# MH2100 Calculus III

## Midterm Examination (Solutions)

AY 2025/2026, Semester 1

*Quantitative Research Society @NTU*

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**Overview**

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- **Coverage:** limits and total differentiability; directional derivatives and the chain rule; extrema and Lagrange multipliers; double integrals and polar coordinates.
- **Format:** four problems, 100 marks in total; 110 minutes.
- **Source:** MH2100 Midterm Examination, AY2025/2026 Semester 1. The questions below reproduce the paper; the solutions and marking schemes are unofficial QRS@NTU material.

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# Instructions

Answer **all** questions. Show all working clearly; answers without justification may receive zero marks.

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# Question 1 (25 Marks)

**PROBLEM 1.**

(a) Evaluate

$$\lim_{(x,y)\to(2,1)} \frac{x^2y - x^2 - 4y + 4}{xy - x - 2y + 2}. \qquad \textbf{(5 points)}$$

(b) Determine whether or not the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by

$$\forall x, y \in \mathbb{R} : \qquad f(x, y) \stackrel{\text{df}}{=} \sqrt[3]{x^3 + y^3},$$

is totally differentiable at $(0, 0)$. **(20 points)**

## Solution

**Method 1: Factorisation; definition of total differentiability with a path test**

(a) Let

$$q(x, y) = \frac{x^2y - x^2 - 4y + 4}{xy - x - 2y + 2}.$$

Factor the numerator by grouping the terms with $y$ and without $y$:

$$\begin{aligned} x^2y - x^2 - 4y + 4 &= x^2(y - 1) - 4(y - 1) \\ &= (y - 1)(x^2 - 4) \\ &= (y - 1)(x - 2)(x + 2). \end{aligned}$$

Factor the denominator in the same way:

$$\begin{aligned} xy - x - 2y + 2 &= x(y - 1) - 2(y - 1) \\ &= (x - 2)(y - 1). \end{aligned}$$

Hence $q$ is defined exactly on

$$D = \{(x, y) \in \mathbb{R}^2 \mid x \neq 2 \text{ and } y \neq 1\},$$

and $(2, 1)$ is a limit point of $D$, since $(2 + t, 1 + t) \in D$ for every $t \neq 0$. For $(x, y) \in D$ the factor $(x - 2)(y - 1)$ is nonzero and cancels:

$$q(x, y) = \frac{(y - 1)(x - 2)(x + 2)}{(x - 2)(y - 1)} = x + 2.$$

The polynomial $(x, y) \mapsto x + 2$ is continuous on $\mathbb{R}^2$, so

$$\lim_{\substack{(x,y)\to(2,1) \\ (x,y)\in D}} q(x, y) = \lim_{(x,y)\to(2,1)} (x + 2) = 2 + 2 = 4.$$

Therefore

$$\boxed{\lim_{(x,y)\to(2,1)} \frac{x^2y - x^2 - 4y + 4}{xy - x - 2y + 2} = 4.}$$

(b) Recall the definition: $f$ is totally differentiable at $(0,0)$ if there is a linear map $L(h,k) = ah + bk$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f(0,0) - L(h,k)}{\sqrt{h^2+k^2}} = 0.$$

*Partial derivatives at the origin.* First, $f(0,0) = \sqrt[3]{0} = 0$. For $h \neq 0$, $f(h,0) = \sqrt[3]{h^3} = h$, so

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

For $k \neq 0$, $f(0,k) = \sqrt[3]{k^3} = k$, so

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k-0}{k} = 1.$$

*The only possible derivative.* Suppose $L(h,k) = ah + bk$ satisfies the definition. Along the axis $(h,0)$ with $h \neq 0$,

$$\frac{f(h,0) - f(0,0) - L(h,0)}{\sqrt{h^2}} = \frac{h - ah}{|h|} = \frac{(1-a)h}{|h|},$$

whose absolute value is $|1-a|$ for every $h \neq 0$; this tends to 0 only if $a = 1$. The same computation along $(0,k)$ forces $b = 1$. Hence the only candidate is

$$L(h,k) = f_x(0,0)\,h + f_y(0,0)\,k = h + k.$$

*Path test for the remainder.* Let $E(h,k) = f(h,k) - f(0,0) - (h+k)$. Along the path $(h,k) = (t,t)$ with $t > 0$,

$$f(t,t) = \sqrt[3]{t^3+t^3} = \sqrt[3]{2t^3} = \sqrt[3]{2}\,t, \qquad L(t,t) = t + t = 2t,$$

so

$$\frac{E(t,t)}{\sqrt{t^2+t^2}} = \frac{\sqrt[3]{2}\,t - 2t}{\sqrt{2}\,t} = \frac{\sqrt[3]{2} - 2}{\sqrt{2}}.$$

This is a constant, and it is nonzero because $\sqrt[3]{2} < 2$ (numerically it is about $-0.523$). Hence, as $t \to 0^+$ along $(t,t)$,

$$\frac{E(t,t)}{\sqrt{t^2+t^2}} \to \frac{\sqrt[3]{2} - 2}{\sqrt{2}} \neq 0.$$

If the two-variable limit of $E(h,k)/\sqrt{h^2+k^2}$ were 0, its limit along every path into $(0,0)$ would be 0. So the candidate $L$ fails, and no other candidate exists. Therefore

$$\boxed{f \text{ is not totally differentiable at } (0,0).}$$

**Method 2: Polar coordinates (shifted polar squeeze; polar remainder)**

(a) Write $x = 2 + \rho\cos\varphi$, $y = 1 + \rho\sin\varphi$ with $\rho > 0$. Then $(x, y) \to (2, 1)$ if and only if $\rho \to 0$, and

$$x - 2 = \rho\cos\varphi, \qquad y - 1 = \rho\sin\varphi, \qquad x + 2 = 4 + \rho\cos\varphi.$$

