MH2100 Calculus III

Midterm Exam 1 (Solutions)

AY 2024/2025, Semester 1

Quantitative Research Society @NTU

September 19, 2024

Overview

- **Coverage:** curves and tangent vectors; limits and differentiability; linear approximation; the Multivariable Chain Rule; tangent planes; implicit differentiation; and extrema.
- **Historical source:** MH2100 Midterm Exam 1, AY2024/2025 Semester 1. Every problem below is transcribed from the official paper. Solutions and marking schemes are unofficial QRS@NTU material.
- **Current administration (AY2026/2027):** Tuesday 6 October 2026, 16:30–18:20 (110 minutes), venue TBA. One double-sided typed A4 sheet is permitted; attachments and sticky notes are not.

Instructions to candidates (historical, verbatim)

(1) DO NOT TURN OVER THE EXAM PAPER UNTIL INSTRUCTED TO DO SO.

(2) This midterm exam consists of ONLY THREE problems, each consisting of multiple parts.

(3) Candidates are required to attempt ALL THREE problems.

(4) The total number of points for each problem is indicated at the top right-hand corner of its page.

(5) Candidates may write anywhere on the blank pages reserved for showing work.

(6) A SINGLE double-sided A4-sized help-sheet with handwritten notes is permitted.

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Question 1 (15 Points)

Problem 1

(a) A curve C in $\mathbb{R}^2$ is parametrized by the vector function $\mathbf{r} : \mathbb{R} \to \mathbb{R}^2$, defined by

$$\mathbf{r}(t) = (t^2 - 1, t^3 - t).$$

[4 points] Find all the unit tangent vectors to C produced by $\mathbf{r}$ at the point $(0, 0)$ on C.

(b) For each of the following limits, either evaluate it or explain why it does not exist.

(i) [2 points]

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 + y^2}.$$

(ii) [2 points]

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^6}.$$

(iii) [2 points]

$$\lim_{(x,y)\to(1,2)} \frac{\ln(x)}{4x^2 - y^2}.$$

(c) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = x^2 + \sin(xy).$$

[5 points] Find all the unit vectors $\mathbf{u}$ in $\mathbb{R}^2$ such that $(D_{\mathbf{u}}(f))(1, 0) = 1$.

Solution

Method 1: Velocity vectors, Squeeze Theorem, two-path test, and the gradient formula

(a) First find every parameter value that produces the point $(0, 0)$. The first coordinate gives

$$t^2 - 1 = 0 \iff t = 1 \text{ or } t = -1.$$

The second coordinate factors as $t^3 - t = t(t^2 - 1)$, so it also vanishes at $t = \pm 1$:

$$\mathbf{r}(1) = (1 - 1,\ 1 - 1) = (0, 0), \qquad \mathbf{r}(-1) = (1 - 1,\ -1 + 1) = (0, 0).$$

Hence C passes through $(0, 0)$ twice, at $t = 1$ and at $t = -1$, and each passage gives its own unit tangent vector. Differentiate componentwise:

$$\mathbf{r}'(t) = \left\langle \frac{d}{dt}(t^2 - 1),\ \frac{d}{dt}(t^3 - t) \right\rangle = \langle 2t,\ 3t^2 - 1 \rangle.$$

At $t = 1$,

$$\mathbf{r}'(1) = \langle 2,\ 3 - 1 \rangle = \langle 2, 2 \rangle, \qquad \|\mathbf{r}'(1)\| = \sqrt{2^2 + 2^2} = 2\sqrt{2} \neq 0,$$

so

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{1}{2\sqrt{2}} \langle 2, 2 \rangle = \frac{1}{\sqrt{2}} \langle 1, 1 \rangle.$$

At $t = -1$,

$$\mathbf{r}'(-1) = \langle -2,\ 3-1\rangle = \langle -2, 2\rangle, \qquad \|\mathbf{r}'(-1)\| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2} \neq 0,$$

so

$$\mathbf{T}(-1) = \frac{\mathbf{r}'(-1)}{\|\mathbf{r}'(-1)\|} = \frac{1}{2\sqrt{2}}\langle -2, 2\rangle = \frac{1}{\sqrt{2}}\langle -1, 1\rangle.$$

Therefore the unit tangent vectors at $(0,0)$ are

$$\boxed{\frac{1}{\sqrt{2}}\langle 1, 1\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}}\langle -1, 1\rangle}$$

(b) (i) Let $q(x,y) = \dfrac{x^2 y}{x^2+y^2}$ for $(x,y) \neq (0,0)$. Since $y^2 \geq 0$, we have $x^2 \leq x^2 + y^2$, so

$$0 \leq \frac{x^2}{x^2+y^2} \leq 1.$$

Hence

$$0 \leq |q(x,y)| = |y| \cdot \frac{x^2}{x^2+y^2} \leq |y|.$$

As $(x,y) \to (0,0)$, $|y| \to 0$. By the Squeeze Theorem $|q(x,y)| \to 0$, and therefore $q(x,y) \to 0$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2+y^2} = \boxed{0}$$

(ii) Let $q(x,y) = \dfrac{xy^3}{x^2+y^6}$. Two-Path Test: if q has different limits along two paths into $(0,0)$, then the limit does not exist.

Path 1, the y-axis $x = 0$, $y \neq 0$: the denominator is $y^6 \neq 0$ and

$$q(0,y) = \frac{0 \cdot y^3}{0+y^6} = 0 \ \longrightarrow\ 0 \qquad (y \to 0).$$

Path 2, the curve $x = y^3$, $y \neq 0$, which passes through the origin since $(y^3, y) \to (0,0)$ as $y \to 0$:

$$q(y^3, y) = \frac{y^3 \cdot y^3}{(y^3)^2 + y^6} = \frac{y^6}{2y^6} = \frac{1}{2} \ \longrightarrow\ \frac{1}{2} \qquad (y \to 0).$$

Since $0 \neq \frac{1}{2}$,

$$\boxed{\text{the limit does not exist}}$$

(iii) Let $q(x,y) = \dfrac{\ln x}{4x^2 - y^2}$, defined for $x > 0$ and $y \neq \pm 2x$. At $(1,2)$ the numerator is $\ln 1 = 0$ and the denominator is $4 - 4 = 0$, so substitution is impossible. Use the Two-Path Test.

