MH2100 Calculus III

Midterm Exam 2 (Solutions)

AY 2024/2025, Semester 1

Quantitative Research Society @NTU

October 24, 2024

Overview

- **Coverage:** Lagrange multipliers; extrema on disks; and double integrals.
- **Historical source:** MH2100 Midterm Exam 2, AY2024/2025 Semester 1. Every problem below is transcribed from the official paper. Solutions and marking schemes are unofficial QRS@NTU material.
- **Current administration (AY2026/2027):** Tuesday 6 October 2026, 16:30–18:20 (110 minutes), venue TBA. One double-sided typed A4 sheet is permitted; attachments and sticky notes are not.
- **2026 scope note:** spherical coordinates, general Jacobian transformations, and line integrals are beyond the current midterm scope. They are retained and labelled for historical coverage.

Instructions to candidates (historical, verbatim)

(1) Do not turn over the exam paper until instructed to do so.

(2) This midterm exam consists of only three problems, each consisting of two parts.

(3) Candidates are required to attempt all three problems.

(4) The point allocation for each problem is shown at the top right-hand corner of its page.

(5) Candidates may write anywhere on the blank pages reserved for showing work.

(6) A single double-sided A4-sized help-sheet with only handwritten notes is permitted.

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Question 1 (20 Points)

Problem 1

(a) Define a function $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = y^2 - 10z.$$

Let S denote the sphere with center $(0, 0, 0)$ and radius 6, i.e.,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 36\}.$$

[10 points] Use the Method of Lagrange Multipliers to find the absolute extrema of $f|_S$. **Remark:** $f|_S$ denotes the restriction of f to S.

(b) Let D denote the closed disk in $\mathbb{R}^2$ with center $(0, 0)$ and radius 1, i.e.,

$$D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

Define a function $f : D \to \mathbb{R}$ by

$$f(x, y) = 2x^2 + y^2 - y + 3.$$

(i) [4 points] Find all the critical points of f, as well as the values of f at these points. **Remark:** Any critical point of f must belong to the interior of D.

(ii) [4 points] Using the standard parametrization $\mathbf{r}$ of the unit circle, given by

$$\forall t \in [0, 2\pi] : \qquad \mathbf{r}(t) = (\cos(t), \sin(t)),$$

find the absolute extrema of f on the boundary of D (which is the unit circle).

(iii) [2 points] Use your work for (b)(i) and (b)(ii) to identify the absolute extrema of f.

Solution

Method 1: Lagrange multipliers and the Closed Interval Method on the boundary

(a) Let $g(x, y, z) = x^2 + y^2 + z^2$, so that $S = \{(x, y, z) \mid g(x, y, z) = 36\}$.

The sphere S is closed and bounded, and f is a polynomial, hence continuous on S. By the Extreme Value Theorem, $f|_S$ attains an absolute maximum and an absolute minimum.

The partial derivatives are

$$f_x = 0, \qquad f_y = 2y, \qquad f_z = -10, \qquad g_x = 2x, \qquad g_y = 2y, \qquad g_z = 2z,$$

so

$$\nabla f = \langle 0, 2y, -10 \rangle, \qquad \nabla g = \langle 2x, 2y, 2z \rangle.$$

The only zero of ∇g is $(0, 0, 0)$, and $(0, 0, 0) \notin S$ because $0^2 + 0^2 + 0^2 = 0 \neq 36$. Hence $\nabla g \neq \mathbf{0}$ at every point of S, and by the Method of Lagrange Multipliers every extreme point of $f|_S$ satisfies $\nabla f = \lambda \nabla g$ for some $\lambda \in \mathbb{R}$:

$$0 = 2\lambda x, \tag{1}$$
$$2y = 2\lambda y, \tag{2}$$
$$-10 = 2\lambda z, \tag{3}$$
$$x^2 + y^2 + z^2 = 36. \tag{4}$$

Equation (3) gives $\lambda z = -5$, so $\lambda \neq 0$ and $z \neq 0$. Since $\lambda \neq 0$, equation (1) gives

$$x = 0.$$

Equation (2) rearranges to

$$2y(1 - \lambda) = 0,$$

so $y = 0$ or $\lambda = 1$.

Case 1: $y = 0$. With $x = 0$, equation (4) becomes

$$z^2 = 36 \implies z = 6 \text{ or } z = -6,$$

and (3) gives $\lambda = -\frac{5}{6}$ or $\lambda = \frac{5}{6}$ respectively. The values are

$$f(0, 0, 6) = 0^2 - 10(6) = -60,$$
$$f(0, 0, -6) = 0^2 - 10(-6) = 60.$$

Case 2: $\lambda = 1$. Equation (3) gives $2z = -10$, so $z = -5$. With $x = 0$, equation (4) becomes

$$y^2 = 36 - 0^2 - (-5)^2 = 11 \implies y = \sqrt{11} \text{ or } y = -\sqrt{11}.$$

The value at both points is

$$f(0, \pm\sqrt{11}, -5) = 11 - 10(-5) = 61.$$

The two cases exhaust the solutions of (1)–(4). Comparing the candidate values $-60 < 60 < 61$:

$$\therefore \quad \boxed{\max_S f = 61 \text{ at } (0, \sqrt{11}, -5), (0, -\sqrt{11}, -5); \qquad \min_S f = -60 \text{ at } (0, 0, 6).}$$

(b) (i) The partial derivatives are

$$f_x = 4x, \qquad f_y = 2y - 1.$$

Since f is a polynomial, it is differentiable everywhere, so its critical points are the solutions of $\nabla f = \mathbf{0}$:

$$4x = 0 \quad \text{and} \quad 2y - 1 = 0 \iff x = 0,\ y = \tfrac{1}{2}.$$

The point lies in the interior of D because $0^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} < 1$. Its value is

$$f\left(0, \tfrac{1}{2}\right) = 2(0)^2 + \left(\tfrac{1}{2}\right)^2 - \tfrac{1}{2} + 3 = \tfrac{1}{4} - \tfrac{1}{2} + 3 = \tfrac{11}{4}.$$

Hence the only critical point is $(0, \frac{1}{2})$, and

$$\boxed{f(0, 1/2) = 11/4.}$$

(ii) Let $h(t) = f(\mathbf{r}(t))$ for $t \in [0, 2\pi]$. Then

$$h(t) = 2\cos^2 t + \sin^2 t - \sin t + 3.$$

By the Chain Rule,

$$h'(t) = 4\cos t\,(-\sin t) + 2\sin t \cos t - \cos t$$

$$= -2\sin t\cos t - \cos t$$
$$= -\cos t\,(2\sin t + 1).$$

Hence $h'(t) = 0$ if and only if $\cos t = 0$ or $\sin t = -\frac{1}{2}$. On $[0, 2\pi]$:

$$\cos t = 0 \iff t = \tfrac{\pi}{2},\ \tfrac{3\pi}{2}; \qquad \sin t = -\tfrac{1}{2} \iff t = \tfrac{7\pi}{6},\ \tfrac{11\pi}{6}.$$

