda y} \implies \frac{y}{x} = \frac{4x}{y} \implies y^2 = 4x^2.$$

Substituting $y^2 = 4x^2$ into (3),

$$4x^2 + 4x^2 = 8 \implies x^2 = 1 \implies x = \pm 1,$$

and then $y^2 = 4$, so $y = \pm 2$. The candidate points are $(1, 2)$, $(1, -2)$, $(-1, 2)$, $(-1, -2)$. For each, (2) gives $\lambda = x\,\mathrm{e}^{2xy}/y$, a finite nonzero number, so all four are genuine solutions.

Evaluate T at every candidate.

$$\begin{aligned} T(1, 2) &= \mathrm{e}^{2(1)(2)} = \mathrm{e}^4, \\ T(-1, -2) &= \mathrm{e}^{2(-1)(-2)} = \mathrm{e}^4, \\ T(1, -2) &= \mathrm{e}^{2(1)(-2)} = \mathrm{e}^{-4}, \\ T(-1, 2) &= \mathrm{e}^{2(-1)(2)} = \mathrm{e}^{-4}. \end{aligned}$$

By the EVT the largest of these values is the maximum and the smallest is the minimum. Therefore

$$\boxed{T_{\max} = \mathrm{e}^4 \text{ at } (1, 2), (-1, -2); \qquad T_{\min} = \mathrm{e}^{-4} \text{ at } (1, -2), (-1, 2).}$$

(ii) The rectangle is $R = [0, 2] \times [0, \pi]$ and $3x^2 \sin y$ is continuous on R, so by Fubini's Theorem the double integral equals the iterated integral in either order. Integrating in y first,

$$\iint_R 3x^2 \sin y \,\mathrm{d}A = \int_0^2 \left(\int_0^\pi 3x^2 \sin y \,\mathrm{d}y \right) \mathrm{d}x.$$

For fixed x, the inner integral is

$$\int_0^\pi 3x^2 \sin y \,\mathrm{d}y = 3x^2 \left[-\cos y \right]_0^\pi = 3x^2 \big(-\cos \pi + \cos 0 \big) = 3x^2 (1 + 1) = 6x^2.$$

The outer integral is

$$\int_0^2 6x^2 \,\mathrm{d}x = \left[2x^3 \right]_0^2 = 2(8) - 0 = 16.$$

Therefore

$$\boxed{16}$$

(iii) The region of integration is

$$D = \{(x, y) \mid 0 \leq x \leq 1,\ x \leq y \leq 1\},$$

the triangle with vertices $(0,0)$, $(0,1)$, $(1,1)$. Read with y as the outer variable: y runs from 0 to 1, and for fixed y the variable x runs from the y-axis $x=0$ to the line $x=y$. Hence

$$D=\{(x,y)\mid 0\le y\le 1,\ 0\le x\le y\}.$$

The integrand $(2+6x)\cos(y^2+y^3)$ is continuous on the closed triangle D, so Fubini's Theorem allows the order to be reversed:

$$\int_0^1\int_x^1 (2+6x)\cos(y^2+y^3)\,dy\,dx=\int_0^1\int_0^y (2+6x)\cos(y^2+y^3)\,dx\,dy.$$

For fixed y, the factor $\cos(y^2+y^3)$ is a constant in the inner integral:

$$\int_0^y (2+6x)\cos(y^2+y^3)\,dx=\cos(y^2+y^3)\left[2x+3x^2\right]_0^y=(2y+3y^2)\cos(y^2+y^3).$$

So the integral is

$$\int_0^1 (2y+3y^2)\cos(y^2+y^3)\,dy.$$

Let $u=y^2+y^3$. Then

$$du=(2y+3y^2)\,dy,\qquad y=0\mapsto u=0,\qquad y=1\mapsto u=1+1=2.$$

Hence

$$\int_0^1 (2y+3y^2)\cos(y^2+y^3)\,dy=\int_0^2 \cos u\,du=[\sin u]_0^2=\sin 2-\sin 0=\sin 2.$$

Therefore

$$\boxed{\int_0^1\int_x^1 (2+6x)\cos(y^2+y^3)\,dy\,dx=\sin 2}$$

(iv) The triangle in the xy-plane with vertices $(0,0)$, $(4,0)$, $(0,4)$ is

$$\{(x,y)\mid x\ge 0,\ y\ge 0,\ x+y\le 4\}.$$

The solid lies above this triangle ($z\ge 0$) and below the plane ($z\le 2-x$). Both hold for some z only when

$$0\le 2-x\iff x\le 2.$$

So the footprint of E is the part of the triangle with $0\le x\le 2$, and

$$E=\{(x,y,z)\mid 0\le x\le 2,\ 0\le y\le 4-x,\ 0\le z\le 2-x\}.$$

The integrand x is continuous on E, so by Fubini's Theorem

$$\iiint_E x\,dV=\int_0^2\int_0^{4-x}\int_0^{2-x} x\,dz\,dy\,dx.$$

Innermost integral (in z):

$$\int_0^{2-x} x\,dz=x\,[z]_0^{2-x}=x(2-x).$$

Middle integral (in y; the integrand does not depend on y):

$$\int_0^{4-x} x(2-x)\,\mathrm{d}y = x(2-x)\,[y]_0^{4-x} = x(2-x)(4-x).$$

Expand:

$$x(2-x)(4-x) = x\left(8-6x+x^2\right) = 8x-6x^2+x^3.$$

Outer integral:

$$\int_0^2 \left(8x-6x^2+x^3\right)\,\mathrm{d}x = \left[4x^2-2x^3+\frac{x^4}{4}\right]_0^2 = 16-16+4 = 4.$$

Therefore

$$\boxed{4}$$

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Method 2: Parametrising the constraint

This method applies to part (i). Note: the question specifies the method of Lagrange multipliers, so this method serves only as a check of Method 1.

The ellipse $4x^2 + y^2 = 8$, that is $\dfrac{x^2}{2} + \dfrac{y^2}{8} = 1$, is parametrised by

$$x = \sqrt{2}\cos t, \qquad y = 2\sqrt{2}\sin t, \qquad 0 \le t \le 2\pi,$$

because

$$4x^2 + y^2 = 4(2\cos^2 t) + 8\sin^2 t = 8(\cos^2 t + \sin^2 t) = 8.$$

Along the wire,

$$2xy = 2\left(\sqrt{2}\cos t\right)\left(2\sqrt{2}\sin t\right) = 8\sin t\cos t = 4\sin 2t,$$

so

$$T(t) = \mathrm{e}^{4\sin 2t}, \qquad 0 \le t \le 2\pi.$$

$T(t)$ is differentiable on the closed interval $[0, 2\pi]$, so by the Extreme Value Theorem its extreme values exist and occur at critical points in $(0, 2\pi)$ or at the endpoints $t = 0$, $t = 2\pi$ (the same point of the wire). By the Chain Rule,

$$T'(t) = \mathrm{e}^{4\sin 2t} \cdot 4\cos 2t \cdot 2 = 8\cos 2t\, \mathrm{e}^{4\sin 2t}.$$