Using the factorisations $x^2y - x^2 - 4y + 4 = (y - 1)(x - 2)(x + 2)$ and $xy - x - 2y + 2 = (x - 2)(y - 1)$, for $(x, y)$ in the domain $(\cos\varphi\sin\varphi \neq 0)$:

$$q(x, y) = \frac{\rho\sin\varphi \cdot \rho\cos\varphi \cdot (4 + \rho\cos\varphi)}{\rho\cos\varphi \cdot \rho\sin\varphi} = 4 + \rho\cos\varphi.$$

Hence

$$0 \leq |q(x, y) - 4| = \rho\,|\cos\varphi| \leq \rho,$$

a bound independent of $\varphi$. Since $\rho \to 0$, the Squeeze Theorem gives $q(x, y) \to 4$:

$$\boxed{\lim_{(x,y)\to(2,1)} \frac{x^2y - x^2 - 4y + 4}{xy - x - 2y + 2} = 4.}$$

(b) As in Method 1, $f(0, 0) = 0$ and $f_x(0, 0) = f_y(0, 0) = 1$, so the only candidate derivative is $L(h, k) = h + k$. Write $(h, k) = (r\cos\theta, r\sin\theta)$ with $r > 0$. The real cube root satisfies $\sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b}$ for all real $a, b$, so

$$f(r\cos\theta, r\sin\theta) = \sqrt[3]{r^3(\cos^3\theta + \sin^3\theta)} = r\sqrt[3]{\cos^3\theta + \sin^3\theta}.$$

Hence the normalised remainder is

$$\frac{f(h, k) - 0 - (h + k)}{\sqrt{h^2 + k^2}} = \frac{r\sqrt[3]{\cos^3\theta + \sin^3\theta} - r\cos\theta - r\sin\theta}{r} = G(\theta),$$

where

$$G(\theta) = \sqrt[3]{\cos^3\theta + \sin^3\theta} - \cos\theta - \sin\theta.$$

The quotient depends on $\theta$ only. If its limit as $(h, k) \to (0, 0)$ were 0, then letting $r \to 0$ along each fixed ray would give $G(\theta) = 0$ for every $\theta$. At $\theta = \frac{\pi}{4}$, $\cos\theta = \sin\theta = 2^{-1/2}$, so

$$\cos^3\theta + \sin^3\theta = 2 \cdot 2^{-3/2} = 2^{-1/2}, \qquad \cos\theta + \sin\theta = 2 \cdot 2^{-1/2} = 2^{1/2},$$

and

$$G\Big(\frac{\pi}{4}\Big) = 2^{-1/6} - 2^{1/2} = 2^{-1/6}\big(1 - 2^{2/3}\big) < 0.$$

(This is the constant $(\sqrt[3]{2} - 2)/\sqrt{2}$ of Method 1.) Hence the remainder condition fails, and

$$\boxed{f \text{ is not totally differentiable at } (0, 0).}$$

**Method 3: Epsilon–delta definition; directional-derivative linearity test**

(a) Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. If $(x, y) \in D$ and $0 < \sqrt{(x-2)^2 + (y-1)^2} < \delta$, then, by the cancellation $q(x, y) = x + 2$ on $D$,

$$|q(x, y) - 4| = |x - 2| \leq \sqrt{(x-2)^2 + (y-1)^2} < \delta = \varepsilon.$$

By the definition of the limit,

$$\boxed{\lim_{(x,y)\to(2,1)} \frac{x^2y - x^2 - 4y + 4}{xy - x - 2y + 2} = 4.}$$

(b) Recall the theorem: if $f$ is totally differentiable at $p$, then for every unit vector $\mathbf{u}$ the directional derivative exists and $(D_{\mathbf{u}}f)(p) = \operatorname{grad} f(p) \cdot \mathbf{u}$. Since $f(h, 0) = h$ and $f(0, k) = k$,

$$\operatorname{grad} f(0, 0) = \big(f_x(0, 0), f_y(0, 0)\big) = (1, 1).$$

Take $\mathbf{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. The theorem would give

$$\operatorname{grad} f(0, 0) \cdot \mathbf{u} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}.$$

Compute the directional derivative from its definition. For $t \neq 0$,

$$f\Big(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\Big) = \sqrt[3]{\frac{2t^3}{2\sqrt{2}}} = \sqrt[3]{\frac{t^3}{2^{1/2}}} = 2^{-1/6}\, t,$$

so

$$(D_{\mathbf{u}}f)(0, 0) = \lim_{t\to 0} \frac{f(t\mathbf{u}) - f(0, 0)}{t} = \lim_{t\to 0} \frac{2^{-1/6}t}{t} = 2^{-1/6}.$$

Since $2^{-1/6} < 1 < \sqrt{2}$, the equality $(D_{\mathbf{u}}f)(0, 0) = \operatorname{grad} f(0, 0) \cdot \mathbf{u}$ fails. By the contrapositive of the theorem,

$$\boxed{f \text{ is not totally differentiable at } (0, 0).}$$

**Marking scheme (25 marks)**

(a) **(5 marks)**

- 2 marks: factors the numerator correctly.
- 1 mark: factors the denominator correctly.
- 1 mark: cancels the common factors on the domain of the quotient.
- 1 mark: obtains the limit 4.

(b) **(20 marks)**

- 4 marks: computes $f_x(0, 0) = f_y(0, 0) = 1$.
- 4 marks: identifies the only possible derivative $L(h, k) = h + k$.
- 6 marks: chooses and evaluates a valid path, such as $(t, t)$.
- 4 marks: shows that the normalized remainder does not tend to zero.
- 2 marks: states the correct conclusion about total differentiability.