The function h is continuous on the closed interval $[0, 2\pi]$, so by the Closed Interval Method its extrema occur among these points and the endpoints $0, 2\pi$. At $t = \frac{7\pi}{6}, \frac{11\pi}{6}$ we have $\cos^2 t = \frac{3}{4}$, $\sin^2 t = \frac{1}{4}$, $\sin t = -\frac{1}{2}$. Therefore

$$\begin{aligned}
h(0) = h(2\pi) &= 2(1)^2 + 0^2 - 0 + 3 = 5, & \mathbf{r}(0) &= (1, 0),\\
h\left(\tfrac{\pi}{2}\right) &= 2(0)^2 + 1^2 - 1 + 3 = 3, & \mathbf{r}\left(\tfrac{\pi}{2}\right) &= (0, 1),\\
h\left(\tfrac{7\pi}{6}\right) &= 2\cdot\tfrac{3}{4} + \tfrac{1}{4} + \tfrac{1}{2} + 3 = \tfrac{21}{4}, & \mathbf{r}\left(\tfrac{7\pi}{6}\right) &= \left(-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right),\\
h\left(\tfrac{3\pi}{2}\right) &= 2(0)^2 + (-1)^2 + 1 + 3 = 5, & \mathbf{r}\left(\tfrac{3\pi}{2}\right) &= (0, -1),\\
h\left(\tfrac{11\pi}{6}\right) &= 2\cdot\tfrac{3}{4} + \tfrac{1}{4} + \tfrac{1}{2} + 3 = \tfrac{21}{4}, & \mathbf{r}\left(\tfrac{11\pi}{6}\right) &= \left(\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}\right).
\end{aligned}$$

Since $\mathbf{r}$ maps $[0, 2\pi]$ onto the unit circle ∂D, the extrema of f on ∂D are the extrema of h:

$$\therefore\ \boxed{\max_{\partial D} f = 21/4 \text{ at } (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2); \quad \min_{\partial D} f = 3 \text{ at } (0, 1).}$$

(iii) The disk D is closed and bounded and f is continuous, so by the Extreme Value Theorem f attains an absolute maximum and an absolute minimum on D. An absolute extremum at an interior point is a local extremum of the differentiable function f, hence a critical point; otherwise it lies on ∂D. The interior candidate from (b)(i) has value $\frac{11}{4}$; every value of f on ∂D lies between 3 and $\frac{21}{4}$ by (b)(ii). Comparing

$$\tfrac{11}{4} < 3 < \tfrac{21}{4},$$

the absolute minimum is $\frac{11}{4}$ and the absolute maximum is $\frac{21}{4}$:

$$\therefore\ \boxed{\min_{D} f = 11/4 \text{ at } (0, 1/2); \qquad \max_{D} f = 21/4 \text{ at } (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2).}$$

Method 2: Completing the square

Parts (a) and (b)(ii) of this method should be used only as a check: (a) asks for the Method of Lagrange Multipliers and (b)(ii) for the standard parametrization.

(a) On S we have $y^2 = 36 - x^2 - z^2$. Substituting into f and completing the square in z:

$$\begin{aligned} f(x,y,z) &= 36 - x^2 - z^2 - 10z \\ &= 36 - x^2 - \left(z^2 + 10z + 25\right) + 25 \\ &= 61 - x^2 - (z+5)^2. \end{aligned}$$

Since $x^2 \geq 0$ and $(z+5)^2 \geq 0$, we get $f \leq 61$ on S, with equality if and only if $x = 0$ and $z = -5$. Then

$$y^2 = 36 - 0^2 - (-5)^2 = 11 \implies y = \pm\sqrt{11},$$

and the points $(0, \pm\sqrt{11}, -5)$ lie on S because $0 + 11 + 25 = 36$. So the maximum is 61.

For the lower bound, every point of S satisfies $z^2 \leq x^2 + y^2 + z^2 = 36$, so $z \leq 6$. Together with $y^2 \geq 0$,

$$f(x,y,z) = y^2 - 10z \geq 0 - 10z \geq -10(6) = -60.$$

Equality holds if and only if $y = 0$ and $z = 6$; then $x^2 = 36 - 0^2 - 6^2 = 0$, so $x = 0$, and $f(0,0,6) = -60$. Therefore

$$\boxed{\max_S f = 61 \text{ at } (0, \sqrt{11}, -5), (0, -\sqrt{11}, -5); \qquad \min_S f = -60 \text{ at } (0,0,6).}$$

(b) (i) Complete the square in y:

$$f(x,y) = 2x^2 + \left(y^2 - y + \tfrac{1}{4}\right) - \tfrac{1}{4} + 3 = 2x^2 + \left(y - \tfrac{1}{2}\right)^2 + \tfrac{11}{4}.$$

From this form,

$$f_x = 4x, \qquad f_y = 2\left(y - \tfrac{1}{2}\right),$$

which vanish together only at $(0, \frac{1}{2})$, an interior point of D since $\frac{1}{4} < 1$. Both squares vanish there, so

$$f\left(0, \tfrac{1}{2}\right) = 0 + 0 + \tfrac{11}{4},$$

and the same form shows $f \geq \frac{11}{4}$ everywhere. Hence

$$\boxed{f(0, 1/2) = 11/4.}$$

(ii) On the circle $\mathbf{r}(t) = (\cos t, \sin t)$, put $s = \sin t$. As t runs over $[0, 2\pi]$, s runs over $[-1, 1]$, and $\cos^2 t = 1 - s^2$. Then

$$\begin{aligned} f(\mathbf{r}(t)) &= 2(1 - s^2) + s^2 - s + 3 \\ &= 5 - s^2 - s \\ &= 5 - \left(s^2 + s + \tfrac{1}{4}\right) + \tfrac{1}{4} \\ &= \tfrac{21}{4} - \left(s + \tfrac{1}{2}\right)^2. \end{aligned}$$

Maximum: $\left(s + \frac{1}{2}\right)^2 \geq 0$, so $f \leq \frac{21}{4}$, with equality if and only if $s = -\frac{1}{2}$, i.e. $\sin t = -\frac{1}{2}$ and $\cos t = \pm\sqrt{1 - \frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$.

Minimum: for $s \in [-1, 1]$ we have $s + \frac{1}{2} \in \left[-\frac{1}{2}, \frac{3}{2}\right]$, so $\left(s + \frac{1}{2}\right)^2 \leq \frac{9}{4}$, with equality if and only if $s = 1$. Hence $f \geq \frac{21}{4} - \frac{9}{4} = 3$, with equality if and only if $\sin t = 1$, $\cos t = 0$. Therefore

$$\boxed{\max_{\partial D} f = 21/4 \text{ at } (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2); \quad \min_{\partial D} f = 3 \text{ at } (0,1).}$$

(iii) Lower bound on D: by the completed square in (b)(i), $f \geq \frac{11}{4}$ on D, with equality at $(0, \frac{1}{2}) \in D$.
Upper bound on D: every point of D satisfies $x^2 \leq 1 - y^2$, so

$$\begin{aligned} f(x,y) &= 2x^2 + y^2 - y + 3 \\ &\leq 2(1-y^2) + y^2 - y + 3 \\ &= 5 - y^2 - y \\ &= \tfrac{21}{4} - \left(y + \tfrac{1}{2}\right)^2 \leq \tfrac{21}{4}. \end{aligned}$$

Equality holds if and only if $x^2 = 1 - y^2$ and $y = -\frac{1}{2}$, i.e. $x^2 = \frac{3}{4}$, $x = \pm\frac{\sqrt{3}}{2}$. Therefore

$$\boxed{\min_D f = 11/4 \text{ at } (0, 1/2); \qquad \max_D f = 21/4 \text{ at } (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2).}$$

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Method 3: Parametrising the sphere by height and angle

This method should be used only as a check: the question asks for the Method of Lagrange Multipliers.