Path 1, the vertical line $x = 1$, $y \neq 2$ and y close to 2 (so $y \neq -2$ as well): the denominator is $4 - y^2 \neq 0$ and

$$q(1,y) = \frac{\ln 1}{4-y^2} = \frac{0}{4-y^2} = 0 \ \longrightarrow\ 0 \qquad (y \to 2).$$

Path 2, the horizontal line $y = 2$, $x \neq 1$, $x > 0$:

$$q(x, 2) = \frac{\ln x}{4x^2 - 4}.$$

As $x \to 1$ both numerator and denominator tend to 0. Both are differentiable near $x = 1$ and the derivative of the denominator, $8x$, is non-zero near $x = 1$, so L'Hôpital's Rule gives

$$\lim_{x\to 1} \frac{\ln x}{4x^2 - 4} = \lim_{x\to 1} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx}(4x^2 - 4)} = \lim_{x\to 1} \frac{1/x}{8x} = \frac{1/1}{8 \cdot 1} = \frac{1}{8}.$$

Since $0 \neq \frac{1}{8}$,

$$\boxed{\text{the limit does not exist}}$$

(c) Let $f(x, y) = x^2 + \sin(xy)$. Compute the partial derivatives, using the Chain Rule on $\sin(xy)$:

$$f_x(x, y) = 2x + \cos(xy) \cdot y = 2x + y\cos(xy), \qquad f_y(x, y) = \cos(xy) \cdot x = x\cos(xy).$$

Both partial derivatives are continuous on $\mathbb{R}^2$ (products and compositions of polynomials and cos), so f is totally differentiable on $\mathbb{R}^2$. Hence, for every unit vector $\mathbf{u}$, the directional derivative is given by the gradient formula

$$(D_{\mathbf{u}} f)(1, 0) = \nabla f(1, 0) \cdot \mathbf{u}.$$

Evaluate the gradient at $(1, 0)$:

$$f_x(1, 0) = 2(1) + 0 \cdot \cos 0 = 2, \qquad f_y(1, 0) = 1 \cdot \cos 0 = 1, \qquad \nabla f(1, 0) = \langle 2, 1\rangle.$$

Write $\mathbf{u} = \langle a, b\rangle$. The two conditions are

$$\nabla f(1, 0) \cdot \mathbf{u} = 2a + b = 1, \qquad \|\mathbf{u}\|^2 = a^2 + b^2 = 1.$$

From the first equation $b = 1 - 2a$. Substitute into the second:

$$\begin{aligned} a^2 + (1 - 2a)^2 &= 1 \\ a^2 + 1 - 4a + 4a^2 &= 1 \\ 5a^2 - 4a &= 0 \\ a(5a - 4) &= 0. \end{aligned}$$

Case $a = 0$: $b = 1 - 0 = 1$, so $\mathbf{u} = \langle 0, 1\rangle$. Check:

$$0^2 + 1^2 = 1, \qquad 2(0) + 1 = 1.$$

Case $a = \frac{4}{5}$: $b = 1 - \frac{8}{5} = -\frac{3}{5}$, so $\mathbf{u} = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle$. Check:

$$\frac{16}{25} + \frac{9}{25} = 1, \qquad 2 \cdot \frac{4}{5} - \frac{3}{5} = \frac{8}{5} - \frac{3}{5} = 1.$$

Therefore

$$\boxed{\mathbf{u} = \langle 0, 1\rangle \quad \text{or} \quad \mathbf{u} = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle}$$

Method 2: Difference quotients, polar coordinates, families of curves, and angle parametrisation

(a) Use the definition of the velocity as a limit of difference quotients,

$$\mathbf{r}'(t_0) = \lim_{t\to t_0} \frac{\mathbf{r}(t) - \mathbf{r}(t_0)}{t - t_0}.$$

Factor both coordinates:

$$\mathbf{r}(t) = \big((t-1)(t+1),\ t(t-1)(t+1)\big).$$

At $t_0 = 1$, where $\mathbf{r}(1) = (0,0)$, for $t \neq 1$,

$$\begin{aligned}\frac{\mathbf{r}(t) - \mathbf{r}(1)}{t-1} &= \frac{\big((t-1)(t+1),\ t(t-1)(t+1)\big)}{t-1} \\ &= \big(t+1,\ t(t+1)\big) \ \longrightarrow\ (2,\ 2) \qquad (t \to 1).\end{aligned}$$

At $t_0 = -1$, where $\mathbf{r}(-1) = (0,0)$, for $t \neq -1$,

$$\begin{aligned}\frac{\mathbf{r}(t) - \mathbf{r}(-1)}{t+1} &= \frac{\big((t-1)(t+1),\ t(t-1)(t+1)\big)}{t+1} \\ &= \big(t-1,\ t(t-1)\big) \ \longrightarrow\ (-2,\ 2) \qquad (t \to -1).\end{aligned}$$

Both limits have length $\sqrt{4+4} = 2\sqrt{2}$. Dividing by it,

$$\frac{1}{2\sqrt{2}}\langle 2,2\rangle = \frac{1}{\sqrt{2}}\langle 1,1\rangle, \qquad \frac{1}{2\sqrt{2}}\langle -2,2\rangle = \frac{1}{\sqrt{2}}\langle -1,1\rangle.$$

Therefore

$$\boxed{\frac{1}{\sqrt{2}}\langle 1,1\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}}\langle -1,1\rangle}$$

(b)(i) Put $x = r\cos\theta$, $y = r\sin\theta$ with $r > 0$; then $x^2 + y^2 = r^2$, and $(x,y) \to (0,0)$ if and only if $r \to 0^+$, uniformly in θ. Then

$$\frac{x^2 y}{x^2+y^2} = \frac{r^2\cos^2\theta \cdot r\sin\theta}{r^2} = r\cos^2\theta\sin\theta.$$

Since $|\cos\theta| \le 1$ and $|\sin\theta| \le 1$,

$$\left|\frac{x^2y}{x^2+y^2} - 0\right| = r\,|\cos\theta|^2\,|\sin\theta| \le r \ \longrightarrow\ 0 \qquad (r \to 0^+).$$

The bound depends on r alone, so

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \boxed{0}$$

(b)(ii) Let $q(x,y) = \dfrac{xy^3}{x^2+y^6}$. Straight lines do not decide this limit: along $y = kx$, $x \neq 0$,

$$q(x,kx) = \frac{x\cdot k^3x^3}{x^2+k^6x^6} = \frac{k^3x^2}{1+k^6x^4} \ \longrightarrow\ 0 \qquad (x \to 0),$$

and along $x = 0$ the value is 0. Use instead the family of curves $x = my^3$ ($m \in \mathbb{R}$ fixed), each passing through the origin. For $y \neq 0$,

$$q(my^3, y) = \frac{my^3 \cdot y^3}{m^2y^6 + y^6} = \frac{m\,y^6}{(m^2+1)\,y^6} = \frac{m}{m^2+1}.$$

The value is constant along each curve but depends on m: $m = 0$ gives 0 and $m = 1$ gives $\frac{1}{2}$. Hence

$$\boxed{\text{the limit does not exist}}$$

(b)(iii) Let $q(x, y) = \dfrac{\ln x}{4x^2 - y^2}$ and use the family of lines $y = 2 + m(x-1)$ ($m \in \mathbb{R}$ fixed) through $(1, 2)$. Factor the denominator as a difference of squares, $4x^2 - y^2 = (2x - y)(2x + y)$. On the line,

$$2x - y = 2x - 2 - m(x-1) = (2-m)(x-1), \qquad 2x + y = 2x + 2 + m(x-1).$$

The line $m = 2$ is $y = 2x$, on which the denominator vanishes, so take $m \neq 2$. For $x \neq 1$ close to 1,

$$\begin{aligned} q\big(x, 2 + m(x-1)\big) &= \frac{\ln x}{(2-m)(x-1)\big(2x + 2 + m(x-1)\big)} \\ &= \frac{\ln x}{x-1} \cdot \frac{1}{(2-m)\big(2x + 2 + m(x-1)\big)}. \end{aligned}$$