# Question 2 (25 Marks)

**PROBLEM 2.**

(a) A function $f : \mathbb{R}^3 \to \mathbb{R}$ is defined by

$$\forall x, y, z \in \mathbb{R} : \qquad f(x, y, z) \stackrel{\text{df}}{=} z^2 - x^3y^2.$$

Let $\mathbf{u}$ denote the unit vector pointing from $(1, -1, 1)$ toward $(2, 1, 0)$. Compute $(D_{\mathbf{u}}f)(1, -1, 1)$ – with justification. **(10 points)**

(b) Two functions, $f, g : \mathbb{R}^2 \to \mathbb{R}$, satisfy the following four properties:

- $f$ is totally differentiable at $(1, 4)$.
- The tangent plane to the graph of $f$ at $(1, 4, f(1, 4))$ is

$$\{(x, y, z) \in \mathbb{R}^3 \mid 3x + 5y = z + 26\}.$$

- $f_{xy}(1, 4) = f_{yx}(1, 4) = 1$ and $f_{xx}(1, 4) = f_{yy}(1, 4) = -1$.
- For all $x, y \in \mathbb{R}$, we have $g(x, y) = x \cdot f(x^2 - y, x + 2y)$.

Compute $g_{xy}(-2, 3)$ – with justification. **(15 points)**

## Solution

**Method 1: Gradient dot product; second-order Chain Rule with a regularity check**

(a) The displacement from $(1, -1, 1)$ to $(2, 1, 0)$ is

$$\mathbf{v} = (2, 1, 0) - (1, -1, 1) = (1, 2, -1), \qquad \|\mathbf{v}\| = \sqrt{1^2 + 2^2 + (-1)^2} = \sqrt{6},$$

so

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\sqrt{6}}(1, 2, -1).$$

The partial derivatives of $f(x, y, z) = z^2 - x^3y^2$ are

$$f_x = -3x^2y^2, \qquad f_y = -2x^3y, \qquad f_z = 2z.$$

They are polynomials, hence continuous on $\mathbb{R}^3$; by the $C^1$ criterion $f$ is totally differentiable on $\mathbb{R}^3$, and for a differentiable $f$ the directional derivative is $(D_{\mathbf{u}}f)(p) = \operatorname{grad} f(p) \cdot \mathbf{u}$. At $(1, -1, 1)$:

$$\begin{aligned} f_x(1, -1, 1) &= -3(1)^2(-1)^2 = -3, \\ f_y(1, -1, 1) &= -2(1)^3(-1) = 2, \\ f_z(1, -1, 1) &= 2(1) = 2. \end{aligned}$$

Hence

$$(D_{\mathbf{u}}f)(1, -1, 1) = (-3, 2, 2) \cdot \frac{(1, 2, -1)}{\sqrt{6}} = \frac{-3 + 4 - 2}{\sqrt{6}} = -\frac{1}{\sqrt{6}}.$$

Therefore

$$\boxed{(D_{\mathbf{u}}f)(1, -1, 1) = -\frac{1}{\sqrt{6}}.}$$

(b) Write $f = f(s,t)$, and let $f_x, f_y$ denote the partial derivatives of $f$ in its first and second slot. Set

$$s = x^2 - y, \qquad t = x + 2y, \qquad g(x,y) = x\, f(s,t),$$

so that

$$s_x = 2x, \qquad s_y = -1, \qquad t_x = 1, \qquad t_y = 2.$$

At $(x,y) = (-2,3)$:

$$s = (-2)^2 - 3 = 1, \qquad t = -2 + 2 \cdot 3 = 4.$$

*Data from the tangent plane.* Since $f$ is differentiable at $(1,4)$, its tangent plane is $z = f(1,4) + f_x(1,4)(x-1) + f_y(1,4)(y-4)$. The given plane is

$$z = 3x + 5y - 26 = -3 + 3(x-1) + 5(y-4),$$

because $3 \cdot 1 + 5 \cdot 4 - 26 = -3$. Comparing coefficients,

$$f(1,4) = -3, \qquad f_x(1,4) = 3, \qquad f_y(1,4) = 5.$$

*Regularity check.* Two applications of the Chain Rule are needed.

- To write $g_x(x,y)$ by the Chain Rule at points $(x,y)$ near $(-2,3)$, $f$ must be differentiable at the points $(s,t)$ near $(1,4)$; the data give differentiability at $(1,4)$ only.
- To differentiate $f_x(s,t)$ and $f_y(s,t)$ with respect to $y$ at $(-2,3)$ by the Chain Rule, $f_x$ and $f_y$ must be differentiable at $(1,4)$; the data give only the existence of the four iterated second partials at $(1,4)$, and existence of partial derivatives does not imply differentiability.

So the literal data do not justify the second-order Chain Rule. We compute under the hypothesis

$$\text{(H)} \qquad f \in C^2 \text{ on an open neighbourhood of } (1,4),$$

which makes $f$, $f_x$, $f_y$ of class $C^1$, hence differentiable, near $(1,4)$.

*Computation under (H).* By the Product Rule and the Chain Rule,

$$\begin{aligned} g_x &= 1 \cdot f(s,t) + x\big[f_x(s,t)\, s_x + f_y(s,t)\, t_x\big] \\ &= f(s,t) + x\big[2x\, f_x(s,t) + f_y(s,t)\big] \\ &= f(s,t) + 2x^2 f_x(s,t) + x\, f_y(s,t). \end{aligned}$$

Differentiate with respect to $y$ ($x$ fixed). The Chain Rule applied to $f$, $f_x$, $f_y$ gives

$$\begin{aligned} \frac{\partial}{\partial y} f(s,t) &= f_x\, s_y + f_y\, t_y = -f_x + 2f_y, \\ \frac{\partial}{\partial y} f_x(s,t) &= f_{xx}\, s_y + f_{xy}\, t_y = -f_{xx} + 2f_{xy}, \\ \frac{\partial}{\partial y} f_y(s,t) &= f_{yx}\, s_y + f_{yy}\, t_y = -f_{yx} + 2f_{yy}. \end{aligned}$$