(a) A point of S with height $z = t$ satisfies $x^2 + y^2 = 36 - t^2$, so $t \in [-6, 6]$ and (x, y) lies on the circle of radius $\sqrt{36 - t^2}$. Hence

$$\mathbf{R}(t, \theta) = \left(\sqrt{36 - t^2}\cos\theta,\ \sqrt{36 - t^2}\sin\theta,\ t\right), \qquad t \in [-6, 6],\ \theta \in [0, 2\pi],$$

maps onto S, and

$$F(t, \theta) = f(\mathbf{R}(t, \theta)) = (36 - t^2)\sin^2\theta - 10t.$$

For fixed t, we have $36 - t^2 \geq 0$ and $0 \leq \sin^2\theta \leq 1$, so

$$-10t \leq F(t, \theta) \leq 36 - t^2 - 10t,$$

with the lower bound attained at $\theta = 0$ and the upper bound at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$. Therefore

$$\max_S f = \max_{t\in[-6,6]} p(t), \quad p(t) = 36 - t^2 - 10t; \qquad \min_S f = \min_{t\in[-6,6]} q(t), \quad q(t) = -10t.$$

For p: $p'(t) = -2t - 10 = 0 \iff t = -5 \in [-6, 6]$. By the Closed Interval Method,

$$\begin{aligned} p(-6) &= 36 - 36 + 60 = 60, \\ p(-5) &= 36 - 25 + 50 = 61, \\ p(6) &= 36 - 36 - 60 = -60. \end{aligned}$$

So $\max p = 61$ at $t = -5$, and with $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ the points are $\mathbf{R} = (0, \pm\sqrt{36 - 25}, -5) = (0, \pm\sqrt{11}, -5)$.

For q: $q'(t) = -10 < 0$, so q is decreasing and its minimum on $[-6, 6]$ is $q(6) = -60$, attained at $\mathbf{R}(6, \theta) = (0, 0, 6)$. Therefore

$$\boxed{\max_S f = 61 \text{ at } (0, \sqrt{11}, -5), (0, -\sqrt{11}, -5); \qquad \min_S f = -60 \text{ at } (0, 0, 6).}$$

Method 4: Lagrange multipliers on the boundary circle

This method should be used only as a check: (b)(ii) asks for the standard parametrization of the unit circle.

(b) (ii) Let $g(x, y) = x^2 + y^2$, so $\partial D = \{g = 1\}$. The circle is closed and bounded and f is continuous, so by the Extreme Value Theorem f has extrema on ∂D. The partial derivatives are

$$f_x = 4x, \quad f_y = 2y - 1, \quad g_x = 2x, \quad g_y = 2y,$$

and $\nabla g = \langle 2x, 2y\rangle \neq \mathbf{0}$ on ∂D because $(0, 0) \notin \partial D$. The Lagrange system $\nabla f = \lambda\nabla g$, $g = 1$ is

$$\begin{aligned} 4x &= 2\lambda x, && (1) \\ 2y - 1 &= 2\lambda y, && (2) \\ x^2 + y^2 &= 1. && (3) \end{aligned}$$

Equation (1) rearranges to $2x(2-\lambda)=0$, so $x=0$ or $\lambda=2$.
Case 1: $x=0$. Equation (3) gives $y=\pm1$; equation (2) gives $\lambda=\frac{1}{2}$ for $y=1$ and $\lambda=\frac{3}{2}$ for $y=-1$. The values are

$$f(0,1)=0+1-1+3=3,$$
$$f(0,-1)=0+1+1+3=5.$$

Case 2: $\lambda=2$. Equation (2) becomes $2y-1=4y$, so $y=-\frac{1}{2}$. Equation (3) gives $x^2=1-\frac{1}{4}=\frac{3}{4}$, so $x=\pm\frac{\sqrt{3}}{2}$. The value is

$$f\left(\pm\tfrac{\sqrt{3}}{2},-\tfrac{1}{2}\right)=2\cdot\tfrac{3}{4}+\tfrac{1}{4}+\tfrac{1}{2}+3=\tfrac{21}{4}.$$

Comparing $3<5<\frac{21}{4}$:

$$\boxed{\max_{\partial D} f=21/4 \text{ at } (\sqrt{3}/2,-1/2),(-\sqrt{3}/2,-1/2); \quad \min_{\partial D} f=3 \text{ at } (0,1).}$$

(iii) By the Extreme Value Theorem on the closed bounded disk D, the absolute extrema exist and lie at the interior critical point $(0,\frac{1}{2})$, where $f=\frac{11}{4}$ (from (b)(i)), or on ∂D, where $3\le f\le\frac{21}{4}$. Since $\frac{11}{4}<3$:

$$\boxed{\min_{D} f=11/4 \text{ at } (0,1/2); \qquad \max_{D} f=21/4 \text{ at } (\sqrt{3}/2,-1/2),(-\sqrt{3}/2,-1/2).}$$

UNOFFICIAL QRS marking scheme (20 points)

(a) **(10 points)** 2: invokes EVT; 3: Lagrange equations and branches; 3: solves all branches; 2: correct extrema and points.

(b) **(10 points)**

(i) **(4 points)** 2: critical equations; 1: point; 1: value.

(ii) **(4 points)** 2: boundary parametrisation; 1: extrema; 1: points.

(iii) **(2 points)** compares all candidates and states global extrema.

Question 2 (15 Points)

Problem 2

(a) [5 points] Evaluate

$$\int_0^1 \left[\int_{\sqrt{y}}^1 \frac{1}{\sqrt{x^3+1}} \, \mathrm{d}x \right] \mathrm{d}y.$$

Remark: First sketch the region of integration.

(b) *(beyond 2026 midterm scope)* Let E denote the region in the first octant of $\mathbb{R}^3$ (i.e., $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$) that is

- bounded above by the sphere given by the equation $x^2 + y^2 + z^2 = 1$;
- bounded below by the closed cone given by the equation $z = \sqrt{\dfrac{x^2+y^2}{3}}$.

(i) [3 points] Describe E in spherical coordinates, using inequalities for ρ, θ, and ϕ.

(ii) [7 points] Evaluate $\iiint_E 3z \, \mathrm{d}V(x,y,z)$ using your work for (b)(i). **Caution:** Work that does not use spherical coordinates will not receive credit.

Solution

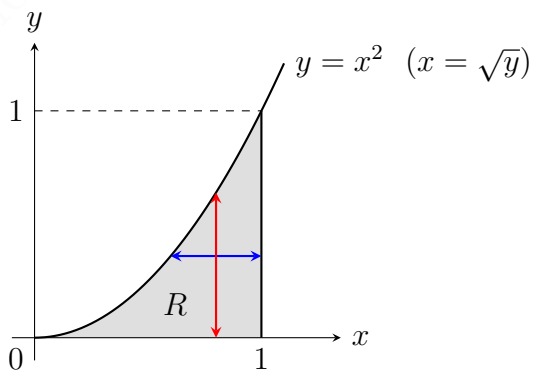
Method 1: Reversing the order of integration and spherical coordinates

(a) The region of integration, read from the given limits, is

$$R = \{(x,y) \mid 0 \leq y \leq 1,\ \sqrt{y} \leq x \leq 1\}.$$

For $x \geq 0$ and $y \geq 0$, $\sqrt{y} \leq x \iff y \leq x^2$. Every point of R has $0 \leq \sqrt{y} \leq x \leq 1$, and for each $x \in [0,1]$ the admissible y run from 0 to x^2 (and $x^2 \leq 1$, so $y \leq 1$ holds automatically). Hence

$$R = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x^2\}.$$



The blue horizontal strip is the original order ($\sqrt{y} \leq x \leq 1$ at height y); the red vertical strip is the reversed order ($0 \leq y \leq x^2$ at abscissa x).