Recall that $\dfrac{\ln x}{x-1} = \dfrac{\ln x - \ln 1}{x-1} \to (\ln)'(1) = \dfrac{1}{1} = 1$ as $x \to 1$, by the definition of the derivative. The second factor is continuous at $x = 1$ with value $\dfrac{1}{(2-m)\cdot 4}$. By the Product Law,

$$\lim_{x \to 1} q\big(x, 2 + m(x-1)\big) = 1 \cdot \frac{1}{4(2-m)} = \frac{1}{4(2-m)}.$$

The line $m = 0$ gives $\frac{1}{8}$ and the line $m = 1$ gives $\frac{1}{4}$. Two lines give different limits, so

$$\boxed{\text{the limit does not exist}}$$

(c) Use the definition of the directional derivative as a one-variable derivative. For a unit vector $\mathbf{u} = \langle a, b \rangle$ let

$$\varphi(t) = f\big((1,0) + t\langle a, b\rangle\big) = f(1 + ta,\ tb) = (1+ta)^2 + \sin\big((1+ta)tb\big),$$

so that $(D_{\mathbf{u}}f)(1,0) = \varphi'(0)$. Since $(1+ta)tb = tb + t^2ab$ has derivative $b + 2tab$,

$$\varphi'(t) = 2a(1+ta) + \cos\big((1+ta)tb\big)\,(b + 2tab),$$

and therefore

$$(D_{\mathbf{u}}f)(1,0) = \varphi'(0) = 2a \cdot 1 + \cos 0 \cdot b = 2a + b.$$

Parametrise the unit vectors by angle: $a = \cos\theta$, $b = \sin\theta$. The condition becomes

$$2\cos\theta + \sin\theta = 1.$$

Let ϕ be the angle with $\cos\phi = \frac{2}{\sqrt{5}}$, $\sin\phi = \frac{1}{\sqrt{5}}$. Then

$$\sqrt{5}\cos(\theta - \phi) = \sqrt{5}\left(\cos\theta \cdot \frac{2}{\sqrt{5}} + \sin\theta \cdot \frac{1}{\sqrt{5}}\right) = 2\cos\theta + \sin\theta,$$

so the condition is $\cos(\theta - \phi) = \frac{1}{\sqrt{5}}$. Let $\alpha \in (0, \frac{\pi}{2})$ with $\cos\alpha = \frac{1}{\sqrt{5}}$; then $\sin\alpha = \sqrt{1 - \frac{1}{5}} = \frac{2}{\sqrt{5}}$, and $\theta - \phi = \pm\alpha$ up to multiples of 2π.

Case $\theta = \phi + \alpha$:

$$\cos\theta = \cos\phi\cos\alpha - \sin\phi\sin\alpha = \frac{2}{5} - \frac{2}{5} = 0,$$
$$\sin\theta = \sin\phi\cos\alpha + \cos\phi\sin\alpha = \frac{1}{5} + \frac{4}{5} = 1.$$

Case $\theta = \phi - \alpha$:

$$\cos\theta = \cos\phi\cos\alpha + \sin\phi\sin\alpha = \frac{2}{5} + \frac{2}{5} = \frac{4}{5},$$
$$\sin\theta = \sin\phi\cos\alpha - \cos\phi\sin\alpha = \frac{1}{5} - \frac{4}{5} = -\frac{3}{5}.$$

Therefore

$$\boxed{\mathbf{u} = \langle 0, 1\rangle \quad \text{or} \quad \mathbf{u} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle}$$

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Method 3: Cartesian equation of the curve, the epsilon–delta definition, and orthogonal decomposition along the gradient

(a) Eliminate the parameter. From $x = t^2 - 1$ and $y = t^3 - t = t(t^2 - 1) = tx$,

$$y^2 = t^2x^2 = (x+1)x^2.$$

So C lies on the cubic $y^2 = x^2(1+x)$. Its gradient $\nabla(y^2 - x^2 - x^3) = \langle -2x - 3x^2,\ 2y\rangle$ is $\mathbf{0}$ at $(0,0)$, so it gives no normal there: the curve crosses itself. For x near 0 the cubic splits into the two branches

$$y = x\sqrt{1+x} \qquad \text{and} \qquad y = -x\sqrt{1+x},$$

and, since $t = y/x = \pm\sqrt{1+x}$ for $x \neq 0$, with $\sqrt{1+x}$ near 1, the branch with t near 1 is $y = x\sqrt{1+x}$ and the branch with t near -1 is $y = -x\sqrt{1+x}$. Their slopes at $x = 0$ are

$$\frac{d}{dx}\left(\pm x\sqrt{1+x}\right)\Big|_{x=0} = \pm\left(\sqrt{1+x} + \frac{x}{2\sqrt{1+x}}\right)\Big|_{x=0} = \pm 1.$$

So the unit tangent directions are $\pm\frac{1}{\sqrt{2}}\langle 1,1\rangle$ on the first branch and $\pm\frac{1}{\sqrt{2}}\langle 1,-1\rangle$ on the second. The orientation comes from $x(t) = (t-1)(t+1)$. Near $t = 1$ the factor $t+1$ is positive, so $x(t)$ has the sign of $t-1$: x increases from negative to positive as t passes 1, and the tangent has positive x-component, $\frac{1}{\sqrt{2}}\langle 1,1\rangle$. Near $t = -1$ the factor $t-1$ is negative, so $x(t)$ has the sign of $-(t+1)$: x decreases from positive to negative as t passes -1, and the tangent has negative x-component, $\frac{1}{\sqrt{2}}\langle -1,1\rangle$. Therefore

$$\boxed{\frac{1}{\sqrt{2}}\langle 1,1\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}}\langle -1,1\rangle}$$

(b)(i) Use the ε–δ definition. Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. If $0 < \sqrt{x^2+y^2} < \delta$, then by the bound $\frac{x^2}{x^2+y^2} \leq 1$ from Method 1,

$$\left|\frac{x^2y}{x^2+y^2} - 0\right| \leq |y| \leq \sqrt{x^2+y^2} < \delta = \varepsilon.$$

Hence

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \boxed{0}$$

(c) From Method 1, $\nabla f(1,0) = \langle 2,1\rangle$ and $(D_{\mathbf{u}}f)(1,0) = \nabla f(1,0)\cdot\mathbf{u}$. The vectors $\langle 2,1\rangle$ and $\langle -1,2\rangle$ are orthogonal, since $2(-1) + 1(2) = 0$, and each has squared length 5. Write every $\mathbf{u} \in \mathbb{R}^2$ in this orthogonal basis:

$$\mathbf{u} = c\langle 2,1\rangle + d\langle -1,2\rangle.$$

The directional-derivative condition fixes the component along the gradient:

$$\nabla f(1,0)\cdot\mathbf{u} = c\,(2^2+1^2) + d\,(2(-1)+1(2)) = 5c = 1 \iff c = \frac{1}{5}.$$