Hence

$$g_{xy} = (-f_x + 2f_y) + 2x^2(-f_{xx} + 2f_{xy}) + x(-f_{yx} + 2f_{yy}).$$

At $(x, y) = (-2, 3)$, with every derivative of $f$ taken at $(1, 4)$:

$$\begin{aligned} -f_x + 2f_y &= -3 + 2 \cdot 5 = 7, \\ 2x^2(-f_{xx} + 2f_{xy}) &= 2 \cdot 4 \cdot \big(1 + 2 \cdot 1\big) = 8 \cdot 3 = 24, \\ x(-f_{yx} + 2f_{yy}) &= (-2)\big(-1 + 2 \cdot (-1)\big) = (-2)(-3) = 6. \end{aligned}$$

Therefore $g_{xy}(-2, 3) = 7 + 24 + 6 = 37$:

$$\boxed{g_{xy}(-2, 3) = 37 \quad \text{if } f \text{ is } C^2 \text{ near } (1, 4).}$$

*Counterexample without (H).* Write $U = s - 1$, $V = t - 4$ and define $F = F_0 + Q$, where

$$F_0 = -3 + 3U + 5V - \tfrac{1}{2}U^2 + UV - \tfrac{1}{2}V^2, \qquad Q = \begin{cases} \dfrac{U^2V^2}{U^2 + V^2}, & (U, V) \neq (0, 0), \\ 0, & (U, V) = (0, 0). \end{cases}$$

For $(U, V) \neq (0, 0)$ the Quotient Rule gives

$$\begin{aligned} Q_U &= \frac{2UV^2(U^2 + V^2) - U^2V^2 \cdot 2U}{(U^2 + V^2)^2} = \frac{2UV^4}{(U^2 + V^2)^2}, \\ Q_V &= \frac{2U^2V(U^2 + V^2) - U^2V^2 \cdot 2V}{(U^2 + V^2)^2} = \frac{2U^4V}{(U^2 + V^2)^2}. \end{aligned}$$

At the origin, $Q(h, 0) = 0$ and $Q(0, k) = 0$, so $Q_U(0, 0) = \lim_{h \to 0} \frac{0-0}{h} = 0$ and $Q_V(0, 0) = 0$. Since $V^4 \leq (U^2 + V^2)^2$ and $U^4 \leq (U^2 + V^2)^2$,

$$|Q_U| \leq 2|U| \to 0, \qquad |Q_V| \leq 2|V| \to 0,$$

so $Q_U, Q_V$ are continuous on $\mathbb{R}^2$ and $Q \in C^1(\mathbb{R}^2)$. Hence $F \in C^1(\mathbb{R}^2)$, and in particular $F$ is differentiable at $(1, 4)$. Moreover $Q_U(h, 0) = Q_U(0, k) = 0$ and $Q_V(h, 0) = Q_V(0, k) = 0$ for all $h, k$, so the four iterated second partials of $Q$ at $(0, 0)$ are

$$\begin{aligned} Q_{UU}(0, 0) &= \lim_{h \to 0} \frac{Q_U(h, 0) - 0}{h} = 0, & Q_{UV}(0, 0) &= \lim_{k \to 0} \frac{Q_U(0, k) - 0}{k} = 0, \\ Q_{VU}(0, 0) &= \lim_{h \to 0} \frac{Q_V(h, 0) - 0}{h} = 0, & Q_{VV}(0, 0) &= \lim_{k \to 0} \frac{Q_V(0, k) - 0}{k} = 0. \end{aligned}$$

For $F_0$:

$$F_{0,U} = 3 - U + V, \quad F_{0,V} = 5 + U - V, \quad F_{0,UU} = -1, \quad F_{0,UV} = F_{0,VU} = 1, \quad F_{0,VV} = -1.$$

At $(s, t) = (1, 4)$, i.e. $U = V = 0$: $F = -3$, $F_s = 3$, $F_t = 5$, so the tangent plane is $z = -3 + 3(s - 1) + 5(t - 4)$, i.e. $3s + 5t = z + 26$; and $F_{ss} = -1$, $F_{st} = F_{ts} = 1$, $F_{tt} = -1$. So $F$ satisfies every datum.

Let $G(x, y) = x\, F(x^2 - y, x + 2y)$. Since $F \in C^1(\mathbb{R}^2)$, the first-order Chain Rule holds at every point:

$$G_x(x, y) = F + 2x^2 F_s + x\, F_t \qquad \text{(evaluated at } (x^2 - y,\ x + 2y)).$$

On the line $x = -2$, $y = 3 + \tau$:

$$s = 4 - 3 - \tau = 1 - \tau, \qquad t = -2 + 6 + 2\tau = 4 + 2\tau, \qquad (U, V) = (-\tau, 2\tau),$$

and

$$G_x(-2,3+\tau) = \big[F + 8F_U - 2F_V\big]_{(U,V)=(-\tau,2\tau)}.$$

The $F_0$-part:

$$\begin{aligned} F_0(-\tau,2\tau) &= -3 - 3\tau + 10\tau - \tfrac{1}{2}\tau^2 - 2\tau^2 - 2\tau^2 = -3 + 7\tau - \tfrac{9}{2}\tau^2, \\ F_{0,U}(-\tau,2\tau) &= 3 + \tau + 2\tau = 3 + 3\tau, \qquad F_{0,V}(-\tau,2\tau) = 5 - \tau - 2\tau = 5 - 3\tau, \\ F_0 + 8F_{0,U} - 2F_{0,V} &= -3 + 7\tau - \tfrac{9}{2}\tau^2 + 24 + 24\tau - 10 + 6\tau = 11 + 37\tau - \tfrac{9}{2}\tau^2. \end{aligned}$$