Since $\mathrm{e}^{4\sin 2t} > 0$,

$$T'(t) = 0 \iff \cos 2t = 0 \iff 2t \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}\right\} \iff t \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}.$$

Evaluate at the critical points and the endpoint:

$$\begin{aligned}
t = \tfrac{\pi}{4}: &\quad (x,y) = \left(\sqrt{2}\cdot\tfrac{\sqrt{2}}{2},\ 2\sqrt{2}\cdot\tfrac{\sqrt{2}}{2}\right) = (1,2), && T = \mathrm{e}^{4\sin(\pi/2)} = \mathrm{e}^{4},\\
t = \tfrac{3\pi}{4}: &\quad (x,y) = \left(\sqrt{2}\cdot\left(-\tfrac{\sqrt{2}}{2}\right),\ 2\sqrt{2}\cdot\tfrac{\sqrt{2}}{2}\right) = (-1,2), && T = \mathrm{e}^{4\sin(3\pi/2)} = \mathrm{e}^{-4},\\
t = \tfrac{5\pi}{4}: &\quad (x,y) = \left(\sqrt{2}\cdot\left(-\tfrac{\sqrt{2}}{2}\right),\ 2\sqrt{2}\cdot\left(-\tfrac{\sqrt{2}}{2}\right)\right) = (-1,-2), && T = \mathrm{e}^{4\sin(5\pi/2)} = \mathrm{e}^{4},\\
t = \tfrac{7\pi}{4}: &\quad (x,y) = \left(\sqrt{2}\cdot\tfrac{\sqrt{2}}{2},\ 2\sqrt{2}\cdot\left(-\tfrac{\sqrt{2}}{2}\right)\right) = (1,-2), && T = \mathrm{e}^{4\sin(7\pi/2)} = \mathrm{e}^{-4},\\
t = 0,\, 2\pi: &\quad (x,y) = \left(\sqrt{2}, 0\right), && T = \mathrm{e}^{0} = 1.
\end{aligned}$$

Comparing the values $\mathrm{e}^4 > 1 > \mathrm{e}^{-4}$,

$$\boxed{T_{\max} = \mathrm{e}^4 \text{ at } (1,2), (-1,-2); \qquad T_{\min} = \mathrm{e}^{-4} \text{ at } (1,-2), (-1,2).}$$

Method 3: Completing the square (AM–GM)

This method applies to part (i). Note: the question specifies the method of Lagrange multipliers, so this method serves only as a check of Method 1.

Since $u \mapsto \mathrm{e}^{2u}$ is strictly increasing, $T = \mathrm{e}^{2xy}$ is largest exactly where xy is largest and smallest exactly where xy is smallest. So it suffices to find the range of xy on $4x^2 + y^2 = 8$.

Upper bound. Completing the square,

$$4x^2 + y^2 = (2x - y)^2 + 4xy.$$

On the wire the left side is 8, so

$$xy = \frac{8 - (2x - y)^2}{4} = 2 - \frac{(2x - y)^2}{4} \leq 2,$$

with equality if and only if $y = 2x$. Substituting $y = 2x$ into the constraint,

$$4x^2 + 4x^2 = 8 \implies x^2 = 1 \implies x = \pm 1,$$

which gives the points $(1, 2)$ and $(-1, -2)$, where $xy = 2$.

Lower bound. Similarly,

$$4x^2 + y^2 = (2x + y)^2 - 4xy,$$

so on the wire

$$xy = \frac{(2x + y)^2 - 8}{4} = \frac{(2x + y)^2}{4} - 2 \geq -2,$$

with equality if and only if $y = -2x$. Substituting $y = -2x$ into the constraint,

$$4x^2 + 4x^2 = 8 \implies x = \pm 1,$$

which gives the points $(1, -2)$ and $(-1, 2)$, where $xy = -2$.

(Equivalently, AM–GM applied to $(2x)^2$ and y^2 gives $|2x|\,|y| \leq \frac{1}{2}\left((2x)^2 + y^2\right) = 4$, that is $|xy| \leq 2$, with equality iff $|y| = |2x|$.)

Hence on the wire

$$-2 \leq xy \leq 2 \implies \mathrm{e}^{-4} \leq \mathrm{e}^{2xy} \leq \mathrm{e}^{4},$$

with both bounds attained at the points found above. Therefore

$$\boxed{T_{\max} = \mathrm{e}^4 \text{ at } (1, 2), (-1, -2); \qquad T_{\min} = \mathrm{e}^{-4} \text{ at } (1, -2), (-1, 2).}$$

Method 4: Fubini in the other order

This method applies to parts (ii) and (iv).

(ii) Fubini's Theorem (continuous integrand on the rectangle R) also allows integrating in x first:

$$\iint_R 3x^2 \sin y \, \mathrm{d}A = \int_0^{\pi} \left(\int_0^2 3x^2 \sin y \, \mathrm{d}x \right) \mathrm{d}y.$$

For fixed y, the factor $\sin y$ is a constant in the inner integral:

$$\int_0^2 3x^2 \sin y \, \mathrm{d}x = \sin y \left[x^3 \right]_0^2 = 8 \sin y.$$

The outer integral is

$$\int_0^{\pi} 8 \sin y \, \mathrm{d}y = 8 \left[-\cos y \right]_0^{\pi} = 8\big(-\cos \pi + \cos 0 \big) = 8(2) = 16.$$

Therefore

$$\boxed{16}$$

(iv) Describe E with z as the outermost variable. From Method 1,

$$E = \{(x, y, z) \mid 0 \le x \le 2,\ 0 \le y \le 4 - x,\ 0 \le z \le 2 - x\}.$$

The condition $z \le 2 - x$ is the same as $x \le 2 - z$, and $0 \le x \le 2$ forces $0 \le z \le 2$. So for fixed $z \in [0, 2]$, the variable x runs over $0 \le x \le 2 - z$, and for fixed (x, z) the variable y runs over $0 \le y \le 4 - x$:

$$E = \{(x, y, z) \mid 0 \le z \le 2,\ 0 \le x \le 2 - z,\ 0 \le y \le 4 - x\}.$$

By Fubini's Theorem,

$$\iiint_E x \, \mathrm{d}V = \int_0^2 \int_0^{2-z} \int_0^{4-x} x \, \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z.$$

Innermost integral (in y):

$$\int_0^{4-x} x \, \mathrm{d}y = x(4 - x) = 4x - x^2.$$

Middle integral (in x):

$$\int_0^{2-z} \left(4x - x^2\right) \mathrm{d}x = \left[2x^2 - \frac{x^3}{3} \right]_0^{2-z} = 2(2 - z)^2 - \frac{(2 - z)^3}{3}.$$