The unit condition fixes the other component; the cross term vanishes by orthogonality:

$$\|\mathbf{u}\|^2 = 5c^2 + 5d^2 = \frac{1}{5} + 5d^2 = 1 \iff d^2 = \frac{4}{25} \iff d = \pm\frac{2}{5}.$$

For $d = \frac{2}{5}$:

$$\mathbf{u} = \frac{1}{5}\langle 2,1\rangle + \frac{2}{5}\langle -1,2\rangle = \left\langle \frac{2}{5} - \frac{2}{5},\ \frac{1}{5} + \frac{4}{5} \right\rangle = \langle 0,1\rangle.$$

For $d = -\frac{2}{5}$:

$$\mathbf{u} = \frac{1}{5}\langle 2,1\rangle - \frac{2}{5}\langle -1,2\rangle = \left\langle \frac{2}{5} + \frac{2}{5},\ \frac{1}{5} - \frac{4}{5} \right\rangle = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle.$$

Therefore

$$\boxed{\mathbf{u} = \langle 0,1\rangle \quad \text{or} \quad \mathbf{u} = \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle}$$

UNOFFICIAL QRS marking scheme (15 points)

(a) **(4 points)** 1: finds both parameters; 1: differentiates; 1: evaluates both velocities; 1: gives the two oriented unit tangents.

(b) **(6 points)**

(i) **(2 points)** 1: valid bound; 1: limit 0.

(ii) **(2 points)** 1: two paths; 1: nonexistence conclusion.

(iii) **(2 points)** 1: two valid path limits; 1: nonexistence conclusion.

(c) **(5 points)** 1: gradient; 1: evaluates it; 1: directional-derivative condition; 1: unit condition; 1: both vectors.

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Question 2 (15 Points)

Problem 2

(a) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{2x^2y + y^3}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0), \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

(i) [3 points] Determine whether f is continuous at $(0,0)$. Justify your answer.

(ii) [2 points] Evaluate $f_x(0,0)$ and $f_y(0,0)$, if these quantities exist.

(iii) [4 points] Determine whether f is totally differentiable at $(0,0)$. Justify your answer.

(b) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \mathrm{e}^{2x^2+3y}.$$

[3 points] Use a linear approximation of f at $(0,0)$ to estimate the value of $f(0.01, -0.02)$.

(c) Suppose that the functions $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ satisfy the following properties:

- f is totally differentiable throughout $\mathbb{R}^3$.
- For all $x, y, z \in \mathbb{R}$, we have $g(x,y,z) = f(x-y, y-z, z-x)$.

[3 points] Use the Multivariable Chain Rule to show, for all $x, y, z \in \mathbb{R}$, that

$$g_x(x,y,z) + g_y(x,y,z) + g_z(x,y,z) = 0.$$

Solution

Method 1: Squeeze Theorem, limit definitions, definition of differentiability, linearisation, and the Chain Rule

(a) (i) For $(x,y) \neq (0,0)$ let $r = \sqrt{x^2+y^2} > 0$. Then $|x| \leq r$ and $|y| \leq r$, so by the triangle inequality

$$|2x^2y + y^3| \leq 2x^2|y| + |y|^3 \leq 2r^2 \cdot r + r^3 = 3r^3.$$

Hence

$$0 \leq |f(x,y) - f(0,0)| = \frac{|2x^2y + y^3|}{x^2+y^2} \leq \frac{3r^3}{r^2} = 3r \longrightarrow 0 \qquad ((x,y) \to (0,0)).$$

By the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$. Therefore

$$\boxed{f \text{ is continuous at } (0,0).}$$

(ii) On the axes, for $h \neq 0$,

$$f(h,0) = \frac{2h^2 \cdot 0 + 0^3}{h^2 + 0} = 0, \qquad f(0,h) = \frac{2 \cdot 0 \cdot h + h^3}{0 + h^2} = h.$$

From the definitions of the partial derivatives,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

Therefore

$$\boxed{f_x(0,0) = 0, \qquad f_y(0,0) = 1}$$

(iii) By definition, f is totally differentiable at $(0,0)$ if and only if its partial derivatives exist there and

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\sqrt{h^2+k^2}} = 0.$$

By (ii) the linear part is $L(h,k) = 0\cdot h + 1\cdot k = k$, so the quotient to examine is

$$E(h,k) = \frac{f(h,k) - k}{\sqrt{h^2+k^2}}.$$

Along the diagonal $h = k = t$ with $t > 0$,

$$f(t,t) = \frac{2t^2\cdot t + t^3}{t^2+t^2} = \frac{3t^3}{2t^2} = \frac{3t}{2}, \qquad f(t,t) - t = \frac{t}{2}, \qquad \sqrt{t^2+t^2} = \sqrt{2}\,t,$$

so

$$E(t,t) = \frac{t/2}{\sqrt{2}\,t} = \frac{1}{2\sqrt{2}} \longrightarrow \frac{1}{2\sqrt{2}} \neq 0 \qquad (t\to 0^+).$$

If $E(h,k) \to 0$ as $(h,k)\to(0,0)$, then E would tend to 0 along every path; the diagonal gives $\frac{1}{2\sqrt{2}}$. Hence $E(h,k) \not\to 0$, and

$$\boxed{f \text{ is not totally differentiable at } (0,0).}$$

(b) Let $f(x,y) = \mathrm{e}^{2x^2+3y}$. By the Chain Rule on the exponential,

$$f_x(x,y) = \mathrm{e}^{2x^2+3y}\cdot\frac{\partial}{\partial x}(2x^2+3y) = 4x\,\mathrm{e}^{2x^2+3y},$$
$$f_y(x,y) = \mathrm{e}^{2x^2+3y}\cdot\frac{\partial}{\partial y}(2x^2+3y) = 3\,\mathrm{e}^{2x^2+3y}.$$

Both are continuous on $\mathbb{R}^2$, so f is totally differentiable at $(0,0)$ and its linearisation there is a valid approximation near $(0,0)$. At $(0,0)$,

$$f(0,0) = \mathrm{e}^0 = 1, \qquad f_x(0,0) = 4\cdot 0\cdot \mathrm{e}^0 = 0, \qquad f_y(0,0) = 3\mathrm{e}^0 = 3.$$

The linearisation is

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + 0\cdot x + 3y = 1 + 3y.$$

At $(0.01, -0.02)$,

$$L(0.01,-0.02) = 1 + 3(-0.02) = 1 - 0.06 = 0.94.$$

Therefore

$$\boxed{f(0.01,-0.02) \approx L(0.01,-0.02) = 0.94}$$

(c) Let $u = x - y$, $v = y - z$, $w = z - x$, so that $g(x, y, z) = f(u, v, w)$. The inner partial derivatives are

$$u_x = 1,\ u_y = -1,\ u_z = 0; \qquad v_x = 0,\ v_y = 1,\ v_z = -1; \qquad w_x = -1,\ w_y = 0,\ w_z = 1.$$