The integrand $\dfrac{1}{\sqrt{x^3+1}}$ is continuous on R because $x^3 + 1 \geq 1 > 0$ for $x \geq 0$, and R is both of Type I and of Type II. By Fubini's Theorem both iterated integrals equal the double integral over R:

$$\int_0^1 \left[\int_{\sqrt{y}}^1 \frac{\mathrm{d}x}{\sqrt{x^3+1}} \right] \mathrm{d}y = \iint_R \frac{\mathrm{d}A}{\sqrt{x^3+1}} = \int_0^1 \left[\int_0^{x^2} \frac{\mathrm{d}y}{\sqrt{x^3+1}} \right] \mathrm{d}x.$$

The inner integrand does not depend on y, so

$$\int_0^{x^2} \frac{\mathrm{d}y}{\sqrt{x^3+1}} = \frac{1}{\sqrt{x^3+1}}\Big[y\Big]_0^{x^2} = \frac{x^2}{\sqrt{x^3+1}}.$$

For the outer integral substitute $w = x^3+1$, so $\mathrm{d}w = 3x^2\,\mathrm{d}x$ and $x^2\,\mathrm{d}x = \frac{1}{3}\,\mathrm{d}w$. The new limits are $x = 0 \mapsto w = 1$ and $x = 1 \mapsto w = 2$. Then

$$\begin{aligned}\int_0^1 \frac{x^2}{\sqrt{x^3+1}}\,\mathrm{d}x &= \int_1^2 \frac{1}{\sqrt{w}}\cdot\frac{\mathrm{d}w}{3} = \frac{1}{3}\int_1^2 w^{-1/2}\,\mathrm{d}w \\ &= \frac{1}{3}\Big[2w^{1/2}\Big]_1^2 = \frac{1}{3}\left(2\sqrt{2}-2\right).\end{aligned}$$

Therefore

$$\boxed{\int_0^1 \frac{x^2}{\sqrt{x^3+1}}\,\mathrm{d}x = \frac{2}{3}\left(\sqrt{2}-1\right).}$$

(b) *Beyond 2026 midterm scope.* Use spherical coordinates

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi,$$

with $\rho \geq 0$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$, for which $x^2+y^2+z^2 = \rho^2$ and $x^2+y^2 = \rho^2\sin^2\phi$.

(i) First octant: $x \geq 0$ and $y \geq 0$ with $\sin\phi \geq 0$ give $\cos\theta \geq 0$ and $\sin\theta \geq 0$, i.e. $0 \leq \theta \leq \frac{\pi}{2}$; $z \geq 0$ gives $\cos\phi \geq 0$, i.e. $0 \leq \phi \leq \frac{\pi}{2}$.
Below the sphere: $\rho^2 \leq 1$, i.e. $0 \leq \rho \leq 1$.
Above the cone: since $\sqrt{\rho^2\sin^2\phi} = \rho\sin\phi$ for $\phi \in [0,\pi]$,

$$z \geq \sqrt{\frac{x^2+y^2}{3}} \iff \rho\cos\phi \geq \frac{\rho\sin\phi}{\sqrt{3}}.$$

For $\rho > 0$ and $0 \leq \phi < \frac{\pi}{2}$ divide by $\rho\cos\phi > 0$: $\tan\phi \leq \sqrt{3}$, i.e. $\phi \leq \frac{\pi}{3}$. The value $\phi = \frac{\pi}{2}$ fails, since $0 \geq \frac{1}{\sqrt{3}}$ is false. The cone itself is $\phi = \frac{\pi}{3}$. Hence

$$\boxed{0 \leq \rho \leq 1, \qquad 0 \leq \theta \leq \frac{\pi}{2}, \qquad 0 \leq \phi \leq \frac{\pi}{3}.}$$

(ii) The Jacobian matrix of $(\rho,\phi,\theta) \mapsto (x,y,z)$ has determinant

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \det\begin{pmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{pmatrix}.$$

Expanding along the third row:

$$\begin{aligned}&\cos\phi\left(\rho\cos\phi\cos\theta\cdot\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta\cdot\rho\cos\phi\sin\theta\right) \\ &\quad + \rho\sin\phi\left(\sin\phi\cos\theta\cdot\rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta\cdot\sin\phi\sin\theta\right) \\ &= \rho^2\sin\phi\cos^2\phi\,(\cos^2\theta+\sin^2\theta) + \rho^2\sin^3\phi\,(\cos^2\theta+\sin^2\theta) \\ &= \rho^2\sin\phi\,(\cos^2\phi+\sin^2\phi) = \rho^2\sin\phi.\end{aligned}$$

On E, $\sin\phi \geq 0$, so $\mathrm{d}V = \rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$, and the integrand is $3z = 3\rho\cos\phi$. The region is a box in (ρ,ϕ,θ) and the integrand is continuous, so by Fubini's Theorem

$$\iiint_E 3z\,\mathrm{d}V = \int_0^{\pi/2}\int_0^{\pi/3}\int_0^1 3\rho\cos\phi\cdot\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$= \left(\int_0^{\pi/2} \mathrm{d}\theta\right)\left(\int_0^{\pi/3} \cos\phi \sin\phi \,\mathrm{d}\phi\right)\left(\int_0^1 3\rho^3 \,\mathrm{d}\rho\right).$$

The three factors:

$$\int_0^{\pi/2} \mathrm{d}\theta = \Big[\theta\Big]_0^{\pi/2} = \frac{\pi}{2}; \qquad \int_0^1 3\rho^3 \,\mathrm{d}\rho = \left[\frac{3\rho^4}{4}\right]_0^1 = \frac{3}{4}.$$

For the ϕ-factor substitute $w = \sin\phi$, $\mathrm{d}w = \cos\phi \,\mathrm{d}\phi$, with $\phi = 0 \mapsto w = 0$ and $\phi = \frac{\pi}{3} \mapsto w = \frac{\sqrt{3}}{2}$:

$$\int_0^{\pi/3} \cos\phi \sin\phi \,\mathrm{d}\phi = \int_0^{\sqrt{3}/2} w \,\mathrm{d}w = \left[\frac{w^2}{2}\right]_0^{\sqrt{3}/2} = \frac{1}{2}\cdot\frac{3}{4} = \frac{3}{8}.$$

Therefore

$$\iiint_E 3z \,\mathrm{d}V = \frac{\pi}{2}\cdot\frac{3}{8}\cdot\frac{3}{4} = \boxed{\frac{9\pi}{64}}$$

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Method 2: Integration by parts in the original order

(a) Let $h(x) = (x^3+1)^{-1/2}$ and

$$F(y) = \int_{\sqrt{y}}^{1} h(x)\,\mathrm{d}x, \qquad y \in [0,1].$$

Since h is continuous on $[0,1]$, F is continuous on $[0,1]$, and for $0 < y \le 1$ the Fundamental Theorem of Calculus with the Chain Rule gives

$$F'(y) = -h(\sqrt{y}) \cdot \frac{\mathrm{d}}{\mathrm{d}y}\sqrt{y} = -\frac{1}{\sqrt{y^{3/2}+1}} \cdot \frac{1}{2\sqrt{y}}.$$

Also $F(1) = \int_1^1 h\,\mathrm{d}x = 0$, and $0 < h \le 1$ on $[0,1]$ gives $0 \le F(y) \le 1$.