The $Q$-part, using $U^2 + V^2 = 5\tau^2$ for $\tau \neq 0$:

$$\begin{aligned} Q(-\tau,2\tau) &= \frac{\tau^2 \cdot 4\tau^2}{5\tau^2} = \frac{4\tau^2}{5}, \\ Q_U(-\tau,2\tau) &= \frac{2(-\tau)(16\tau^4)}{25\tau^4} = -\frac{32\tau}{25}, \\ Q_V(-\tau,2\tau) &= \frac{2\tau^4(2\tau)}{25\tau^4} = \frac{4\tau}{25}, \\ Q + 8Q_U - 2Q_V &= \frac{4\tau^2}{5} - \frac{256\tau}{25} - \frac{8\tau}{25} = \frac{4\tau^2}{5} - \frac{264\tau}{25}, \end{aligned}$$

which also holds at $\tau = 0$. Adding the two parts,

$$G_x(-2,3+\tau) = 11 + \Big(37 - \frac{264}{25}\Big)\tau + \Big(\frac{4}{5} - \frac{9}{2}\Big)\tau^2 = 11 + \frac{661}{25}\tau - \frac{37}{10}\tau^2,$$

so

$$G_{xy}(-2,3) = \frac{\mathrm{d}}{\mathrm{d}\tau} G_x(-2,3+\tau)\Big|_{\tau=0} = \frac{661}{25} \neq 37.$$

The polynomial $F_0$ is $C^2$ and also satisfies every datum, and it gives 37. Two functions satisfying all the stated properties give different values, so on the literal data

$$\boxed{\text{underdetermined}}.$$

Unofficial solutions

**Method 2: Restriction to a line; $g_y$ first and Clairaut's Theorem**

(a) By definition, $(D_{\mathbf{u}}f)(p) = \varphi'(0)$ where $\varphi(\tau) = f(p + \tau\mathbf{u})$. Let $a = \frac{1}{\sqrt{6}}$, so $\mathbf{u} = (a, 2a, -a)$ and $p + \tau\mathbf{u} = (1 + a\tau,\ -1 + 2a\tau,\ 1 - a\tau)$. Then

$$\varphi(\tau) = (1 - a\tau)^2 - (1 + a\tau)^3(-1 + 2a\tau)^2.$$

By the Product Rule and the one-variable Chain Rule,

$$\varphi'(\tau) = -2a(1 - a\tau) - \Big[3a(1 + a\tau)^2(-1 + 2a\tau)^2 + (1 + a\tau)^3 \cdot 2(-1 + 2a\tau) \cdot 2a\Big].$$

At $\tau = 0$:

$$\varphi'(0) = -2a - \big[3a \cdot 1 \cdot 1 + 1 \cdot 2 \cdot (-1) \cdot 2a\big] = -2a - (3a - 4a) = -a.$$

Therefore

$$\boxed{(D_{\mathbf{u}}f)(1, -1, 1) = -\frac{1}{\sqrt{6}}.}$$

(b) Use the notation of Method 1: $s = x^2 - y$, $t = x + 2y$, $(s, t) = (1, 4)$ at $(-2, 3)$, and $f(1, 4) = -3$, $f_x(1, 4) = 3$, $f_y(1, 4) = 5$ from the tangent plane. The regularity check of Method 1 applies unchanged, so work under (H): $f \in C^2$ near $(1, 4)$.

*Differentiate in $y$ first.* By the Chain Rule with $s_y = -1$, $t_y = 2$,

$$g_y = x\big[f_x(s, t)\, s_y + f_y(s, t)\, t_y\big] = x\big(-f_x(s, t) + 2f_y(s, t)\big).$$

*Then in $x$.* By the Product Rule, and the Chain Rule on $f_x$, $f_y$ with $s_x = 2x$, $t_x = 1$:

$$\frac{\partial}{\partial x} f_x(s, t) = f_{xx} \cdot 2x + f_{xy} \cdot 1, \qquad \frac{\partial}{\partial x} f_y(s, t) = f_{yx} \cdot 2x + f_{yy} \cdot 1,$$
$$g_{yx} = \big(-f_x + 2f_y\big) + x\Big[-\big(2xf_{xx} + f_{xy}\big) + 2\big(2xf_{yx} + f_{yy}\big)\Big].$$

At $(-2, 3)$:

$$2xf_{xx} + f_{xy} = (-4)(-1) + 1 = 5, \qquad 2xf_{yx} + f_{yy} = (-4)(1) + (-1) = -5,$$
$$g_{yx}(-2, 3) = 7 + (-2)\big[-5 + 2(-5)\big] = 7 + (-2)(-15) = 37.$$

*Clairaut's Theorem.* Under (H), $f \in C^2$ near $(1, 4)$ and $s, t$ are polynomials, so $g = x\, f(s, t)$ is $C^2$ on an open neighbourhood of $(-2, 3)$. Clairaut's Theorem then gives $g_{xy} = g_{yx}$ there, so $g_{xy}(-2, 3) = 37$:

$$\boxed{g_{xy}(-2, 3) = 37 \quad \text{if } f \text{ is } C^2 \text{ near } (1, 4).}$$

Without (H), the function $F = F_0 + Q$ of Method 1 satisfies every datum and gives $G_{xy}(-2, 3) = \frac{661}{25}$, while $F_0$ gives 37. Hence on the literal data

$$\boxed{\text{underdetermined}}.$$

**Method 3: Second-order Taylor polynomial and coefficient comparison**

This method applies to part (b) (uses the second-order Taylor expansion of Chapter 3 at a non-stationary point).