Chain Rule hypotheses: f is totally differentiable on $\mathbb{R}^3$ (given), and u, v, w are polynomials, hence differentiable. Write f_1, f_2, f_3 for the partial derivatives of f with respect to its first, second and third arguments, each evaluated at $(x - y,\ y - z,\ z - x)$. The Multivariable Chain Rule gives

$$\begin{aligned} g_x &= f_1 u_x + f_2 v_x + f_3 w_x = f_1 \cdot 1 + f_2 \cdot 0 + f_3 \cdot (-1) = f_1 - f_3, \\ g_y &= f_1 u_y + f_2 v_y + f_3 w_y = f_1 \cdot (-1) + f_2 \cdot 1 + f_3 \cdot 0 = -f_1 + f_2, \\ g_z &= f_1 u_z + f_2 v_z + f_3 w_z = f_1 \cdot 0 + f_2 \cdot (-1) + f_3 \cdot 1 = -f_2 + f_3. \end{aligned}$$

Adding the three lines,

$$g_x + g_y + g_z = (f_1 - f_3) + (-f_1 + f_2) + (-f_2 + f_3) = 0.$$

Therefore, for all $x, y, z \in \mathbb{R}$,

$$\boxed{g_x(x, y, z) + g_y(x, y, z) + g_z(x, y, z) = 0}$$

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Method 2: Polar coordinates, axis restrictions, the directional-derivative test, one-variable linearisation, and translation invariance

(a)(i) Put $x = r\cos\theta$, $y = r\sin\theta$ with $r > 0$. The numerator is

$$2x^2y + y^3 = r^3\big(2\cos^2\theta\sin\theta + \sin^3\theta\big) = r^3\sin\theta\big(2\cos^2\theta + \sin^2\theta\big) = r^3\sin\theta\big(1 + \cos^2\theta\big),$$

so

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\sin\theta(1+\cos^2\theta)}{r^2} = r\sin\theta\big(1+\cos^2\theta\big).$$

Since $|\sin\theta| \le 1$ and $1 + \cos^2\theta \le 2$,

$$|f(x,y) - f(0,0)| \le 2r \;\longrightarrow\; 0 \qquad (r \to 0^+),$$

with a bound depending on r alone. Hence $f(x,y) \to 0 = f(0,0)$ and

$$\boxed{f \text{ is continuous at } (0,0).}$$

(a)(ii) A partial derivative at $(0,0)$ is the ordinary derivative of the restriction of f to the corresponding axis. Let $\varphi(x) = f(x,0)$ and $\psi(y) = f(0,y)$. For $x \neq 0$, $\varphi(x) = \frac{0}{x^2} = 0$, and $\varphi(0) = f(0,0) = 0$; so $\varphi(x) = 0$ for every x. For $y \neq 0$, $\psi(y) = \frac{y^3}{y^2} = y$, and $\psi(0) = 0$; so $\psi(y) = y$ for every y. Hence

$$f_x(0,0) = \varphi'(0) = 0, \qquad f_y(0,0) = \psi'(0) = 1,$$

and

$$\boxed{f_x(0,0) = 0, \qquad f_y(0,0) = 1}$$

(a)(iii) Theorem: if f is totally differentiable at $(0,0)$, then for every unit vector $\mathbf{u} = \langle a, b\rangle$ the directional derivative exists and

$$(D_{\mathbf{u}}f)(0,0) = \nabla f(0,0)\cdot\mathbf{u} = 0\cdot a + 1\cdot b = b.$$

Compute $(D_{\mathbf{u}}f)(0,0)$ directly. For $t \neq 0$, using $a^2 + b^2 = 1$, so that $2a^2 + b^2 = 1 + a^2$,

$$f(ta, tb) = \frac{2t^2a^2\cdot tb + t^3b^3}{t^2(a^2+b^2)} = tb\,(2a^2 + b^2) = tb\,(1 + a^2),$$

so

$$(D_{\mathbf{u}}f)(0,0) = \lim_{t\to 0}\frac{f(ta,tb) - f(0,0)}{t} = \lim_{t\to 0}\frac{tb(1+a^2)}{t} = b(1+a^2).$$

Take $\mathbf{u} = \big\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\big\rangle$:

$$(D_{\mathbf{u}}f)(0,0) = \frac{1}{\sqrt{2}}\left(1 + \frac{1}{2}\right) = \frac{3}{2\sqrt{2}}, \qquad \nabla f(0,0)\cdot\mathbf{u} = \frac{1}{\sqrt{2}} = \frac{2}{2\sqrt{2}}.$$

These differ, so by the contrapositive of the theorem

$$\boxed{f \text{ is not totally differentiable at } (0,0).}$$

(b) Write $f = \mathrm{e}^w$ with $w(x, y) = 2x^2 + 3y$, so $w(0, 0) = 0$. The one-variable linearisation of e^w at $w = 0$ is

$$\mathrm{e}^w \approx \mathrm{e}^0 + \mathrm{e}^0(w - 0) = 1 + w.$$

The linearisation of w at $(0, 0)$ uses $w_x = 4x$, $w_y = 3$, so $w_x(0, 0) = 0$, $w_y(0, 0) = 3$ and

$$w(x, y) \approx w(0, 0) + 0 \cdot x + 3y = 3y;$$

the term $2x^2$ is quadratic and does not enter. Composing the two linearisations (Chain Rule: the linearisation of a composition is the composition of the linearisations),

$$f(x, y) \approx 1 + 3y, \qquad f(0.01, -0.02) \approx 1 + 3(-0.02) = 0.94.$$

Therefore

$$\boxed{f(0.01, -0.02) \approx L(0.01, -0.02) = 0.94}$$

(c) Fix $(x, y, z) \in \mathbb{R}^3$ and let $\varphi(s) = g(x + s,\ y + s,\ z + s)$ for $s \in \mathbb{R}$. By the definition of g,

$$\varphi(s) = f\big((x + s) - (y + s),\ (y + s) - (z + s),\ (z + s) - (x + s)\big) = f(x - y,\ y - z,\ z - x),$$

which does not depend on s. Hence $\varphi'(s) = 0$ for all s, and in particular $\varphi'(0) = 0$.