Outer integral: let $w = 2 - z$. Then

$$\mathrm{d}w = -\,\mathrm{d}z, \qquad z = 0 \mapsto w = 2, \qquad z = 2 \mapsto w = 0,$$

so

$$\int_0^2 \left(2(2 - z)^2 - \frac{(2 - z)^3}{3} \right) \mathrm{d}z = \int_2^0 \left(2w^2 - \frac{w^3}{3} \right) (-\,\mathrm{d}w) = \int_0^2 \left(2w^2 - \frac{w^3}{3} \right) \mathrm{d}w.$$

Evaluate:

$$\int_0^2 \left(2w^2 - \frac{w^3}{3} \right) \mathrm{d}w = \left[\frac{2w^3}{3} - \frac{w^4}{12} \right]_0^2 = \frac{16}{3} - \frac{16}{12} = \frac{16}{3} - \frac{4}{3} = 4.$$

Therefore

$$\boxed{4}$$

Method 5: Original order via the Fundamental Theorem of Calculus and integration by parts

This method applies to part (iii). Note: the question specifies reversing the order of integration, so this method serves only as a check of Method 1.

The integrand $\cos(y^2+y^3)$ of the inner integral has no elementary antiderivative, so instead of evaluating the inner integral, define

$$G(x) = \int_x^1 \cos(y^2+y^3)\,\mathrm{d}y, \qquad 0 \le x \le 1.$$

Since $y \mapsto \cos(y^2+y^3)$ is continuous, the Fundamental Theorem of Calculus gives

$$G'(x) = -\cos(x^2+x^3), \qquad G(1) = \int_1^1 \cos(y^2+y^3)\,\mathrm{d}y = 0.$$

Because $(2+6x)$ does not depend on y, the given integral is

$$\int_0^1 \int_x^1 (2+6x)\cos(y^2+y^3)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 (2+6x)\,G(x)\,\mathrm{d}x.$$

Integrate by parts with

$$U = G(x), \quad \mathrm{d}V = (2+6x)\,\mathrm{d}x, \qquad \mathrm{d}U = -\cos(x^2+x^3)\,\mathrm{d}x, \quad V = 2x+3x^2.$$

Then

$$\begin{aligned}\int_0^1 (2+6x)\,G(x)\,\mathrm{d}x &= \Big[(2x+3x^2)\,G(x)\Big]_0^1 - \int_0^1 (2x+3x^2)\big(-\cos(x^2+x^3)\big)\,\mathrm{d}x \\ &= \big(5\,G(1) - 0\cdot G(0)\big) + \int_0^1 (2x+3x^2)\cos(x^2+x^3)\,\mathrm{d}x \\ &= 0 + \int_0^1 (2x+3x^2)\cos(x^2+x^3)\,\mathrm{d}x.\end{aligned}$$

Let $u = x^2+x^3$. Then

$$\mathrm{d}u = (2x+3x^2)\,\mathrm{d}x, \qquad x=0 \mapsto u=0, \qquad x=1 \mapsto u=2,$$

so

$$\int_0^1 (2x+3x^2)\cos(x^2+x^3)\,\mathrm{d}x = \int_0^2 \cos u\,\mathrm{d}u = [\sin u]_0^2 = \sin 2.$$

Therefore

$$\boxed{\int_0^1 \int_x^1 (2+6x)\cos(y^2+y^3)\,\mathrm{d}y\,\mathrm{d}x = \sin 2}$$

UNOFFICIAL QRS marking scheme (17 marks)

(i) **(5 marks)** 1: correct objective and constraint; 2: Lagrange equations and solutions; 1: classifies the signs; 1: states both extrema and all points.

(ii) **(3 marks)** 1: correct iterated integral; 1: evaluates both factors; 1: obtains 16.

(iii) **(4 marks)** 2: correct reversed bounds; 1: recognises the derivative; 1: obtains $\sin 2$.

(iv) **(5 marks)** 1: identifies the nonnegative-height footprint; 2: correct height integral; 1: evaluates it; 1: obtains 4.

Question 2 (17 Marks)

(beyond 2026 midterm scope)

QUESTION 2.

(i) [4 marks] Use cylindrical coordinates to evaluate the triple integral

$$\iiint_E 2z \, \mathrm{d}V,$$

where E is the solid above the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and below the ellipsoid $z = \sqrt{8 - 2x^2 - 2y^2}$.

(ii) [6 marks] The solid E is a spherical wedge such that

$$\iiint_E f(x, y, z) \, \mathrm{d}V = \int_0^1 \int_x^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} f(x, y, z) \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x.$$

The solid E can be described in spherical coordinates by $T_1 \leq \theta \leq T_2$, $P_1 \leq \phi \leq P_2$, and $R_1 \leq \rho \leq R_2$. Find the values of T_1, T_2, P_1, P_2, R_1, and R_2.

(iii) Consider the transformation $u = x + y$ and $v = y^2$.

(a) [2 marks] Calculate the Jacobian $\dfrac{\partial(u, v)}{\partial(x, y)}$.

(b) Using the transformation, one can evaluate the double integral

$$\iint_R 2y \cos\big((x + y)^3 - 3(x + y)\big) \, \mathrm{d}A,$$

where R is the region bounded by the curves $x = 0$, $y = 1$, and $x + y = 2$. One finds that

$$\iint_R 2y \cos\big((x + y)^3 - 3(x + y)\big) \, \mathrm{d}A = \int_a^b \int_c^d F(u, v) \, \mathrm{d}v \, \mathrm{d}u.$$

(I) [2 marks] Find the values of a, b, c, d. [Hint: they are not all constants].

(II) [1 mark] Find the integrand $F(u, v)$.

(III) [2 marks] It turns out that

$$\iint_R 2y \cos\big((x + y)^3 - 3(x + y)\big) \, \mathrm{d}A = C_1 \sin(C_2)$$

for some constants C_1 and C_2. Find the values of C_1 and C_2.