Fix $0 < \varepsilon < 1$ and integrate by parts on $[\varepsilon, 1]$ with $u = F(y),\ \mathrm{d}v = \mathrm{d}y,\ v = y$:

$$\begin{aligned}\int_\varepsilon^1 F(y)\,\mathrm{d}y &= \Big[yF(y)\Big]_\varepsilon^1 - \int_\varepsilon^1 yF'(y)\,\mathrm{d}y \\ &= 1\cdot F(1) - \varepsilon F(\varepsilon) + \int_\varepsilon^1 \frac{y}{2\sqrt{y}\sqrt{y^{3/2}+1}}\,\mathrm{d}y \\ &= -\varepsilon F(\varepsilon) + \int_\varepsilon^1 \frac{\sqrt{y}}{2\sqrt{y^{3/2}+1}}\,\mathrm{d}y.\end{aligned}$$

As $\varepsilon \to 0^+$, $0 \le \varepsilon F(\varepsilon) \le \varepsilon \to 0$, and both integrands are continuous on $[0,1]$, so

$$\int_0^1 F(y)\,\mathrm{d}y = \int_0^1 \frac{\sqrt{y}}{2\sqrt{y^{3/2}+1}}\,\mathrm{d}y.$$

Substitute $y = x^2$ with $x \in [0,1]$, so $\mathrm{d}y = 2x\,\mathrm{d}x$, $\sqrt{y} = x$, $y^{3/2} = x^3$, and $y = 0 \mapsto x = 0$, $y = 1 \mapsto x = 1$:

$$\int_0^1 \frac{\sqrt{y}}{2\sqrt{y^{3/2}+1}}\,\mathrm{d}y = \int_0^1 \frac{x}{2\sqrt{x^3+1}} \cdot 2x\,\mathrm{d}x = \int_0^1 \frac{x^2}{\sqrt{x^3+1}}\,\mathrm{d}x.$$

Now substitute $w = x^3+1,\ \mathrm{d}w = 3x^2\,\mathrm{d}x$, with $x = 0 \mapsto w = 1$, $x = 1 \mapsto w = 2$:

$$\int_0^1 \frac{x^2}{\sqrt{x^3+1}}\,\mathrm{d}x = \frac{1}{3}\int_1^2 w^{-1/2}\,\mathrm{d}w = \frac{1}{3}\Big[2w^{1/2}\Big]_1^2 = \frac{1}{3}\left(2\sqrt{2}-2\right).$$

Therefore

$$\boxed{\int_0^1 \frac{x^2}{\sqrt{x^3+1}}\,\mathrm{d}x = \frac{2}{3}\left(\sqrt{2}-1\right).}$$

Method 3: Cylindrical coordinates

This method should be used only as a check: the question gives credit only to work in spherical coordinates.

(b) *Beyond 2026 midterm scope.*

(ii) Use $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. The Jacobian is

$$\det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r,$$

so $\mathrm{d}V = r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta$. The cone is $z = \frac{r}{\sqrt{3}}$ and the upper sphere is $z = \sqrt{1-r^2}$. They meet where

$$\frac{r^2}{3} = 1 - r^2 \iff r^2 = \frac{3}{4} \iff r = \frac{\sqrt{3}}{2},$$

and for $0 \le r \le \frac{\sqrt{3}}{2}$ the cone lies below the sphere. Hence

$$E = \left\{0 \le \theta \le \tfrac{\pi}{2},\ 0 \le r \le \tfrac{\sqrt{3}}{2},\ \tfrac{r}{\sqrt{3}} \le z \le \sqrt{1-r^2}\right\}.$$

Inner integral:

$$\int_{r/\sqrt{3}}^{\sqrt{1-r^2}} 3zr\,\mathrm{d}z = \frac{3r}{2}\Big[z^2\Big]_{r/\sqrt{3}}^{\sqrt{1-r^2}} = \frac{3r}{2}\left(1 - r^2 - \frac{r^2}{3}\right) = \frac{3r}{2} - 2r^3.$$

Middle integral:

$$\int_0^{\sqrt{3}/2} \left(\frac{3r}{2} - 2r^3\right)\mathrm{d}r = \left[\frac{3r^2}{4} - \frac{r^4}{2}\right]_0^{\sqrt{3}/2} = \frac{3}{4}\cdot\frac{3}{4} - \frac{1}{2}\cdot\frac{9}{16} = \frac{9}{16} - \frac{9}{32} = \frac{9}{32}.$$

Outer integral:

$$\iiint_E 3z\,\mathrm{d}V = \int_0^{\pi/2} \frac{9}{32}\,\mathrm{d}\theta = \frac{\pi}{2}\cdot\frac{9}{32} = \boxed{\frac{9\pi}{64}}$$

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Method 4: Horizontal cross-sections

This method should be used only as a check: the question gives credit only to work in spherical coordinates.

(b) *Beyond 2026 midterm scope.*

(ii) For $z \geq 0$, the cone condition $z \geq \sqrt{(x^2+y^2)/3}$ is $x^2+y^2 \leq 3z^2$, and the sphere condition is $x^2+y^2 \leq 1-z^2$. So the cross-section E_z at height $z \in [0,1]$ is the quarter disk $(x, y \geq 0)$ of squared radius $\min(3z^2, 1-z^2)$. Since

$$3z^2 \leq 1-z^2 \iff z^2 \leq \tfrac{1}{4} \iff z \leq \tfrac{1}{2},$$

its area is

$$A(z) = \frac{\pi}{4} \cdot 3z^2 \quad (0 \leq z \leq \tfrac{1}{2}), \qquad A(z) = \frac{\pi}{4}(1-z^2) \quad (\tfrac{1}{2} \leq z \leq 1).$$

The integrand $3z$ is constant on each cross-section, so by Fubini's Theorem

$$\iiint_E 3z \, \mathrm{d}V = \int_0^1 3z \, A(z) \, \mathrm{d}z.$$

Lower piece:

$$\int_0^{1/2} 3z \cdot \frac{3\pi}{4} z^2 \, \mathrm{d}z = \frac{9\pi}{4} \left[\frac{z^4}{4}\right]_0^{1/2} = \frac{9\pi}{4} \cdot \frac{1}{64} = \frac{9\pi}{256}.$$

Upper piece:

$$\begin{aligned}\int_{1/2}^1 3z \cdot \frac{\pi}{4}(1-z^2) \, \mathrm{d}z &= \frac{3\pi}{4} \left[\frac{z^2}{2} - \frac{z^4}{4}\right]_{1/2}^1 = \frac{3\pi}{4} \left[\left(\frac{1}{2} - \frac{1}{4}\right) - \left(\frac{1}{8} - \frac{1}{64}\right)\right] \\ &= \frac{3\pi}{4} \left(\frac{1}{4} - \frac{7}{64}\right) = \frac{3\pi}{4} \cdot \frac{9}{64} = \frac{27\pi}{256}.\end{aligned}$$

Adding,

$$\iiint_E 3z \, \mathrm{d}V = \frac{9\pi}{256} + \frac{27\pi}{256} = \frac{36\pi}{256} = \boxed{\frac{9\pi}{64}}$$

UNOFFICIAL QRS marking scheme (15 points)

(a) **(5 points)** 2: sketches/reverses the region; 2: integrates in y; 1: substitutes and obtains $\frac{2}{3}(\sqrt{2}-1)$.

