Nanyang Technological University
AY2024-25, Semester 1
MH2100 (Calculus III)
Midterm Exam 1

DATE: September 19, 2024 **TIME ALLOWED:** 90 minutes

FULL NAME _____

MATRICULATION NUMBER _____

TUTORIAL GROUP _____

INSTRUCTIONS TO CANDIDATES

(1) **DO NOT TURN OVER THE EXAM PAPER UNTIL INSTRUCTED TO DO SO.**

(2) This midterm exam consists of **ONLY THREE** problems, each consisting of multiple parts.

(3) Candidates are required to attempt **ALL THREE** problems.

(4) The total number of points for each problem is indicated at the top right-hand corner of its page.

(5) Candidates may write anywhere on the blank pages reserved for showing work.

(6) A **SINGLE** double-sided A4-sized help-sheet with handwritten notes is permitted.

Problem 1 **Point allocation:** 15 points

(a) A curve C in $\mathbb{R}^2$ is parametrized by the vector function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$, defined by

$$\mathbf{r}(t) \stackrel{\text{df}}{=} \big(t^2 - 1, t^3 - t\big).$$

[**4 points**] Find all the unit tangent vectors to C produced by $\mathbf{r}$ at the point $(0, 0)$ on C.

(b) For each of the following limits, either evaluate it or explain why it does not exist.

(i) [**2 points**] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}$.

(ii) [**2 points**] $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}$.

(iii) [**2 points**] $\displaystyle\lim_{(x,y)\to(1,2)} \frac{\ln(x)}{4x^2 - y^2}$.

(c) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) \stackrel{\text{df}}{=} x^2 + \sin(xy).$$

[**5 points**] Find all the unit vectors $\mathbf{u}$ in $\mathbb{R}^2$ such that $(D_{\mathbf{u}}(f))(1, 0) = 1$.

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Problem 2 **Point allocation:** 15 points

(a) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) \stackrel{\text{df}}{=} \begin{cases} \dfrac{2x^2y + y^3}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0); \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(i) [**3 points**] Determine whether f is continuous at $(0,0)$. Justify your answer.

(ii) [**2 points**] Evaluate $f_{\text{x}}(0,0)$ and $f_{\text{y}}(0,0)$, if these quantities exist.

(iii) [**4 points**] Determine whether f is totally differentiable at $(0,0)$. Justify your answer.

(b) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) \stackrel{\text{df}}{=} e^{2x^2+3y}.$$

[**3 points**] Use a linear approximation of f at $(0,0)$ to estimate the value of $f(0.01, -0.02)$.

(c) Suppose that the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ satisfy the following properties:

- f is totally differentiable throughout $\mathbb{R}^3$.
- For all $x, y, z \in \mathbb{R}$, we have $g(x,y,z) = f(x-y, y-z, z-x)$.

[**3 points**] Use the Multivariable Chain Rule to show, for all $x, y, z \in \mathbb{R}$, that

$$g_{\text{x}}(x,y,z) + g_{\text{y}}(x,y,z) + g_{\text{z}}(x,y,z) = 0.$$

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Problem 3 **Point allocation:** 20 points

(a) [**4 points**] Find a linear equation of the tangent plane to the surface S in $\mathbb{R}^3$, given by

$$xy^2z^3 + x^4y^3z^2 = 4,$$

at the point $(1, -1, 2)$.

(b) Define a function $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x, y, z) \stackrel{\text{df}}{=} x^3 + z^2 + ye^{xz} + z\cos(y).$$

Let $S \stackrel{\text{df}}{=} \big\{(x, y, z) \in \mathbb{R}^3 \;\big|\; F(x, y, z) = 0\big\}$, which is the level surface of F with value 0.

(i) [**2 points**] Show that the quantity $F_z(0, 0, 0)$ exists and is non-zero.
Remark (safe to ignore): By the Implicit Function Theorem, we can thus find

- a sufficiently small open disk D in $\mathbb{R}^2$ centered at $(0, 0)$,
- a sufficiently small bounded open interval I in $\mathbb{R}$ centered at 0, and
- a function $z : D \to I$ that is totally differentiable throughout D,

such that, within the small region $D \times I$, the surface S is actually the graph surface of z.

(ii) [**4 points**] Use the method of implicit differentiation to evaluate $\dfrac{\partial z}{\partial x}(0, 0)$ and $\dfrac{\partial z}{\partial y}(0, 0)$.

(c) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) \stackrel{\text{df}}{=} 12x^2 + y^3 - 12xy.$$

(i) [**3 points**] Find all the critical points of f.

(ii) [**4 points**] Use the Second-Derivative Test to determine the nature of each critical point of f.

(iii) [**3 points**] Determine whether f has any absolute extremum. Justify your answer.

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