Solution

Method 1: Cylindrical and spherical coordinates, and the given transformation

(i) In cylindrical coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z, \qquad x^2 + y^2 = r^2, \qquad \mathrm{d}V = r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta.$$

The two surfaces become

$$\text{hemisphere: } z = \sqrt{4 - r^2}, \qquad \text{ellipsoid: } z = \sqrt{8 - 2r^2}.$$

They meet where

$$4 - r^2 = 8 - 2r^2 \iff r^2 = 4 \iff r = 2 \quad (\text{at } z = 0).$$

For $0 \le r < 2$,

$$(8 - 2r^2) - (4 - r^2) = 4 - r^2 > 0,$$

so the ellipsoid lies above the hemisphere there. Both surfaces are symmetric about the z-axis, so θ runs over a full turn. Hence

$$E = \left\{ (r, \theta, z) \;\middle|\; 0 \le \theta \le 2\pi,\ 0 \le r \le 2,\ \sqrt{4 - r^2} \le z \le \sqrt{8 - 2r^2} \right\}$$

and, by Fubini's Theorem,

$$\iiint_E 2z \, \mathrm{d}V = \int_0^{2\pi} \int_0^2 \int_{\sqrt{4-r^2}}^{\sqrt{8-2r^2}} 2z \, r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta.$$

Innermost integral (in z):

$$\int_{\sqrt{4-r^2}}^{\sqrt{8-2r^2}} 2z \, r \, \mathrm{d}z = r \left[z^2 \right]_{\sqrt{4-r^2}}^{\sqrt{8-2r^2}} = r\big((8 - 2r^2) - (4 - r^2)\big) = r(4 - r^2) = 4r - r^3.$$

Middle integral (in r):

$$\int_0^2 \left(4r - r^3\right) \mathrm{d}r = \left[2r^2 - \frac{r^4}{4} \right]_0^2 = 8 - 4 = 4.$$

Outer integral (in θ):

$$\int_0^{2\pi} 4 \, \mathrm{d}\theta = 8\pi.$$

Therefore

$$\boxed{8\pi}$$

(ii) The Cartesian description is

$$0 \le x \le 1, \qquad x \le y \le \sqrt{2 - x^2}, \qquad \sqrt{x^2 + y^2} \le z \le \sqrt{4 - x^2 - y^2}.$$

Projection onto the xy-plane. The outer two bounds describe

$$D = \{(x, y) \mid 0 \le x \le 1,\ x \le y \le \sqrt{2 - x^2}\}.$$

The curve $y = \sqrt{2 - x^2}$ is the upper half of the circle $x^2 + y^2 = 2$, of radius $\sqrt{2}$. The line $y = x$ meets this circle where

$$x^2 + x^2 = 2 \iff x = 1 \quad (x \ge 0),$$

so D is the sector of the disc $x^2 + y^2 \le 2$ between the ray $y = x$ (polar angle $\theta = \pi/4$) and the positive y-axis ($\theta = \pi/2$). In polar form

$$D = \left\{ (r, \theta) \;\middle|\; \tfrac{\pi}{4} \le \theta \le \tfrac{\pi}{2},\ 0 \le r \le \sqrt{2} \right\}.$$

In spherical coordinates θ is the same polar angle, so

$$T_1 = \frac{\pi}{4}, \qquad T_2 = \frac{\pi}{2}.$$

Lower surface. $z = \sqrt{x^2+y^2}$ is the upper cone $z = r$. In spherical coordinates

$$z = \rho\cos\phi, \qquad r = \sqrt{x^2+y^2} = \rho\sin\phi \quad (0 \le \phi \le \pi),$$

so the cone is $\cos\phi = \sin\phi$, that is $\phi = \pi/4$. The solid lies above the cone ($z \ge r$), that is, between the positive z-axis ($\phi = 0$) and the cone:

$$P_1 = 0, \qquad P_2 = \frac{\pi}{4}.$$

Upper surface. $z = \sqrt{4-x^2-y^2}$ is the upper hemisphere $x^2+y^2+z^2 = 4$, that is $\rho = 2$. The solid lies inside it and contains points arbitrarily close to the origin:

$$R_1 = 0, \qquad R_2 = 2.$$

Consistency of the projection. The cone $z = r$ meets the sphere $\rho = 2$ where $r^2 + r^2 = 4$, that is $r = \sqrt{2}$. So the region between the cone and the sphere projects onto $r \le \sqrt{2}$, which is exactly the radius of D. Hence the box $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{4}$, $0 \le \rho \le 2$ is exactly E:

$$\boxed{T_1 = \frac{\pi}{4}, \quad T_2 = \frac{\pi}{2}, \quad P_1 = 0, \quad P_2 = \frac{\pi}{4}, \quad R_1 = 0, \quad R_2 = 2.}$$

(iii) (a) With $u = x + y$ and $v = y^2$, the partial derivatives are

$$\frac{\partial u}{\partial x} = 1, \qquad \frac{\partial u}{\partial y} = 1, \qquad \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial y} = 2y.$$

Expanding the determinant,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2y \end{vmatrix} = 1 \cdot 2y - 1 \cdot 0 = \boxed{2y}$$

(b)(I) The lines $x = 0$, $y = 1$, $x + y = 2$ meet pairwise at

$$(0,1), \qquad (1,1), \qquad (0,2),$$

so R is the triangle

$$R = \{(x,y) \mid 1 \le y \le 2,\ 0 \le x \le 2 - y\}.$$

Change of Variables hypotheses. The map $(x,y) \mapsto (u,v) = (x+y, y^2)$ is C^1. On R we have $y \ge 1 > 0$, so $v = y^2$ determines $y = \sqrt{v}$ and then $x = u - \sqrt{v}$; hence the map is one-to-one on R. Its Jacobian $2y \ge 2$ is nonzero on R.
Image of each boundary piece.

$$\begin{aligned} y = 1 &\implies v = 1^2 = 1, \\ x = 0 &\implies u = y,\ v = y^2 \implies v = u^2, \\ x + y = 2 &\implies u = 2. \end{aligned}$$

The vertices map to

$$(0,1) \mapsto (1,1), \qquad (1,1) \mapsto (2,1), \qquad (0,2) \mapsto (2,4).$$

So the image region S is bounded below by $v = 1$, above by $v = u^2$, and on the right by $u = 2$, with u running from 1 to 2:

$$S = \{(u,v) \mid 1 \le u \le 2,\ 1 \le v \le u^2\}.$$

With v inside and u outside,

$$\boxed{a = 1, \qquad b = 2, \qquad c = 1, \qquad d = u^2.}$$

(b)(II) By the Change of Variables Theorem (hypotheses checked in (b)(I)),

$$\mathrm{d}A = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \mathrm{d}u\,\mathrm{d}v = \frac{1}{\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|}\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{|2y|}\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{2y}\,\mathrm{d}u\,\mathrm{d}v \quad (y>0).$$

Also $x+y=u$, so

$$2y\cos\big((x+y)^3-3(x+y)\big)\,\mathrm{d}A = 2y\cos(u^3-3u)\cdot\frac{1}{2y}\,\mathrm{d}u\,\mathrm{d}v = \cos(u^3-3u)\,\mathrm{d}u\,\mathrm{d}v.$$

Therefore

$$\boxed{F(u,v)=\cos(u^3-3u)}$$

(b)(III) Using (I) and (II),

$$\iint_R 2y\cos\big((x+y)^3-3(x+y)\big)\,\mathrm{d}A = \int_1^2\int_1^{u^2}\cos(u^3-3u)\,\mathrm{d}v\,\mathrm{d}u.$$

Inner integral (the integrand does not depend on v):

$$\int_1^{u^2}\cos(u^3-3u)\,\mathrm{d}v = \cos(u^3-3u)\,[v]_1^{u^2} = (u^2-1)\cos(u^3-3u).$$

Outer integral: let $w=u^3-3u$. Then

$$\mathrm{d}w = (3u^2-3)\,\mathrm{d}u = 3(u^2-1)\,\mathrm{d}u,$$

$$u=1\mapsto w=1-3=-2, \qquad u=2\mapsto w=8-6=2.$$

Hence

$$\begin{aligned}\int_1^2 (u^2-1)\cos(u^3-3u)\,\mathrm{d}u &= \frac{1}{3}\int_{-2}^{2}\cos w\,\mathrm{d}w = \frac{1}{3}\,[\sin w]_{-2}^{2}\\ &= \frac{1}{3}\big(\sin 2-\sin(-2)\big) = \frac{1}{3}(2\sin 2) = \frac{2}{3}\sin 2.\end{aligned}$$

Comparing with $C_1\sin(C_2)$,

$$\boxed{C_1=\frac{2}{3},\quad C_2=2}$$

Method 2: Fubini with z outermost (horizontal slices)

This method applies to part (i).