(b) **(10 points)**

(i) **(3 points)** 1: each spherical bound.

(ii) **(7 points)** 2: correct integrand and Jacobian; 3: integral; 2: value $9\pi/64$.

Unofficial QRS solutions

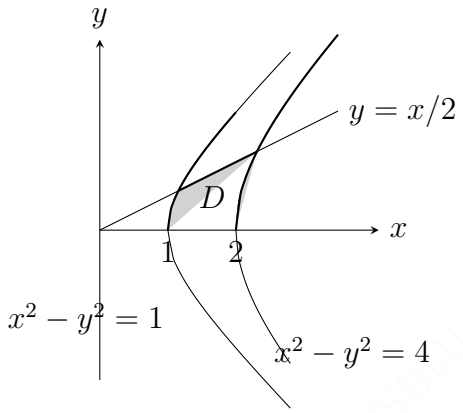
Question 3 (15 Points)

(beyond 2026 midterm scope)

Problem 3

(a) Let D denote the region in the first quadrant of $\mathbb{R}^2$ bounded by the following four curves:

- The two hyperbolas given by the equations $x^2 - y^2 = 1$ and $x^2 - y^2 = 4$.
- The two lines given by the equations $y = 0$ and $y = x/2$.



[7 points] Evaluate $\iint_D \frac{y}{x}\,\mathrm{d}A(x,y)$ using a change of variables.

(b) (i) Define a function $f : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{R}$ by

$$f(x,y,z) = \frac{1}{x^2+y^2+z^2}.$$

Let C denote the curve in $\mathbb{R}^3$ having the parametrization $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$ defined by

$$\mathbf{r}(t) = (\cos t, \sin t, t).$$

[4 points] Evaluate $\int_C f\,\mathrm{d}s$.

(ii) Define a vector field $\mathbf{F}$ on $\mathbb{R}^3$ by

$$\mathbf{F}(x,y,z) = (x, 3xy, -(x+z)).$$

Let C denote the directed line segment going from $(1,4,2)$ to $(0,5,1)$.
[4 points] Evaluate $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

Solution

Method 1: Change of variables and direct parametrisation

(a) *Beyond 2026 midterm scope.* On $x > 0$ define

$$T(x,y) = (u,v) = \left(x^2 - y^2,\ \frac{y}{x}\right).$$

The four boundary curves map to

$$x^2 - y^2 = 1 \mapsto u = 1, \qquad x^2 - y^2 = 4 \mapsto u = 4, \qquad y = 0 \mapsto v = 0, \qquad y = \tfrac{x}{2} \mapsto v = \tfrac{1}{2}.$$

For $x > 0$, the conditions $1 \le x^2 - y^2 \le 4$ and $0 \le y \le \frac{x}{2}$ are $1 \le u \le 4$ and $0 \le v \le \frac{1}{2}$, so

$$T(D) = G = \{(u, v) \mid 1 \le u \le 4,\ 0 \le v \le \tfrac{1}{2}\}.$$

Inverse: $y = vx$ gives $u = x^2 - v^2x^2 = x^2(1 - v^2)$. Since $1 - v^2 \ge \frac{3}{4} > 0$ and $x > 0$,

$$x = \sqrt{\frac{u}{1 - v^2}}, \qquad y = v\sqrt{\frac{u}{1 - v^2}},$$

so T is one-to-one from D onto G.

The partial derivatives are

$$u_x = 2x, \qquad u_y = -2y, \qquad v_x = -\frac{y}{x^2}, \qquad v_y = \frac{1}{x},$$

so

$$\frac{\partial(u, v)}{\partial(x, y)} = \det \begin{pmatrix} 2x & -2y \\ -y/x^2 & 1/x \end{pmatrix} = 2x \cdot \frac{1}{x} - (-2y)\left(-\frac{y}{x^2}\right) = 2 - \frac{2y^2}{x^2} = 2(1 - v^2).$$

This is nonzero on D since $v \le \frac{1}{2}$, so the inverse map has Jacobian

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \frac{1}{2(1 - v^2)}, \qquad \mathrm{d}A(x, y) = \frac{\mathrm{d}u\,\mathrm{d}v}{2(1 - v^2)}.$$

The integrand is $\frac{y}{x} = v$. By the Change of Variables Theorem and Fubini's Theorem on the rectangle G,

$$\iint_D \frac{y}{x}\,\mathrm{d}A = \int_1^4 \int_0^{1/2} \frac{v}{2(1 - v^2)}\,\mathrm{d}v\,\mathrm{d}u.$$

For the inner integral substitute $w = 1 - v^2$, $\mathrm{d}w = -2v\,\mathrm{d}v$, so $v\,\mathrm{d}v = -\frac{1}{2}\,\mathrm{d}w$, with $v = 0 \mapsto w = 1$ and $v = \frac{1}{2} \mapsto w = \frac{3}{4}$:

$$\begin{aligned}\int_0^{1/2} \frac{v}{2(1 - v^2)}\,\mathrm{d}v &= \int_1^{3/4} \frac{1}{2w}\left(-\frac{\mathrm{d}w}{2}\right) = \frac{1}{4}\int_{3/4}^1 \frac{\mathrm{d}w}{w} \\ &= \frac{1}{4}\Big[\ln w\Big]_{3/4}^1 = \frac{1}{4}\left(0 - \ln\frac{3}{4}\right) = \frac{1}{4}\ln\frac{4}{3}.\end{aligned}$$

The outer integral multiplies by the length $4 - 1 = 3$:

$$\iint_D \frac{y}{x}\,\mathrm{d}A = \int_1^4 \frac{1}{4}\ln\frac{4}{3}\,\mathrm{d}u = 3 \cdot \frac{1}{4}\ln\frac{4}{3} = \boxed{\frac{3}{4}\ln\left(\frac{4}{3}\right)}$$

(b) *Beyond 2026 midterm scope.*

(i) For $\mathbf{r}(t) = (\cos t, \sin t, t)$,

$$\mathbf{r}'(t) = (-\sin t, \cos t, 1), \qquad \|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}.$$

Along C, $\|\mathbf{r}(t)\|^2 = \cos^2 t + \sin^2 t + t^2 = 1 + t^2 \ge 1$, so C avoids the origin and

$$f(\mathbf{r}(t)) = \frac{1}{1 + t^2}.$$

With $\mathrm{d}s = \|\mathbf{r}'(t)\|\,\mathrm{d}t = \sqrt{2}\,\mathrm{d}t$,

$$\int_C f\,\mathrm{d}s = \int_0^{2\pi} \frac{1}{1+t^2}\cdot\sqrt{2}\,\mathrm{d}t = \sqrt{2}\Big[\arctan t\Big]_0^{2\pi} = \sqrt{2}\,(\arctan(2\pi) - 0)\,.$$

Therefore

$$\boxed{\int_C f\,\mathrm{d}s = \sqrt{2}\int_0^{2\pi} \frac{\mathrm{d}t}{1+t^2} = \sqrt{2}\arctan(2\pi).}$$