From the tangent plane, $f(1,4) = -3$, $f_x(1,4) = 3$, $f_y(1,4) = 5$. Work under (H): $f \in C^2$ near $(1,4)$ (Method 1 explains why some such hypothesis is needed). Taylor's Theorem at $(1,4)$ gives

$$\begin{aligned} f(1+U,4+V) &= f(1,4) + f_x U + f_y V + \tfrac{1}{2}\big(f_{xx}U^2 + 2f_{xy}UV + f_{yy}V^2\big) + o(U^2+V^2) \\ &= -3 + 3U + 5V - \tfrac{1}{2}U^2 + UV - \tfrac{1}{2}V^2 + o(U^2+V^2). \end{aligned}$$

Let $x = -2+\sigma$, $y = 3+\tau$. Then

$$\begin{aligned} U &= s - 1 = (-2+\sigma)^2 - (3+\tau) - 1 = -4\sigma - \tau + \sigma^2, \\ V &= t - 4 = (-2+\sigma) + 2(3+\tau) - 4 = \sigma + 2\tau. \end{aligned}$$

Since $|U| + |V| \le C(|\sigma| + |\tau|)$ near $(0,0)$, every $o(U^2+V^2)$ term is $o(\sigma^2+\tau^2)$, and so is every term of degree at least 3 in $\sigma, \tau$. Up to such terms:

$$\begin{aligned} 3U + 5V &= -12\sigma - 3\tau + 3\sigma^2 + 5\sigma + 10\tau = -7\sigma + 7\tau + 3\sigma^2, \\ U^2 &= 16\sigma^2 + 8\sigma\tau + \tau^2, \\ UV &= (-4\sigma - \tau)(\sigma + 2\tau) = -4\sigma^2 - 9\sigma\tau - 2\tau^2, \\ V^2 &= \sigma^2 + 4\sigma\tau + 4\tau^2, \\ -\tfrac{1}{2}U^2 + UV - \tfrac{1}{2}V^2 &= -\tfrac{25}{2}\sigma^2 - 15\sigma\tau - \tfrac{9}{2}\tau^2. \end{aligned}$$

Hence

$$f(s,t) = -3 - 7\sigma + 7\tau - \tfrac{19}{2}\sigma^2 - 15\sigma\tau - \tfrac{9}{2}\tau^2 + o(\sigma^2+\tau^2).$$

Multiply by $x = -2+\sigma$:

$$\begin{aligned} -2f(s,t) &= 6 + 14\sigma - 14\tau + 19\sigma^2 + 30\sigma\tau + 9\tau^2 + o(\sigma^2+\tau^2), \\ \sigma f(s,t) &= -3\sigma - 7\sigma^2 + 7\sigma\tau + o(\sigma^2+\tau^2), \\ g(-2+\sigma, 3+\tau) &= 6 + 11\sigma - 14\tau + 12\sigma^2 + 37\sigma\tau + 9\tau^2 + o(\sigma^2+\tau^2). \end{aligned}$$

Under (H), $g$ is $C^2$ near $(-2,3)$ (so $g_{xy} = g_{yx}$ by Clairaut), and its own Taylor expansion is

$$g(-2+\sigma, 3+\tau) = g + g_x\sigma + g_y\tau + \tfrac{1}{2}\big(g_{xx}\sigma^2 + 2g_{xy}\sigma\tau + g_{yy}\tau^2\big) + o(\sigma^2+\tau^2),$$

with derivatives at $(-2,3)$. Two quadratic polynomials whose difference is $o(\sigma^2+\tau^2)$ coincide, so comparing the coefficients of $\sigma\tau$ gives $g_{xy}(-2,3) = 37$ (and, as a check, $g_x(-2,3) = 11$, $g_y(-2,3) = -14$):

$$\boxed{g_{xy}(-2,3) = 37 \quad \text{if } f \text{ is } C^2 \text{ near } (1,4).}$$

Without (H), the function $F = F_0 + Q$ of Method 1 satisfies every datum and gives $\frac{661}{25}$ instead of 37, so on the literal data

$$\boxed{\text{underdetermined}}.$$

**Marking scheme (25 marks)**

(a) **(10 marks)**

- 3 marks: finds and normalizes the direction vector.

- 4 marks: computes $\operatorname{grad} f(1,-1,1) = (-3,2,2)$.
- 2 marks: applies the directional-derivative formula with justification.
- 1 mark: obtains $-1/\sqrt{6}$.

(b) **(15 marks)**

- 3 marks: extracts $f_x(1,4) = 3$ and $f_y(1,4) = 5$ from the tangent plane.
- 2 marks: identifies $(x^2 - y, x + 2y) = (1,4)$ at $(-2,3)$.
- 3 marks: explains that the stated data do not justify the second-order chain rule.
- 4 marks: gives a valid counterexample or otherwise establishes that the literal question is underdetermined.
- 3 marks: under an explicit $C^2$-neighbourhood assumption, differentiates $g_x$, applies the chain rule correctly, and obtains 37.

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# Question 3 (25 Marks)

**PROBLEM 3.**

A function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\forall x, y \in \mathbb{R} : \qquad f(x, y) \stackrel{\text{df}}{=} \mathrm{e}^{-xy}.$$

Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 \leq 1\}$.

(a) Determine all the critical point(s) of $f|_C$. **(5 points)**

**Note:** Every critical point of $f|_C$ belongs – by definition – to the interior of $C$.

(b) Determine the nature of each critical point of $f|_C$. **(5 points)**

**Note:** Apply the Second-Derivative Test.

(c) Solve the Lagrange-multiplier problem for minimizing/maximizing $f$ on

$$\mathrm{Bd}(C) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 4y^2 = 1\},$$

which is the boundary of $C$ in $\mathbb{R}^2$. **(10 points)**

**Note:** You are required to apply the Method of Lagrange Multipliers.

(d) Explain why $f|_C$ attains an absolute minimum/maximum. **(3 points)**

(e) Determine the absolute minimum and absolute maximum of $f|_C$. **(2 points)**