Fix a height $z \geq 0$ and find the horizontal cross-section of E. In cylindrical coordinates, a point at height z lies in E if and only if

$$\sqrt{4-r^2} \leq z \leq \sqrt{8-2r^2}.$$

The upper inequality (both sides nonnegative) is

$$z^2 \leq 8-2r^2 \iff r^2 \leq \frac{8-z^2}{2},$$

which needs $z \leq 2\sqrt{2}$ and already forces $r^2 \leq 4$. The lower inequality (for $r \leq 2$, both sides nonnegative) is

$$z^2 \geq 4-r^2 \iff r^2 \geq 4-z^2.$$

So the slice at height z is

$$\max\left(0,\, 4-z^2\right) \leq r^2 \leq \frac{8-z^2}{2}, \qquad 0 \leq z \leq 2\sqrt{2}.$$

For $0 \leq z \leq 2$: the slice is the annulus $\sqrt{4-z^2} \leq r \leq \sqrt{(8-z^2)/2}$.

For $2 \leq z \leq 2\sqrt{2}$: $4-z^2 \leq 0$, so the slice is the disc $0 \leq r \leq \sqrt{(8-z^2)/2}$.

By Fubini's Theorem, with z outermost,

$$\iiint_E 2z\,\mathrm{d}V = \int_0^2 \int_0^{2\pi} \int_{\sqrt{4-z^2}}^{\sqrt{(8-z^2)/2}} 2z\, r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z + \int_2^{2\sqrt{2}} \int_0^{2\pi} \int_0^{\sqrt{(8-z^2)/2}} 2z\, r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

Inner integrals (in r, then θ) for $0 \leq z \leq 2$:

$$\int_{\sqrt{4-z^2}}^{\sqrt{(8-z^2)/2}} r\,\mathrm{d}r = \frac{1}{2}\left(\frac{8-z^2}{2} - (4-z^2)\right) = \frac{1}{2}\cdot\frac{z^2}{2} = \frac{z^2}{4},$$

$$\int_0^{2\pi} 2z \cdot \frac{z^2}{4}\,\mathrm{d}\theta = \pi z^3.$$

Inner integrals for $2 \leq z \leq 2\sqrt{2}$:

$$\int_0^{\sqrt{(8-z^2)/2}} r\,\mathrm{d}r = \frac{1}{2}\cdot\frac{8-z^2}{2} = \frac{8-z^2}{4}, \qquad \int_0^{2\pi} 2z\cdot\frac{8-z^2}{4}\,\mathrm{d}\theta = \pi z(8-z^2).$$

Outer integrals:

$$\int_0^2 \pi z^3\,\mathrm{d}z = \pi\left[\frac{z^4}{4}\right]_0^2 = 4\pi,$$

$$\int_2^{2\sqrt{2}} \pi\left(8z - z^3\right)\,\mathrm{d}z = \pi\left[4z^2 - \frac{z^4}{4}\right]_2^{2\sqrt{2}} = \pi\big((32-16)-(16-4)\big) = 4\pi.$$

Adding,

$$\iiint_E 2z\,\mathrm{d}V = 4\pi + 4\pi = \boxed{8\pi}$$

Method 3: Substituting the spherical formulas into each inequality

This method applies to part (ii).

Substitute

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi, \qquad \sqrt{x^2+y^2} = \rho\sin\phi,$$

with $\rho \geq 0$, $0 \leq \phi \leq \pi$, into each Cartesian inequality, for points with $\rho > 0$ (the origin alone does not affect the description).

(1) $z \geq \sqrt{x^2+y^2}$. This reads

$$\rho\cos\phi \geq \rho\sin\phi \iff \cos\phi \geq \sin\phi.$$

Since $\sin\phi \geq 0$ on $[0,\pi]$, this forces $\cos\phi \geq 0$, so $\phi \in [0,\pi/2]$, and there $\cos\phi \geq \sin\phi \iff \tan\phi \leq 1 \iff \phi \leq \pi/4$:

$$0 \leq \phi \leq \frac{\pi}{4}.$$

(2) $z \leq \sqrt{4-x^2-y^2}$. Since $z \geq 0$ by (1), squaring is reversible:

$$z^2 \leq 4 - x^2 - y^2 \iff x^2+y^2+z^2 \leq 4 \iff \rho^2 \leq 4 \iff 0 \leq \rho \leq 2.$$

(3) $0 \leq x \leq y$. For $\phi \in (0,\pi/4]$, $\rho\sin\phi > 0$, so dividing by it,

$$0 \leq \cos\theta \leq \sin\theta \iff \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}.$$

(Points on the z-axis, $\phi = 0$, have $x = y = 0$ and satisfy the inequality for every θ.)

(4) $y \leq \sqrt{2-x^2}$ *and* $x \leq 1$. Both follow from (1)–(3):

$$x^2+y^2 = \rho^2\sin^2\phi \leq 2^2\sin^2\frac{\pi}{4} = 4\cdot\frac{1}{2} = 2 \implies y \leq \sqrt{2-x^2},$$

$$x = \rho\sin\phi\cos\theta \leq 2\cdot\frac{\sqrt{2}}{2}\cdot\cos\frac{\pi}{4} = 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = 1.$$

So these two inequalities add no new restriction. Combining (1)–(4), E is exactly $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$, $0 \leq \phi \leq \frac{\pi}{4}$, $0 \leq \rho \leq 2$.

