Nanyang Technological University
AY2024-25, Semester 1
MH2100 (Calculus III)
Midterm Exam 2

Date: October 24, 2024 **Time allowed:** 90 minutes

Full Name _____

Matriculation Number _____

Tutorial Group _____

INSTRUCTIONS TO CANDIDATES

(1) **Do not turn over the exam paper until instructed to do so.**

(2) This midterm exam consists of **only three** problems, each consisting of two parts.

(3) Candidates are required to attempt **all three** problems.

(4) The point allocation for each problem is shown at the top right-hand corner of its page.

(5) Candidates may write anywhere on the blank pages reserved for showing work.

(6) A **single** double-sided A4-sized help-sheet with only handwritten notes is permitted.

Problem 1	
Problem 2	
Problem 3	
Total	

Problem 1 **Point allocation:** 20 points

(a) Define a function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) \stackrel{\text{df}}{=} y^2 - 10z.$$

Let S denote the sphere with center $(0, 0, 0)$ and radius 6, i.e.,

$$S = \big\{(x, y, z) \in \mathbb{R}^3 \;\big|\; x^2 + y^2 + z^2 = 36\big\}.$$

[**10 points**] Use the Method of Lagrange Multipliers to find the absolute extrema of $f|_S$. **Remark:** $f|_S$ denotes the restriction of f to S.

(b) Let D denote the closed disk in $\mathbb{R}^2$ with center $(0, 0)$ and radius 1, i.e,

$$D = \big\{(x, y) \in \mathbb{R}^2 \;\big|\; x^2 + y^2 \leq 1\big\}.$$

Define a function $f : D \to \mathbb{R}$ by

$$f(x, y) \stackrel{\text{df}}{=} 2x^2 + y^2 - y + 3.$$

(i) [**4 points**] Find all the critical points of f, as well as the values of f at these points. **Remark:** Any critical point of f must belong to the interior of D.

(ii) [**4 points**] Using the standard parametrization $\mathbf{r}$ of the unit circle, given by

$$\forall t \in [0, 2\pi] : \quad \mathbf{r}(t) = (\cos(t), \sin(t)),$$

find the absolute extrema of f on the boundary of D (which is the unit circle).

(iii) [**2 points**] Use your work for (b)(i) and (b)(ii) to identify the absolute extrema of f.

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Problem 2 **Point allocation:** 15 points

(a) [**5 points**] Evaluate $\displaystyle\int_0^1 \left[\int_{\sqrt{y}}^1 \frac{1}{\sqrt{x^3+1}} \, \mathrm{d}x \right] \mathrm{d}y$.
Remark: First sketch the region of integration.

(b) Let E denote the region in the first octant of $\mathbb{R}^3$ (i.e., $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$) that is

- bounded above by the sphere given by the equation $x^2 + y^2 + z^2 = 1$;
- bounded below by the closed cone given by the equation $z = \sqrt{\dfrac{x^2+y^2}{3}}$.

(i) [**3 points**] Describe E in spherical coordinates, using inequalities for ρ, θ, and ϕ.

(ii) [**7 points**] Evaluate $\displaystyle\iiint_E 3z \, \mathrm{d}V(x,y,z)$ using your work for (b)(i).
Caution: Work that does not use spherical coordinates **will not receive credit**.

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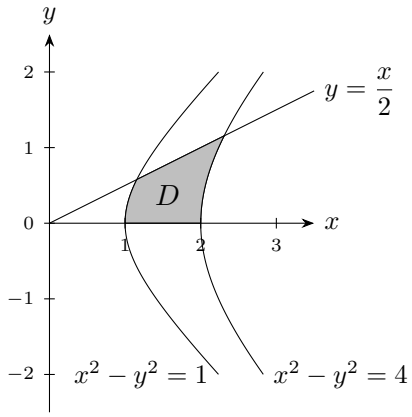
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Problem 3 **Point allocation:** 15 points

(a) Let D denote the region in the first quadrant of $\mathbb{R}^2$ bounded by the following four curves:

- The two hyperbolas given by the equations $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$.
- The two lines given by the equations $y = 0$ and $y = \dfrac{x}{2}$.



[**7 points**] Evaluate $\displaystyle\iint_D \frac{y}{x}\, \mathrm{d}A(x, y)$ using a change of variables.

(b) (i) Define a function $f : \mathbb{R}^3 \setminus \{(0, 0, 0)\} \to \mathbb{R}$ by

$$f(x, y, z) \stackrel{\text{df}}{=} \frac{1}{x^2 + y^2 + z^2}.$$

Let C denote the curve in $\mathbb{R}^3$ having the parametrization $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ defined by

$$\mathbf{r}(t) \stackrel{\text{df}}{=} (\cos(t), \sin(t), t).$$

[**4 points**] Evaluate $\displaystyle\int_C f \, \mathrm{d}s$.

(ii) Define a vector field $\mathbf{F}$ on $\mathbb{R}^3$ by

$$\mathbf{F}(x, y, z) \stackrel{\text{df}}{=} (x, 3xy, -(x + z)).$$

Let C denote the directed line segment going from $(1, 4, 2)$ to $(0, 5, 1)$.

[**4 points**] Evaluate $\displaystyle\int_C \mathbf{F} \bullet \mathrm{d}\mathbf{r}$.

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