(ii) The segment from $(1,4,2)$ to $(0,5,1)$ has direction $(0,5,1)-(1,4,2) = (-1,1,-1)$, so

$$\mathbf{r}(t) = (1-t,\ 4+t,\ 2-t), \qquad \mathbf{r}'(t) = (-1,1,-1), \qquad 0 \le t \le 1.$$

The components of $\mathbf{F}$ along C are

$$\begin{aligned} x &= 1-t, \\ 3xy &= 3(1-t)(4+t) = 3(4-3t-t^2) = 12-9t-3t^2, \\ -(x+z) &= -\big((1-t)+(2-t)\big) = 2t-3. \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) &= (1-t)(-1) + (12-9t-3t^2)(1) + (2t-3)(-1) \\ &= -1+t+12-9t-3t^2-2t+3 \\ &= 14-10t-3t^2, \end{aligned}$$

and

$$\int_C \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \int_0^1 (14-10t-3t^2)\,\mathrm{d}t = \Big[14t-5t^2-t^3\Big]_0^1 = 14-5-1 = \boxed{8}$$

Unofficial Solutions

Method 2: Polar coordinates

(a) *Beyond 2026 midterm scope.* Use $x = r\cos\theta$, $y = r\sin\theta$, with Jacobian

$$\det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r, \qquad \mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Let $\alpha = \arctan\frac{1}{2}$. The lines are $y = 0 \iff \theta = 0$ and $y = \frac{x}{2} \iff \tan\theta = \frac{1}{2} \iff \theta = \alpha$. The hyperbolas involve

$$x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta.$$

For $0 \le \theta \le \alpha$ we have $\tan\theta \le \frac{1}{2} < 1$, so $\theta < \frac{\pi}{4}$ and $\cos 2\theta > 0$. Then

$$1 \le r^2\cos 2\theta \le 4 \iff \frac{1}{\sqrt{\cos 2\theta}} \le r \le \frac{2}{\sqrt{\cos 2\theta}},$$

so

$$D = \left\{0 \le \theta \le \alpha,\ \frac{1}{\sqrt{\cos 2\theta}} \le r \le \frac{2}{\sqrt{\cos 2\theta}}\right\}.$$

The integrand is $\frac{y}{x} = \tan\theta$. The inner integral is

$$\int_{1/\sqrt{\cos 2\theta}}^{2/\sqrt{\cos 2\theta}} \tan\theta \cdot r\,\mathrm{d}r = \tan\theta\left[\frac{r^2}{2}\right]_{1/\sqrt{\cos 2\theta}}^{2/\sqrt{\cos 2\theta}} = \tan\theta\cdot\frac{1}{2}\left(\frac{4}{\cos 2\theta} - \frac{1}{\cos 2\theta}\right) = \frac{3}{2}\cdot\frac{\tan\theta}{\cos 2\theta}.$$

For the outer integral substitute $t = \tan\theta$, $\mathrm{d}t = \sec^2\theta\,\mathrm{d}\theta = (1+t^2)\,\mathrm{d}\theta$. Dividing numerator and denominator of $\cos 2\theta = \dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$ by $\cos^2\theta$ gives $\cos 2\theta = \dfrac{1-t^2}{1+t^2}$. The new limits are $\theta = 0 \mapsto t = 0$ and $\theta = \alpha \mapsto t = \frac{1}{2}$:

$$\frac{3}{2}\int_0^{\alpha}\frac{\tan\theta}{\cos 2\theta}\,\mathrm{d}\theta = \frac{3}{2}\int_0^{1/2} t\cdot\frac{1+t^2}{1-t^2}\cdot\frac{\mathrm{d}t}{1+t^2} = \frac{3}{2}\int_0^{1/2}\frac{t}{1-t^2}\,\mathrm{d}t.$$

Substitute $w = 1 - t^2$, $t\,\mathrm{d}t = -\frac{1}{2}\,\mathrm{d}w$, with $t = 0 \mapsto w = 1$, $t = \frac{1}{2} \mapsto w = \frac{3}{4}$:

$$\int_0^{1/2}\frac{t}{1-t^2}\,\mathrm{d}t = \frac{1}{2}\int_{3/4}^{1}\frac{\mathrm{d}w}{w} = \frac{1}{2}\left(0 - \ln\frac{3}{4}\right) = \frac{1}{2}\ln\frac{4}{3}.$$

Therefore

$$\iint_D \frac{y}{x}\,\mathrm{d}A = \frac{3}{2}\cdot\frac{1}{2}\ln\frac{4}{3} = \boxed{\frac{3}{4}\ln\left(\frac{4}{3}\right)}$$

Method 3: Cartesian iterated integral over two Type II pieces

This method should be used only as a check: the question asks for a change of variables.

(a) *Beyond 2026 midterm scope.* The line $y = \frac{x}{2}$ meets $x^2 - y^2 = 1$ where $\frac{3}{4}x^2 = 1$, i.e. at $\left(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and meets $x^2 - y^2 = 4$ where $\frac{3}{4}x^2 = 4$, i.e. at $\left(\frac{4}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. So $0 \le y \le \frac{2}{\sqrt{3}}$ on D. At height y:

- the right boundary is $x = \sqrt{4 + y^2}$ (from $x^2 - y^2 \le 4$);
- the left boundary is the larger of $\sqrt{1 + y^2}$ (from $x^2 - y^2 \ge 1$) and $2y$ (from $y \le \frac{x}{2}$).

Since $2y \ge \sqrt{1 + y^2} \iff 4y^2 \ge 1 + y^2 \iff y \ge \frac{1}{\sqrt{3}}$, the region splits into two Type II pieces:

$$D = \left\{0 \le y \le \tfrac{1}{\sqrt{3}},\ \sqrt{1 + y^2} \le x \le \sqrt{4 + y^2}\right\} \cup \left\{\tfrac{1}{\sqrt{3}} \le y \le \tfrac{2}{\sqrt{3}},\ 2y \le x \le \sqrt{4 + y^2}\right\}.$$

The integrand $\frac{y}{x}$ is continuous on D (where $x \ge 1$), so Fubini's Theorem applies on each piece. The inner antiderivative is $\int \frac{y}{x}\,\mathrm{d}x = y \ln x$, so the inner integrals are

$$\int_{\sqrt{1+y^2}}^{\sqrt{4+y^2}} \frac{y}{x}\,\mathrm{d}x = \frac{y}{2}\ln(4 + y^2) - \frac{y}{2}\ln(1 + y^2),$$
$$\int_{2y}^{\sqrt{4+y^2}} \frac{y}{x}\,\mathrm{d}x = \frac{y}{2}\ln(4 + y^2) - y\ln(2y).$$

Adding the two pieces,

$$\iint_D \frac{y}{x}\,\mathrm{d}A = \underbrace{\int_0^{2/\sqrt{3}} \frac{y}{2}\ln(4 + y^2)\,\mathrm{d}y}_{I_A} - \underbrace{\int_0^{1/\sqrt{3}} \frac{y}{2}\ln(1 + y^2)\,\mathrm{d}y}_{I_B} - \underbrace{\int_{1/\sqrt{3}}^{2/\sqrt{3}} y\ln(2y)\,\mathrm{d}y}_{I_C}.$$