On the other hand, g is the composition of the totally differentiable f with the linear map $(x, y, z) \mapsto (x - y, y - z, z - x)$, so g is totally differentiable on $\mathbb{R}^3$ by the Chain Rule. The path $s \mapsto (x + s, y + s, z + s)$ has velocity $\langle 1, 1, 1\rangle$, so the Chain Rule gives

$$\varphi'(0) = g_x(x, y, z) \cdot 1 + g_y(x, y, z) \cdot 1 + g_z(x, y, z) \cdot 1.$$

Comparing the two expressions for $\varphi'(0)$,

$$\boxed{g_x(x, y, z) + g_y(x, y, z) + g_z(x, y, z) = 0}$$

Unofficial Midterm Solutions

Method 3: Algebraic splitting, polar form of the remainder, and the matrix form of the Chain Rule

(a)(i) For $(x,y) \neq (0,0)$ split the numerator as $2x^2y + y^3 = y(x^2+y^2) + x^2y$. Then

$$f(x,y) = \frac{y(x^2+y^2)}{x^2+y^2} + \frac{x^2y}{x^2+y^2} = y + \frac{x^2y}{x^2+y^2}.$$

The first term tends to 0. The second term tends to 0 by Question 1(b)(i), via the bound $\left|\frac{x^2y}{x^2+y^2}\right| \leq |y|$. By the Sum Law,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 + 0 = 0 = f(0,0),$$

so

$$\boxed{f \text{ is continuous at } (0,0).}$$

(a)(iii) With $L(x,y) = y$ from (ii), the splitting above gives, for $(x,y) \neq (0,0)$,

$$f(x,y) - L(x,y) = \frac{x^2y}{x^2+y^2}.$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $r > 0$, the remainder quotient is

$$E = \frac{f(x,y) - L(x,y)}{\sqrt{x^2+y^2}} = \frac{r\cos^2\theta\sin\theta}{r} = \cos^2\theta\sin\theta,$$

which does not depend on r. On the ray $\theta = 0$ it is 0; on the ray $\theta = \frac{\pi}{4}$ it is $\frac{1}{2}\cdot\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}$. Two rays give different limits, so E has no limit at $(0,0)$, in particular not the limit 0 required by the definition. Therefore

$$\boxed{f \text{ is not totally differentiable at } (0,0).}$$

(c) Let $T(x,y,z) = (x-y,\ y-z,\ z-x)$, so $g = f \circ T$ and

$$T\begin{pmatrix}x\\y\\z\end{pmatrix} = A\begin{pmatrix}x\\y\\z\end{pmatrix}, \qquad A = \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & -1\\ -1 & 0 & 1\end{pmatrix}.$$

T is linear with derivative matrix A at every point, and f is totally differentiable, so the matrix form of the Chain Rule gives, at every $\mathbf{p} \in \mathbb{R}^3$,

$$\begin{pmatrix} g_x & g_y & g_z\end{pmatrix}(\mathbf{p}) = \begin{pmatrix} f_1 & f_2 & f_3\end{pmatrix}\big(T(\mathbf{p})\big)\, A.$$

The required sum is this row vector times $(1,1,1)^T$:

$$g_x + g_y + g_z = \begin{pmatrix} f_1 & f_2 & f_3\end{pmatrix} A\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad A\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1-1+0\\0+1-1\\-1+0+1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Hence the sum equals $\begin{pmatrix} f_1 & f_2 & f_3\end{pmatrix}\mathbf{0} = 0$, that is,

$$\boxed{g_x(x,y,z) + g_y(x,y,z) + g_z(x,y,z) = 0}$$

UNOFFICIAL QRS marking scheme (15 points)

(a) **(9 points)**

(i) **(3 points)** 2: valid bound; 1: continuity.

(ii) **(2 points)** 1: each partial derivative.

(iii) **(4 points)** 1: candidate derivative; 2: diagonal remainder; 1: non-differentiability conclusion.

(b) **(3 points)** 1: derivatives at the base point; 1: linear approximation; 1: estimate 0.94.

(c) **(3 points)** 2: three Chain Rule derivatives; 1: sums them to zero.

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Question 3 (20 Points)

Problem 3

(a) [4 points] Find a linear equation of the tangent plane to the surface S in $\mathbb{R}^3$, given by

$$xy^2z^3 + x^4y^3z^2 = 4,$$

at the point $(1, -1, 2)$.

(b) Define a function $F : \mathbb{R}^3 \to \mathbb{R}$ by

$$F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos(y).$$

Let $S = \{(x, y, z) \in \mathbb{R}^3 \mid F(x, y, z) = 0\}$, which is the level surface of F with value 0.

(i) [2 points] Show that the quantity $F_z(0, 0, 0)$ exists and is non-zero. **Remark (safe to ignore):** By the Implicit Function Theorem, we can thus find

- a sufficiently small open disk D in $\mathbb{R}^2$ centered at $(0, 0)$,
- a sufficiently small bounded open interval I in $\mathbb{R}$ centered at 0, and
- a function $z : D \to I$ that is totally differentiable throughout D,

such that, within the small region $D \times I$, the surface S is actually the graph surface of z.

(ii) [4 points] Use the method of implicit differentiation to evaluate $\dfrac{\partial z}{\partial x}(0, 0)$ and $\dfrac{\partial z}{\partial y}(0, 0)$.

(c) Define a function $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = 12x^2 + y^3 - 12xy.$$

(i) [3 points] Find all the critical points of f.

(ii) [4 points] Use the Second-Derivative Test to determine the nature of each critical point of f.

(iii) [3 points] Determine whether f has any absolute extremum. Justify your answer.

Solution

Method 1: Gradient normal, implicit differentiation, first-derivative equations, the Second-Derivative Test, and an unbounded path

(a) Let $G(x, y, z) = xy^2z^3 + x^4y^3z^2$, so $S = \{G = 4\}$ is a level surface of G. The point lies on S:

$$G(1, -1, 2) = 1 \cdot 1 \cdot 8 + 1 \cdot (-1) \cdot 4 = 8 - 4 = 4.$$

The partial derivatives are

$$\begin{aligned} G_x &= y^2z^3 + 4x^3y^3z^2, \\ G_y &= 2xyz^3 + 3x^4y^2z^2, \\ G_z &= 3xy^2z^2 + 2x^4y^3z. \end{aligned}$$

At $(1, -1, 2)$ we have $y^2 = 1$, $y^3 = -1$, $z^2 = 4$, $z^3 = 8$, so

$$G_x(1, -1, 2) = 1 \cdot 8 + 4 \cdot 1 \cdot (-1) \cdot 4 = 8 - 16 = -8,$$

$$G_y(1,-1,2) = 2\cdot 1\cdot(-1)\cdot 8 + 3\cdot 1\cdot 1\cdot 4 = -16+12 = -4,$$
$$G_z(1,-1,2) = 3\cdot 1\cdot 1\cdot 4 + 2\cdot 1\cdot(-1)\cdot 2 = 12-4 = 8.$$

G is a polynomial, hence C^1, and $\nabla G(1,-1,2) = \langle -8,-4,8\rangle \neq \mathbf{0}$, so $\nabla G(1,-1,2)$ is a normal vector to the level surface S at $(1,-1,2)$. The tangent plane is

$$\begin{aligned} -8(x-1) - 4(y+1) + 8(z-2) &= 0 \\ -8x + 8 - 4y - 4 + 8z - 16 &= 0 \\ -8x - 4y + 8z &= 12. \end{aligned}$$