## Solution

**Method 1: Gradient, Second-Derivative Test, Lagrange multipliers and the Extreme Value Theorem**

(a) The first partial derivatives of $f(x, y) = \mathrm{e}^{-xy}$ are

$$f_x = \frac{\partial}{\partial x}\mathrm{e}^{-xy} = -y\,\mathrm{e}^{-xy}, \qquad f_y = \frac{\partial}{\partial y}\mathrm{e}^{-xy} = -x\,\mathrm{e}^{-xy}.$$

Since $\mathrm{e}^{-xy} > 0$ for all $(x, y)$,

$$f_x = 0 \iff y = 0, \qquad f_y = 0 \iff x = 0,$$

so $\operatorname{grad} f(x, y) = (0, 0)$ only at $(0, 0)$. As $0^2 + 4 \cdot 0^2 = 0 < 1$, the point $(0, 0)$ lies in $\operatorname{int} C = \{x^2 + 4y^2 < 1\}$. Therefore

$$\boxed{\text{The only critical point of } f|_C \text{ is } (0, 0).}$$

(b) $f$ is a composition of exp with a polynomial, so $f \in C^2(\mathbb{R}^2)$. The second partial derivatives are

$$f_{xx} = \frac{\partial}{\partial x}\big(-y\,\mathrm{e}^{-xy}\big) = -y \cdot (-y)\,\mathrm{e}^{-xy} = y^2\mathrm{e}^{-xy},$$
$$f_{yy} = \frac{\partial}{\partial y}\big(-x\,\mathrm{e}^{-xy}\big) = -x \cdot (-x)\,\mathrm{e}^{-xy} = x^2\mathrm{e}^{-xy},$$
$$f_{xy} = \frac{\partial}{\partial y}\big(-y\,\mathrm{e}^{-xy}\big) = -\mathrm{e}^{-xy} + (-y)(-x)\,\mathrm{e}^{-xy} = (xy - 1)\,\mathrm{e}^{-xy}.$$

At $(0,0)$:

$$f_{xx}(0,0)=0, \qquad f_{yy}(0,0)=0, \qquad f_{xy}(0,0)=(0-1)\mathrm{e}^{0}=-1,$$

so

$$D=f_{xx}(0,0)f_{yy}(0,0)-f_{xy}(0,0)^2=0\cdot 0-(-1)^2=-1<0.$$

The Second-Derivative Test applies ($f\in C^2$ near $(0,0)$ and $\operatorname{grad} f(0,0)=0$), and $D<0$ gives

$$\boxed{(0,0) \text{ is a saddle point of } f|_C.}$$

(c) Let $h(x,y)=x^2+4y^2$, so $\mathrm{Bd}(C)=\{h=1\}$. Then $\operatorname{grad} h=(2x,8y)$, which vanishes only at $(0,0)$; since $h(0,0)=0\neq 1$, $\operatorname{grad} h\neq 0$ on $\mathrm{Bd}(C)$. As $f,h\in C^1$, the Lagrange condition holds at every local extremum of $f$ on $\mathrm{Bd}(C)$: there is $\lambda\in\mathbb{R}$ with $\operatorname{grad} f=\lambda\operatorname{grad} h$. The system is

$$\begin{aligned} -y\,\mathrm{e}^{-xy}&=2\lambda x, && (1)\\ -x\,\mathrm{e}^{-xy}&=8\lambda y, && (2)\\ x^2+4y^2&=1. && (3)\end{aligned}$$

*Case* $x=0$. Then (1) reads $-y=0$, so $y=0$, and (3) reads $0=1$: impossible.

*Case* $y=0$. Then (2) reads $-x=0$, so $x=0$, and (3) reads $0=1$: impossible.

*Case* $x\neq 0$, $y\neq 0$. The left side of (1) is nonzero, so $\lambda\neq 0$. Dividing (1) by (2):

$$\frac{-y\,\mathrm{e}^{-xy}}{-x\,\mathrm{e}^{-xy}}=\frac{2\lambda x}{8\lambda y}\implies \frac{y}{x}=\frac{x}{4y}\implies x^2=4y^2.$$

Substituting into (3):

$$x^2+x^2=1\implies x^2=\frac{1}{2}\implies x=\pm\frac{1}{\sqrt{2}}, \qquad y^2=\frac{x^2}{4}=\frac{1}{8}\implies y=\pm\frac{1}{2\sqrt{2}}.$$

From (1), $\lambda=-\dfrac{y\,\mathrm{e}^{-xy}}{2x}$. The four candidates are:

$$\begin{aligned} P_1&=\left(\tfrac{1}{\sqrt{2}},\tfrac{1}{2\sqrt{2}}\right): && xy=\tfrac{1}{4}, && f(P_1)=\mathrm{e}^{-1/4}, && \lambda=-\tfrac{1}{4}\mathrm{e}^{-1/4},\\ P_2&=\left(-\tfrac{1}{\sqrt{2}},-\tfrac{1}{2\sqrt{2}}\right): && xy=\tfrac{1}{4}, && f(P_2)=\mathrm{e}^{-1/4}, && \lambda=-\tfrac{1}{4}\mathrm{e}^{-1/4},\\ P_3&=\left(\tfrac{1}{\sqrt{2}},-\tfrac{1}{2\sqrt{2}}\right): && xy=-\tfrac{1}{4}, && f(P_3)=\mathrm{e}^{1/4}, && \lambda=\tfrac{1}{4}\mathrm{e}^{1/4},\\ P_4&=\left(-\tfrac{1}{\sqrt{2}},\tfrac{1}{2\sqrt{2}}\right): && xy=-\tfrac{1}{4}, && f(P_4)=\mathrm{e}^{1/4}, && \lambda=\tfrac{1}{4}\mathrm{e}^{1/4}.\end{aligned}$$