Antiderivatives: with $w = c + y^2$, $\mathrm{d}w = 2y\,\mathrm{d}y$, and $\int \ln w\,\mathrm{d}w = w\ln w - w$,

$$\int \frac{y}{2}\ln(c + y^2)\,\mathrm{d}y = \frac{1}{4}\big[(c + y^2)\ln(c + y^2) - (c + y^2)\big];$$

by parts with $u = \ln(2y)$, $\mathrm{d}v = y\,\mathrm{d}y$,

$$\int y\ln(2y)\,\mathrm{d}y = \frac{y^2}{2}\ln(2y) - \int \frac{y^2}{2}\cdot\frac{1}{y}\,\mathrm{d}y = \frac{y^2}{2}\ln(2y) - \frac{y^2}{4}.$$

Evaluating (with $y^2 = \frac{4}{3}$ at $y = \frac{2}{\sqrt{3}}$ and $y^2 = \frac{1}{3}$ at $y = \frac{1}{\sqrt{3}}$):

$$I_A = \frac{1}{4}\left[\frac{16}{3}\ln\frac{16}{3} - \frac{16}{3}\right] - \frac{1}{4}\big[4\ln 4 - 4\big] = \frac{4}{3}\ln\frac{16}{3} - \ln 4 - \frac{1}{3},$$
$$I_B = \frac{1}{4}\left[\frac{4}{3}\ln\frac{4}{3} - \frac{4}{3}\right] - \frac{1}{4}\big[0 - 1\big] = \frac{1}{3}\ln\frac{4}{3} - \frac{1}{12},$$
$$I_C = \left[\frac{2}{3}\ln\frac{4}{\sqrt{3}} - \frac{1}{3}\right] - \left[\frac{1}{6}\ln\frac{2}{\sqrt{3}} - \frac{1}{12}\right] = \frac{2}{3}\ln\frac{4}{\sqrt{3}} - \frac{1}{6}\ln\frac{2}{\sqrt{3}} - \frac{1}{4}.$$

The constants in $I_A - I_B - I_C$ cancel: $-\frac{1}{3} + \frac{1}{12} + \frac{1}{4} = 0$. Write every logarithm in terms of $\ln 2$ and $\ln 3$:

$$\tfrac{4}{3}\ln\tfrac{16}{3} = \tfrac{16}{3}\ln 2 - \tfrac{4}{3}\ln 3, \qquad -\ln 4 = -2\ln 2,$$

$$-\tfrac{1}{3}\ln\tfrac{4}{3} = -\tfrac{2}{3}\ln 2 + \tfrac{1}{3}\ln 3, \qquad -\tfrac{2}{3}\ln\tfrac{4}{\sqrt{3}} = -\tfrac{4}{3}\ln 2 + \tfrac{1}{3}\ln 3,$$
$$\tfrac{1}{6}\ln\tfrac{2}{\sqrt{3}} = \tfrac{1}{6}\ln 2 - \tfrac{1}{12}\ln 3.$$

Collecting coefficients,

$$\begin{aligned} \ln 2: &\quad \tfrac{16}{3} - 2 - \tfrac{2}{3} - \tfrac{4}{3} + \tfrac{1}{6} = \tfrac{32-12-4-8+1}{6} = \tfrac{3}{2}, \\ \ln 3: &\quad -\tfrac{4}{3} + \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{12} = \tfrac{-16+4+4-1}{12} = -\tfrac{3}{4}. \end{aligned}$$

Hence

$$\iint_D \frac{y}{x}\,\mathrm{d}A = \frac{3}{2}\ln 2 - \frac{3}{4}\ln 3 = \frac{3}{4}\big(2\ln 2 - \ln 3\big) = \boxed{\frac{3}{4}\ln\left(\frac{4}{3}\right)}$$

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Method 4: Arc-length reparametrisation

(b) *Beyond 2026 midterm scope.*

(i) Since $\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$, the arc length from $t = 0$ is

$$s(t) = \int_0^t \sqrt{2}\,\mathrm{d}\tau = \sqrt{2}\,t, \qquad t = \frac{s}{\sqrt{2}}, \qquad 0 \le s \le 2\sqrt{2}\,\pi.$$

In terms of s, the point $\mathbf{r}\big(s/\sqrt{2}\big)$ has $x^2 + y^2 + z^2 = 1 + \frac{s^2}{2}$, so along C

$$f = \frac{1}{1 + s^2/2} = \frac{2}{2 + s^2}.$$

With the unit-speed parameter s,

$$\int_C f\,\mathrm{d}s = \int_0^{2\sqrt{2}\,\pi} \frac{2}{2 + s^2}\,\mathrm{d}s.$$

An antiderivative is $\sqrt{2}\arctan\dfrac{s}{\sqrt{2}}$, since

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\sqrt{2}\arctan\frac{s}{\sqrt{2}}\right) = \sqrt{2}\cdot\frac{1/\sqrt{2}}{1 + s^2/2} = \frac{2}{2 + s^2}.$$

Therefore

$$\int_C f\,\mathrm{d}s = \sqrt{2}\left[\arctan\frac{s}{\sqrt{2}}\right]_0^{2\sqrt{2}\,\pi} = \sqrt{2}\big(\arctan(2\pi) - 0\big),$$

$$\boxed{\int_C f\,\mathrm{d}s = \sqrt{2}\arctan(2\pi).}$$

Method 5: Splitting off a gradient field

(uses the Fundamental Theorem for Line Integrals, beyond the 2026 midterm scope)

(b) (ii) The field $\mathbf{F}$ is not conservative, since $\frac{\partial}{\partial y}(x) = 0 \neq 3y = \frac{\partial}{\partial x}(3xy)$. Split off the gradient part: with $\varphi(x, y, z) = \frac{x^2}{2} - \frac{z^2}{2}$,

$$\nabla\varphi = (x, 0, -z), \qquad \mathbf{G} = \mathbf{F} - \nabla\varphi = (0,\, 3xy,\, -x).$$

By the Fundamental Theorem for Line Integrals,

$$\int_C \nabla\varphi \cdot \mathrm{d}\mathbf{r} = \varphi(0, 5, 1) - \varphi(1, 4, 2) = \big(0 - \tfrac{1}{2}\big) - \big(\tfrac{1}{2} - 2\big) = 1.$$

For $\mathbf{G}$ use $\mathbf{r}(t) = (1 - t, 4 + t, 2 - t)$, $\mathbf{r}'(t) = (-1, 1, -1)$, $0 \le t \le 1$:

$$\mathbf{G}(\mathbf{r}(t)) = \big(0,\ 3(1 - t)(4 + t),\ -(1 - t)\big) = \big(0,\ 12 - 9t - 3t^2,\ t - 1\big),$$

$$\mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0 + (12 - 9t - 3t^2) - (t - 1) = 13 - 10t - 3t^2,$$

$$\int_C \mathbf{G} \cdot \mathrm{d}\mathbf{r} = \int_0^1 (13 - 10t - 3t^2)\,\mathrm{d}t = \Big[13t - 5t^2 - t^3\Big]_0^1 = 13 - 5 - 1 = 7.$$

Adding the two parts,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 1 + 7 = \boxed{8}$$

UNOFFICIAL QRS marking scheme (15 points)

(a) **(7 points)** 2: valid transformation; 2: maps region; 2: Jacobian; 1: value.

(b) **(8 points)**

(i) **(4 points)** 1: parametrisation; 1: arc length; 1: integral; 1: value.

(ii) **(4 points)** 1: segment parametrisation; 2: dot product; 1: value 8.

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