As a cross-check with $f \equiv 1$, the volume from the spherical description is

$$\int_{\pi/4}^{\pi/2}\int_0^{\pi/4}\int_0^2 \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta = \frac{\pi}{4}\left[-\cos\phi\right]_0^{\pi/4}\left[\frac{\rho^3}{3}\right]_0^2 = \frac{\pi}{4}\left(1-\frac{\sqrt{2}}{2}\right)\frac{8}{3} = \frac{\pi(2-\sqrt{2})}{3},$$

and from the cylindrical description $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$, $0 \leq r \leq \sqrt{2}$, $r \leq z \leq \sqrt{4-r^2}$ it is

$$\begin{aligned}\frac{\pi}{4}\int_0^{\sqrt{2}}\left(\sqrt{4-r^2}-r\right)r\,\mathrm{d}r &= \frac{\pi}{4}\left[-\frac{(4-r^2)^{3/2}}{3}-\frac{r^3}{3}\right]_0^{\sqrt{2}} \\ &= \frac{\pi}{4}\left(-\frac{2\sqrt{2}}{3}-\frac{2\sqrt{2}}{3}+\frac{8}{3}\right) = \frac{\pi(2-\sqrt{2})}{3}.\end{aligned}$$

The two volumes agree. Therefore

$$\boxed{T_1 = \frac{\pi}{4}, \quad T_2 = \frac{\pi}{2}, \quad P_1 = 0, \quad P_2 = \frac{\pi}{4}, \quad R_1 = 0, \quad R_2 = 2.}$$

Method 4: Inverse transformation

This method applies to part (iii)(a), (b)(I) and (b)(II).

On R we have $y \geq 1 > 0$. Solving $u = x + y$, $v = y^2$ for (x, y) with $y > 0$,

$$y = \sqrt{v}, \qquad x = u - \sqrt{v}.$$

(a) The partial derivatives of the inverse map are

$$\frac{\partial x}{\partial u} = 1, \qquad \frac{\partial x}{\partial v} = -\frac{1}{2\sqrt{v}}, \qquad \frac{\partial y}{\partial u} = 0, \qquad \frac{\partial y}{\partial v} = \frac{1}{2\sqrt{v}}.$$

Expanding the determinant,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -\dfrac{1}{2\sqrt{v}} \\ 0 & \dfrac{1}{2\sqrt{v}} \end{vmatrix} = 1 \cdot \frac{1}{2\sqrt{v}} - \left(-\frac{1}{2\sqrt{v}}\right) \cdot 0 = \frac{1}{2\sqrt{v}} = \frac{1}{2y}.$$

The Jacobians of a map and its inverse are reciprocals, so

$$\frac{\partial(u,v)}{\partial(x,y)} = \left(\frac{1}{2y}\right)^{-1} = \boxed{2y}$$

(b)(I) Write R by inequalities and translate each one:

$$\begin{aligned} x \geq 0 &\iff u - \sqrt{v} \geq 0 \iff \sqrt{v} \leq u \iff v \leq u^2 \quad (u \geq 0), \\ y \geq 1 &\iff \sqrt{v} \geq 1 \iff v \geq 1, \\ x + y \leq 2 &\iff u \leq 2. \end{aligned}$$

Also $u = x + y \geq 0 + 1 = 1$ on R, and $1 \leq v \leq u^2$ is nonempty only for $u \geq 1$. So

$$S = \{(u, v) \mid 1 \leq u \leq 2,\ 1 \leq v \leq u^2\},$$

and

$$\boxed{a = 1, \qquad b = 2, \qquad c = 1, \qquad d = u^2.}$$

(b)(II) Substituting the inverse map and its Jacobian directly,

$$2y \cos\big((x+y)^3 - 3(x+y)\big) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 2\sqrt{v}\cos(u^3 - 3u) \cdot \frac{1}{2\sqrt{v}} = \cos(u^3 - 3u).$$

Therefore

$$\boxed{F(u, v) = \cos(u^3 - 3u)}$$

Method 5: Shear substitution $s = x + y$

This method applies to part (iii)(b)(III). Note: the question evaluates the integral through the transformation $u = x + y$, $v = y^2$, so this method serves only as a check of Method 1.

Use the linear change of variables $s = x + y$, $y = y$. Its Jacobian is

$$\frac{\partial(s,y)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 1 \cdot 0 = 1,$$

so $\mathrm{d}A = \mathrm{d}s\,\mathrm{d}y$, and the map is one-to-one with inverse $x = s - y$. The inequalities of R become

$$x \geq 0 \iff y \leq s, \qquad y \geq 1, \qquad x + y \leq 2 \iff s \leq 2,$$

so the image region is the triangle

$$\{(s,y) \mid 1 \leq y \leq s \leq 2\} = \{(s,y) \mid 1 \leq s \leq 2,\ 1 \leq y \leq s\}.$$

By the Change of Variables Theorem and Fubini's Theorem,

$$\iint_R 2y \cos\big((x+y)^3 - 3(x+y)\big)\,\mathrm{d}A = \int_1^2 \int_1^s 2y \cos(s^3 - 3s)\,\mathrm{d}y\,\mathrm{d}s.$$

Inner integral (in y):

$$\int_1^s 2y \cos(s^3 - 3s)\,\mathrm{d}y = \cos(s^3 - 3s)\,\big[y^2\big]_1^s = (s^2 - 1)\cos(s^3 - 3s).$$

Outer integral: let $w = s^3 - 3s$. Then

$$\mathrm{d}w = 3(s^2 - 1)\,\mathrm{d}s, \qquad s = 1 \mapsto w = -2, \qquad s = 2 \mapsto w = 2,$$

so

$$\int_1^2 (s^2 - 1)\cos(s^3 - 3s)\,\mathrm{d}s = \frac{1}{3}\int_{-2}^2 \cos w\,\mathrm{d}w = \frac{1}{3}\big(\sin 2 - \sin(-2)\big) = \frac{2}{3}\sin 2.$$

Comparing with $C_1 \sin(C_2)$,

$$\boxed{C_1 = \frac{2}{3}, \quad C_2 = 2}$$

UNOFFICIAL QRS marking scheme (17 marks)

(i) **(4 marks)** 1: cylindrical bounds; 1: Jacobian and integrand; 1: reduces the integral; 1: obtains 8π.

(ii) **(6 marks)** 2: azimuthal bounds; 2: polar-angle bounds; 1: radial bounds; 1: all constants correct.

(iii) **(7 marks)**

- (a) **(2 marks)** 1: forms determinant; 1: obtains $2y$.
- (b) **(5 marks)**
  - (I) **(2 marks)** 1: maps the three boundaries; 1: gives all bounds.
  - (II) **(1 mark)** correct transformed integrand.
  - (III) **(2 marks)** 1: evaluates the one-variable integral; 1: gives C_1, C_2.

Unofficial QRS marking scheme

Question 3 (16 Marks)

(beyond 2026 midterm scope)

QUESTION 3.

(i) [4 marks] Let C be the part of the curve $y = x^2$ oriented from $(1, 1)$ to $(2, 4)$. Evaluate

$$\int_C xy\,\mathrm{d}x + (x + y)\,\mathrm{d}y.$$

(ii) [5 marks] Evaluate the line integral of $f(x, y, z) = x + y + z$ over the straight-line segment from $(3, 1, 2)$ to $(2, -3, -1)$.

(iii) Consider the vector field $\mathbf{F} = \langle 4z^2, \cos z, 8xz - y\sin z + 1\rangle$.