Dividing by 4,

$$\boxed{-2x - y + 2z = 3}$$

(b) (i) Differentiate $F(x,y,z) = x^3 + z^2 + y\mathrm{e}^{xz} + z\cos(y)$ term by term in z, holding x, y fixed:

$$F_z(x,y,z) = 0 + 2z + y\cdot x\mathrm{e}^{xz} + \cos y = 2z + xy\mathrm{e}^{xz} + \cos y.$$

Every term is differentiable in z, so F_z exists at every point, and

$$F_z(0,0,0) = 2\cdot 0 + 0\cdot 0\cdot \mathrm{e}^0 + \cos 0 = 1 \neq 0.$$

Therefore

$$\boxed{F_z(0,0,0) \text{ exists and is non-zero.}}$$

(ii) The origin lies on S, since

$$F(0,0,0) = 0 + 0 + 0\cdot \mathrm{e}^0 + 0\cdot\cos 0 = 0,$$

and $0 \in I$, so the implicit function satisfies $z(0,0) = 0$. On D the function $z = z(x,y)$ satisfies the identity

$$x^3 + z^2 + y\mathrm{e}^{xz} + z\cos y = 0.$$

z is totally differentiable on D, so the identity may be differentiated with the Chain Rule.

Differentiate with respect to x (y fixed), using the product rule on $y\mathrm{e}^{xz}$ with $\frac{\partial}{\partial x}(xz) = z + xz_x$:

$$3x^2 + 2zz_x + y\mathrm{e}^{xz}\big(z + xz_x\big) + z_x\cos y = 0.$$

At $(x,y) = (0,0)$, where $z = 0$:

$$0 + 0 + 0 + z_x(0,0)\cdot\cos 0 = 0 \implies z_x(0,0) = 0.$$

Differentiate with respect to y (x fixed), using the product rule twice:

$$\frac{\partial}{\partial y}\big(y\mathrm{e}^{xz}\big) = \mathrm{e}^{xz} + y\mathrm{e}^{xz}\,xz_y, \qquad \frac{\partial}{\partial y}(z\cos y) = z_y\cos y - z\sin y.$$

This gives

$$2zz_y + \mathrm{e}^{xz} + xy\mathrm{e}^{xz}z_y + z_y\cos y - z\sin y = 0.$$

At $(x,y) = (0,0)$, where $z = 0$:

$$0 + \mathrm{e}^0 + 0 + z_y(0,0)\cdot\cos 0 - 0 = 0 \implies 1 + z_y(0,0) = 0 \implies z_y(0,0) = -1.$$

Therefore

$$\boxed{\frac{\partial z}{\partial x}(0,0) = 0, \qquad \frac{\partial z}{\partial y}(0,0) = -1}$$

(c) (i) Let $f(x,y) = 12x^2 + y^3 - 12xy$. Its partial derivatives are

$$f_x(x,y) = 24x - 12y, \qquad f_y(x,y) = 3y^2 - 12x.$$

f is a polynomial, so it is differentiable everywhere and its critical points are exactly the solutions of $f_x = 0$, $f_y = 0$. From the first equation,

$$24x - 12y = 0 \iff y = 2x.$$

Substitute $y = 2x$ into the second equation:

$$3(2x)^2 - 12x = 12x^2 - 12x = 12x(x-1) = 0 \iff x = 0 \text{ or } x = 1.$$

$x = 0$ gives $y = 0$, and $x = 1$ gives $y = 2$. Therefore the critical points are

$$\boxed{(0,0) \quad \text{and} \quad (1,2)}$$

(ii) The second partial derivatives are

$$f_{xx} = \frac{\partial}{\partial x}(24x - 12y) = 24, \qquad f_{xy} = \frac{\partial}{\partial y}(24x - 12y) = -12,$$
$$f_{yy} = \frac{\partial}{\partial y}(3y^2 - 12x) = 6y, \qquad f_{yx} = \frac{\partial}{\partial x}(3y^2 - 12x) = -12,$$

and $f_{yx} = f_{xy}$, as Clairaut's Theorem guarantees for the C^2 function f. The Second-Derivative Test needs continuous second partial derivatives near each critical point, which holds for a polynomial. The discriminant is

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 24 \cdot 6y - (-12)^2 = 144y - 144 = 144(y-1).$$

At $(0,0)$:

$$D(0,0) = 144(0-1) = -144 < 0 \implies (0,0) \text{ is a saddle point.}$$

At $(1,2)$:

$$D(1,2) = 144(2-1) = 144 > 0, \quad f_{xx}(1,2) = 24 > 0 \implies (1,2) \text{ is a local minimum,}$$

with value $f(1,2) = 12 + 8 - 24 = -4$. Therefore

$$\boxed{(0,0) \text{ is a saddle point; } (1,2) \text{ is a local minimum}}$$

(iii) Restrict f to the y-axis:

$$f(0,y) = 12 \cdot 0 + y^3 - 12 \cdot 0 \cdot y = y^3.$$

As $y \to +\infty$, $f(0,y) = y^3 \to +\infty$, so for every $M \in \mathbb{R}$ some point has $f > M$, and f has no absolute maximum. As $y \to -\infty$, $f(0,y) = y^3 \to -\infty$, so for every $M \in \mathbb{R}$ some point has $f < M$, and f has no absolute minimum. Therefore

$$\boxed{f \text{ has no absolute maximum and no absolute minimum}}$$

Method 2: Implicit Function Theorem formulae and completing the square

(a) G is C^1 and $G_z(1,-1,2) = 8 \neq 0$ (computed in Method 1), so by the Implicit Function Theorem S is, near $(1,-1,2)$, the graph of a differentiable function $z = z(x,y)$ with $z(1,-1) = 2$ and

$$z_x(1,-1) = -\frac{G_x}{G_z} = -\frac{-8}{8} = 1, \qquad z_y(1,-1) = -\frac{G_y}{G_z} = -\frac{-4}{8} = \frac{1}{2}.$$

The tangent plane to the graph at $(1,-1,2)$ is

$$\begin{aligned} z - 2 = z_x(1,-1)(x-1) + z_y(1,-1)(y+1) &= (x-1) + \frac{1}{2}(y+1) \\ 2z - 4 &= 2x - 2 + y + 1 \\ -2x - y + 2z &= 3. \end{aligned}$$

Therefore

$$\boxed{-2x - y + 2z = 3}$$

(b)(i) Use the limit definition. On the z-axis,

$$F(0,0,h) = 0 + h^2 + 0 \cdot \mathrm{e}^0 + h\cos 0 = h^2 + h, \qquad F(0,0,0) = 0,$$

so

$$F_z(0,0,0) = \lim_{h\to 0} \frac{F(0,0,h) - F(0,0,0)}{h} = \lim_{h\to 0} \frac{h^2+h}{h} = \lim_{h\to 0}(h+1) = 1 \neq 0.$$