(a) [4 marks] Show that $\mathbf{F}$ is conservative by finding a potential function f for $\mathbf{F}$.

(b) [3 marks] Evaluate

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where C is the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$ for $0 \le t \le \pi/2$.

Solution

Method 1: Parametrisation and the Fundamental Theorem for Line Integrals

(i) Parametrise C by x:

$$\mathbf{r}(t) = \langle t, t^2\rangle, \qquad 1 \le t \le 2.$$

At $t = 1$ the point is $(1, 1)$ and at $t = 2$ it is $(2, 4)$, so increasing t gives the required orientation. Then

$$x = t, \qquad y = t^2, \qquad \mathrm{d}x = \mathrm{d}t, \qquad \mathrm{d}y = 2t\,\mathrm{d}t.$$

Substituting,

$$xy\,\mathrm{d}x = t \cdot t^2\,\mathrm{d}t = t^3\,\mathrm{d}t,$$
$$(x + y)\,\mathrm{d}y = (t + t^2) \cdot 2t\,\mathrm{d}t = \left(2t^2 + 2t^3\right)\,\mathrm{d}t.$$

Hence

$$\int_C xy\,\mathrm{d}x + (x + y)\,\mathrm{d}y = \int_1^2 \left(t^3 + 2t^2 + 2t^3\right)\,\mathrm{d}t = \int_1^2 \left(3t^3 + 2t^2\right)\,\mathrm{d}t.$$

Antiderivative and evaluation:

$$\left[\frac{3t^4}{4} + \frac{2t^3}{3}\right]_1^2 = \left(12 + \frac{16}{3}\right) - \left(\frac{3}{4} + \frac{2}{3}\right) = \frac{45}{4} + \frac{14}{3} = \frac{135}{12} + \frac{56}{12} = \frac{191}{12}.$$

Therefore

$$\boxed{\frac{191}{12}}$$

(ii) The segment from $P = (3, 1, 2)$ to $Q = (2, -3, -1)$ has direction

$$Q - P = \langle 2-3,\ -3-1,\ -1-2\rangle = \langle -1, -4, -3\rangle,$$

so

$$\mathbf{r}(t) = \langle 3, 1, 2\rangle + t\langle -1, -4, -3\rangle = \langle 3-t,\ 1-4t,\ 2-3t\rangle, \qquad 0 \le t \le 1.$$

The speed is

$$\mathbf{r}'(t) = \langle -1, -4, -3\rangle, \qquad \|\mathbf{r}'(t)\| = \sqrt{1+16+9} = \sqrt{26},$$

so $\mathrm{d}s = \sqrt{26}\,\mathrm{d}t$. Along the segment,

$$f(\mathbf{r}(t)) = (3-t) + (1-4t) + (2-3t) = 6 - 8t.$$

Hence

$$\int_C f\,\mathrm{d}s = \int_0^1 (6-8t)\sqrt{26}\,\mathrm{d}t = \sqrt{26}\left[6t - 4t^2\right]_0^1 = \sqrt{26}\,(6-4) = 2\sqrt{26}.$$

Therefore

$$\boxed{2\sqrt{26}}$$

(iii) (a) Look for f with $\nabla f = \mathbf{F}$, that is

$$f_x = 4z^2, \qquad f_y = \cos z, \qquad f_z = 8xz - y\sin z + 1.$$

Integrate the first equation in x (with y, z fixed):

$$f(x, y, z) = \int 4z^2\,\mathrm{d}x = 4xz^2 + g(y, z).$$

Differentiate in y and match the second equation:

$$f_y = g_y(y, z) = \cos z \implies g(y, z) = y\cos z + h(z).$$

So $f = 4xz^2 + y\cos z + h(z)$. Differentiate in z and match the third equation:

$$f_z = 8xz - y\sin z + h'(z) = 8xz - y\sin z + 1 \implies h'(z) = 1 \implies h(z) = z + K.$$

Taking $K = 0$ gives $f = 4xz^2 + y\cos z + z$. Check every component:

$$f_x = 4z^2, \qquad f_y = \cos z, \qquad f_z = 8xz - y\sin z + 1,$$

so $\nabla f = \mathbf{F}$ on $\mathbb{R}^3$ and $\mathbf{F}$ is conservative:

$$\boxed{f(x, y, z) = 4xz^2 + y\cos z + z}$$

(b) The helix starts and ends at

$$\mathbf{r}(0) = \langle \cos 0, \sin 0, 0\rangle = \langle 1, 0, 0\rangle, \qquad \mathbf{r}\left(\tfrac{\pi}{2}\right) = \left\langle \cos\tfrac{\pi}{2}, \sin\tfrac{\pi}{2}, \tfrac{\pi}{2}\right\rangle = \left\langle 0, 1, \tfrac{\pi}{2}\right\rangle.$$

$\mathbf{F} = \nabla f$ with f continuously differentiable on $\mathbb{R}^3$ and C is a smooth curve, so the Fundamental Theorem for Line Integrals applies:

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = f\left(0, 1, \tfrac{\pi}{2}\right) - f(1, 0, 0).$$

Evaluate both values:

$$f\left(0, 1, \tfrac{\pi}{2}\right) = 4(0)\left(\tfrac{\pi}{2}\right)^2 + 1\cdot\cos\tfrac{\pi}{2} + \tfrac{\pi}{2} = 0 + 0 + \tfrac{\pi}{2} = \tfrac{\pi}{2},$$
$$f(1, 0, 0) = 4(1)(0)^2 + 0\cdot\cos 0 + 0 = 0.$$

Therefore

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \frac{\pi}{2} - 0 = \boxed{\frac{\pi}{2}}$$

Method 2: Parametrising by y

This method applies to part (i).

On C we have $x \geq 1 > 0$, so $y = x^2$ can be solved as $x = \sqrt{y}$, with y running from 1 to 4 in the given orientation:

$$x = \sqrt{y}, \qquad \mathrm{d}x = \frac{1}{2\sqrt{y}}\,\mathrm{d}y, \qquad 1 \leq y \leq 4.$$

Substituting,

$$xy\,\mathrm{d}x = \sqrt{y} \cdot y \cdot \frac{1}{2\sqrt{y}}\,\mathrm{d}y = \frac{y}{2}\,\mathrm{d}y,$$
$$(x+y)\,\mathrm{d}y = (\sqrt{y} + y)\ \mathrm{d}y.$$

Hence

$$\int_C xy\,\mathrm{d}x + (x+y)\,\mathrm{d}y = \int_1^4 \left(\frac{y}{2} + y^{1/2} + y\right)\,\mathrm{d}y = \int_1^4 \left(\frac{3y}{2} + y^{1/2}\right)\,\mathrm{d}y.$$

Antiderivative and evaluation:

$$\left[\frac{3y^2}{4} + \frac{2}{3}y^{3/2}\right]_1^4 = \left(12 + \frac{16}{3}\right) - \left(\frac{3}{4} + \frac{2}{3}\right) = \frac{45}{4} + \frac{14}{3} = \frac{191}{12}.$$

Therefore

$$\boxed{\frac{191}{12}}$$

Method 3: Arc-length parametrisation

This method applies to part (ii).

The length of the segment is

$$L = \|Q - P\| = \|\langle -1, -4, -3\rangle\| = \sqrt{26}.$$

Parametrise by arc length s, using the unit vector $\frac{1}{\sqrt{26}}\langle -1, -4, -3\rangle$:

$$\mathbf{r}(s) = \langle 3, 1, 2\rangle + \frac{s}{\sqrt{26}}\langle -1, -4, -3\rangle, \qquad 0 \leq s \leq \sqrt{26}.$$

Then

$$f(\mathbf{r}(s)) = (3 + 1 + 2) + \frac{s}{\sqrt{26}}(-1 - 4 - 3) = 6 - \frac{8s}{\sqrt{26}},$$

and, since $\|\mathbf{r}'(s)\| = 1$,

$$\int_C f\,\mathrm{d}s = \int_0^{\sqrt{26}} \left(6 - \frac{8s}{\sqrt{26}}\right)\,\mathrm{d}s = \left[6s - \frac{4s^2}{\sqrt{26}}\right]_0^{\sqrt{26}} = 6\sqrt{26} - \frac{4 \cdot 26}{\sqrt{26}} = 6\sqrt{26} - 4\sqrt{26} = 2\sqrt{26}.$$

Because the integrand is affine in s, this equals the length times the value of f at the midpoint $\left(\frac{5}{2}, -1, \frac{1}{2}\right)$: $\sqrt{26}\left(\frac{5}{2} - 1 + \frac{1}{2}\right) = 2\sqrt{26}$. Therefore

$$\boxed{2\sqrt{26}}$$

Method 4: Curl test and integration along a polygonal path

This method applies to part (iii)(a).

Write $\mathbf{F} = \langle P, Q, R\rangle$ with

$$P = 4z^2, \qquad Q = \cos z, \qquad R = 8xz - y\sin z + 1.$$

The partial derivatives needed for the curl are

$$P_y = 0, \quad P_z = 8z, \quad Q_x = 0, \quad Q_z = -\sin z, \quad R_x = 8z, \quad R_y = -\sin z.$$

Hence

$$\operatorname{curl}\mathbf{F} = \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\rangle = \langle -\sin z + \sin z,\ 8z - 8z,\ 0 - 0\rangle = \mathbf{0}.$$

$\mathbf{F}$ has continuous partial derivatives on $\mathbb{R}^3$, which is simply connected, so $\operatorname{curl}\mathbf{F} = \mathbf{0}$ implies that $\mathbf{F}$ is conservative and its line integrals are path independent. A potential is therefore

$$f(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r}$$

along any path. Take the polygonal path $(0,0,0) \to (x,0,0) \to (x,y,0) \to (x,y,z)$.

Leg 1: $\mathbf{r}(t) = \langle t, 0, 0\rangle$, $0 \le t \le x$, $\mathbf{r}' = \langle 1, 0, 0\rangle$:

$$\int_0^x P(t,0,0)\,\mathrm{d}t = \int_0^x 4(0)^2\,\mathrm{d}t = 0.$$

Leg 2: $\mathbf{r}(t) = \langle x, t, 0\rangle$, $0 \le t \le y$, $\mathbf{r}' = \langle 0, 1, 0\rangle$:

$$\int_0^y Q(x,t,0)\,\mathrm{d}t = \int_0^y \cos 0\,\mathrm{d}t = y.$$

Leg 3: $\mathbf{r}(t) = \langle x, y, t\rangle$, $0 \le t \le z$, $\mathbf{r}' = \langle 0, 0, 1\rangle$:

$$\int_0^z R(x,y,t)\,\mathrm{d}t = \int_0^z (8xt - y\sin t + 1)\ \mathrm{d}t = \left[4xt^2 + y\cos t + t\right]_0^z = 4xz^2 + y\cos z + z - y.$$

Adding the three legs,

$$f(x,y,z) = 0 + y + \left(4xz^2 + y\cos z + z - y\right) = 4xz^2 + y\cos z + z.$$

Therefore

$$\boxed{f(x,y,z) = 4xz^2 + y\cos z + z}$$

Method 5: Direct parametrisation of the helix

This method applies to part (iii)(b).

For $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le \pi/2$,

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 1 \rangle,$$

and

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle 4t^2,\ \cos t,\ 8t\cos t - \sin t \sin t + 1 \right\rangle.$$

The dot product is

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= -4t^2 \sin t + \cos^2 t + 8t\cos t - \sin^2 t + 1 \\ &= -4t^2 \sin t + 8t \cos t + \cos 2t + 1. \end{aligned}$$

Integrate each term over $[0, \pi/2]$.

Term $-4t^2 \sin t$. Integrating by parts twice,

$$\int t^2 \sin t \, \mathrm{d}t = -t^2 \cos t + \int 2t \cos t \, \mathrm{d}t = -t^2 \cos t + 2t \sin t + 2\cos t,$$

so

$$\int_0^{\pi/2} t^2 \sin t \, \mathrm{d}t = (0 + \pi + 0) - (0 + 0 + 2) = \pi - 2, \qquad \int_0^{\pi/2} -4t^2 \sin t \, \mathrm{d}t = -4\pi + 8.$$

Term $8t\cos t$. By parts,

$$\int t \cos t \, \mathrm{d}t = t \sin t + \cos t,$$

so

$$\int_0^{\pi/2} 8t \cos t \, \mathrm{d}t = 8\left(\left(\tfrac{\pi}{2} + 0\right) - (0 + 1)\right) = 4\pi - 8.$$

Term $\cos 2t$.

$$\int_0^{\pi/2} \cos 2t \, \mathrm{d}t = \left[\frac{\sin 2t}{2}\right]_0^{\pi/2} = \frac{\sin \pi}{2} - 0 = 0.$$

Term 1.

$$\int_0^{\pi/2} 1 \, \mathrm{d}t = \frac{\pi}{2}.$$

Adding,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = (-4\pi + 8) + (4\pi - 8) + 0 + \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$$

UNOFFICIAL QRS marking scheme (16 marks)

(i) **(4 marks)** 1: parametrises/orients the curve; 1: obtains $\mathrm{d}y$; 1: forms the integrand; 1: obtains 191/12.

(ii) **(5 marks)** 1: parametrises the segment; 1: finds $\mathrm{d}s$; 2: substitutes into the scalar line integral; 1: obtains $2\sqrt{26}$.

(iii) **(7 marks)**

(a) **(4 marks)** 1: integrates the first component; 1: matches the second; 1: matches the third; 1: valid potential.

(b) **(3 marks)** 1: identifies endpoints; 1: applies the theorem; 1: obtains $\pi/2$.