Therefore

$$\boxed{F_z(0,0,0) \text{ exists and is non-zero.}}$$

(b)(ii) Differentiating $F(x,y,z(x,y)) = 0$ by the Chain Rule gives, in general,

$$F_x + F_z\, z_x = 0, \qquad F_y + F_z\, z_y = 0,$$

so wherever $F_z \neq 0$,

$$z_x = -\frac{F_x}{F_z}, \qquad z_y = -\frac{F_y}{F_z}.$$

The remaining partial derivatives of F are

$$F_x = 3x^2 + y \cdot z\mathrm{e}^{xz} = 3x^2 + yz\mathrm{e}^{xz}, \qquad F_y = \mathrm{e}^{xz} + z \cdot (-\sin y) = \mathrm{e}^{xz} - z\sin y.$$

At $(0,0,0)$, with $F_z(0,0,0) = 1$ from (i),

$$F_x(0,0,0) = 0 + 0 = 0, \qquad F_y(0,0,0) = \mathrm{e}^0 - 0 = 1,$$

so

$$z_x(0,0) = -\frac{0}{1} = 0, \qquad z_y(0,0) = -\frac{1}{1} = -1.$$

Therefore

$$\boxed{\frac{\partial z}{\partial x}(0,0) = 0, \qquad \frac{\partial z}{\partial y}(0,0) = -1}$$

(c)(i) Complete the square in x:

$$12x^2 - 12xy = 12\left(x^2 - xy\right) = 12\left[\left(x - \frac{y}{2}\right)^2 - \frac{y^2}{4}\right] = 12\left(x - \frac{y}{2}\right)^2 - 3y^2,$$

so

$$f(x,y) = 12\left(x - \frac{y}{2}\right)^2 + h(y), \qquad h(y) = y^3 - 3y^2.$$

Let $u = x - \frac{y}{2}$. By the Chain Rule, with $u_x = 1$ and $u_y = -\frac{1}{2}$,

$$f_x = 24u \cdot 1 = 24u, \qquad f_y = 24u \cdot \left(-\frac{1}{2}\right) + h'(y) = -12u + 3y^2 - 6y.$$

$f_x = 0$ forces $u = 0$, and then $f_y = 0$ becomes

$$3y^2 - 6y = 3y(y-2) = 0 \iff y = 0 \text{ or } y = 2.$$

With $u = 0$, $x = \frac{y}{2}$: $y = 0$ gives $x = 0$ and $y = 2$ gives $x = 1$. Therefore

$$\boxed{(0,0) \quad \text{and} \quad (1,2)}$$

(c)(ii) Note: the question specifies the Second-Derivative Test; this method should be used only if the course allows another classification argument. Classify by the sign of $f - f(\text{point})$ instead of the Hessian.

At $(1,2)$: $f(1,2) = 12 \cdot 0^2 + h(2) = 8 - 12 = -4$, and

$$h(y) - h(2) = y^3 - 3y^2 + 4 = (y-2)^2(y+1),$$

which can be checked by expanding: $(y^2 - 4y + 4)(y+1) = y^3 - 3y^2 + 4$. For every (x,y) with $y \geq -1$,

$$f(x,y) - f(1,2) = 12\left(x - \frac{y}{2}\right)^2 + (y-2)^2(y+1) \geq 0,$$

so $f(x,y) \geq f(1,2)$ on the half-plane $y \geq -1$, which contains a disk around $(1,2)$. Hence $(1,2)$ is a local minimum.

At $(0,0)$: $f(0,0) = 0$. Along the line $y = 0$, for $x \neq 0$,

$$f(x,0) = 12x^2 > 0.$$

Along the line $x = \frac{y}{2}$ (where $u = 0$), for $0 < |y| < 3$,

$$f\left(\frac{y}{2}, y\right) = h(y) = y^2(y-3) < 0.$$

Every disk around $(0,0)$ contains points with $f > 0$ and points with $f < 0$, so $(0,0)$ is neither a local maximum nor a local minimum: it is a saddle point. Therefore

$$\boxed{(0,0) \text{ is a saddle point; } (1,2) \text{ is a local minimum}}$$

(c)(iii) Along the line $x = \frac{y}{2}$,

$$f\left(\frac{y}{2}, y\right) = y^2(y-3) \longrightarrow +\infty \quad (y \to +\infty), \qquad f\left(\frac{y}{2}, y\right) \longrightarrow -\infty \quad (y \to -\infty).$$

So f is unbounded above and below on $\mathbb{R}^2$. Therefore

$$\boxed{f \text{ has no absolute maximum and no absolute minimum}}$$

Method 3: Tangent-plane comparison and critical-point comparison

(b)(ii) Note: the question specifies implicit differentiation; this method should be used only if the course allows another route to the two derivatives. The surface S near the origin can be described in two ways, and the tangent plane at $(0,0,0)$ is the same for both. As a level surface of F, with

$$\nabla F(0,0,0) = \langle F_x, F_y, F_z\rangle(0,0,0) = \langle 0,1,1\rangle \neq \mathbf{0}$$

(values from Method 2), the tangent plane is

$$0\cdot(x-0)+1\cdot(y-0)+1\cdot(z-0)=0 \iff z=-y.$$

As the graph of $z=z(x,y)$ with $z(0,0)=0$, the tangent plane is

$$z = z(0,0) + z_x(0,0)\,x + z_y(0,0)\,y = z_x(0,0)\,x + z_y(0,0)\,y.$$

Comparing the coefficients of x and y in $z = 0\cdot x + (-1)\cdot y$ gives

$$\boxed{\frac{\partial z}{\partial x}(0,0) = 0, \qquad \frac{\partial z}{\partial y}(0,0) = -1}$$

(c)(iii) $\mathbb{R}^2$ is open and f is differentiable, so if f had an absolute extremum at a point $\mathbf{p}$, then $\mathbf{p}$ would be a local extremum and, by Fermat's Theorem, $\nabla f(\mathbf{p}) = \mathbf{0}$. By (i) the only candidates are $(0,0)$ and $(1,2)$.

$(0,0)$ is a saddle point by (ii), so it is not a local extremum and is excluded.

$(1,2)$ has $f(1,2) = -4$. It is not an absolute minimum, because

$$f(0,-2) = 0 + (-2)^3 - 0 = -8 < -4.$$

It is not an absolute maximum, because

$$f(0,1) = 0 + 1^3 - 0 = 1 > -4.$$

No candidate survives. Therefore

$$\boxed{f \text{ has no absolute maximum and no absolute minimum}}$$

UNOFFICIAL QRS marking scheme (20 points)

(a) **(4 points)** 2: gradient; 1: evaluates normal; 1: correct plane.

(b) **(6 points)**

- (i) **(2 points)** 1: computes F_z; 1: evaluates it as nonzero.
- (ii) **(4 points)** 2: computes F_x, F_y; 1: implicit-derivative formulae; 1: both values.

(c) **(10 points)**

- (i) **(3 points)** 2: solves first-derivative equations; 1: both points.
- (ii) **(4 points)** 1: second derivatives; 1: discriminant; 1: saddle; 1: local minimum.
- (iii) **(3 points)** 2: unbounded path; 1: no absolute extrema.