($\lambda$ at $P_1$: $-\frac{1/(2\sqrt{2})}{2/\sqrt{2}}\mathrm{e}^{-1/4}=-\frac{1}{4}\mathrm{e}^{-1/4}$; the others are computed the same way.) $\mathrm{Bd}(C)$ is compact and $f$ is continuous, so $f|_{\mathrm{Bd}(C)}$ attains a minimum $\mathrm{e}^{-1/4}$ (at $P_1,P_2$) and a maximum $\mathrm{e}^{1/4}$ (at $P_3,P_4$). The solutions of the Lagrange problem are

$$\boxed{\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{2\sqrt{2}}\right) \text{ with matched signs, and } \left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{2\sqrt{2}}\right).}$$

(d) $f$ is continuous on $\mathbb{R}^2$ (composition of the continuous maps $(x, y) \mapsto -xy$ and exp). $C = h^{-1}\big((-\infty, 1]\big)$ is closed, as the preimage of a closed set under the continuous $h$. $C$ is bounded, since $x^2 \le 1$ and $y^2 \le \frac{1}{4}$ on $C$, so $C \subseteq [-1, 1] \times [-\frac{1}{2}, \frac{1}{2}]$. Hence $C$ is compact, and the Extreme Value Theorem gives

$$\boxed{f|_C \text{ attains both an absolute minimum and an absolute maximum.}}$$

(e) By (d) the absolute extrema exist. Each lies either at an interior critical point or on $\mathrm{Bd}(C)$. The interior candidate is $(0, 0)$, with

$$f(0, 0) = \mathrm{e}^0 = 1$$

(and by (b) it is a saddle point, not an extremum). On $\mathrm{Bd}(C)$, by (c), the values range between $\mathrm{e}^{-1/4}$ and $\mathrm{e}^{1/4}$. Comparing,

$$\mathrm{e}^{-1/4} < 1 < \mathrm{e}^{1/4}.$$

Therefore

$$\boxed{\min_C f = \mathrm{e}^{-1/4} \text{ at } \left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right),}$$

$$\boxed{\max_C f = \mathrm{e}^{1/4} \text{ at } \left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right).}$$

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**Method 2: Restriction to lines; parametrising the constraint (elliptic polar coordinates)**

For part (c) the paper requires the Method of Lagrange Multipliers; the parametrisation below is a check on the same candidates.

(b) $(0,0)$ is a critical point by (a), and $f(0,0)=1$. For $t\neq 0$,

$$f(t,t)=\mathrm{e}^{-t^2}<1, \qquad f(t,-t)=\mathrm{e}^{t^2}>1.$$

So every disc centred at $(0,0)$ contains points where $f<f(0,0)$ and points where $f>f(0,0)$; $(0,0)$ is neither a local minimum nor a local maximum. Therefore

$$\boxed{(0,0) \text{ is a saddle point of } f|_C.}$$

(c) Parametrise $\mathrm{Bd}(C)$ by $\gamma(\theta)=\left(\cos\theta, \frac{1}{2}\sin\theta\right)$, $0\le\theta<2\pi$: indeed $\cos^2\theta+4\cdot\frac{1}{4}\sin^2\theta=1$, and every $(x,y)$ with $x^2+(2y)^2=1$ is of this form. Along $\gamma$,

$$xy=\tfrac{1}{2}\sin\theta\cos\theta=\tfrac{1}{4}\sin 2\theta, \qquad F(\theta)=f(\gamma(\theta))=\mathrm{e}^{-\frac{1}{4}\sin 2\theta}.$$

By the Chain Rule,

$$F'(\theta)=-\tfrac{1}{4}\cdot 2\cos 2\theta\ \mathrm{e}^{-\frac{1}{4}\sin 2\theta}=-\tfrac{1}{2}\cos 2\theta\ \mathrm{e}^{-\frac{1}{4}\sin 2\theta}.$$

Since the exponential is positive, $F'(\theta)=0 \iff \cos 2\theta=0 \iff \theta\in\left\{\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}\right\}$. Evaluating:

| | | | |
|---|---|---|---|
| $\theta=\frac{\pi}{4}$ : | $\gamma=\left(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$, | $\sin 2\theta=1$, | $F=\mathrm{e}^{-1/4}$, |
| $\theta=\frac{3\pi}{4}$ : | $\gamma=\left(-\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$, | $\sin 2\theta=-1$, | $F=\mathrm{e}^{1/4}$, |
| $\theta=\frac{5\pi}{4}$ : | $\gamma=\left(-\frac{1}{\sqrt{2}},-\frac{1}{2\sqrt{2}}\right)$, | $\sin 2\theta=1$, | $F=\mathrm{e}^{-1/4}$, |
| $\theta=\frac{7\pi}{4}$ : | $\gamma=\left(\frac{1}{\sqrt{2}},-\frac{1}{2\sqrt{2}}\right)$, | $\sin 2\theta=-1$, | $F=\mathrm{e}^{1/4}$. |

$F$ is continuous and $2\pi$-periodic, so its extrema occur at these stationary points. They are the same four points as the Lagrange system:

$$\boxed{\left(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{2\sqrt{2}}\right) \text{ with matched signs, and } \left(\pm\frac{1}{\sqrt{2}},\mp\frac{1}{2\sqrt{2}}\right).}$$

(e) Cover all of $C$ by $x=r\cos\theta$, $y=\frac{r}{2}\sin\theta$ with $0\le r\le 1$, $0\le\theta<2\pi$ (then $x^2+4y^2=r^2\le 1$). Then

$$xy=\tfrac{r^2}{2}\sin\theta\cos\theta=\tfrac{r^2}{4}\sin 2\theta.$$

Since $0\le r^2\le 1$ and $-1\le\sin 2\theta\le 1$,

$$-\tfrac{1}{4}\le xy\le\tfrac{1}{4},$$

with $xy=\frac{1}{4}$ exactly when $r=1$ and $\sin 2\theta=1$ ($\theta=\frac{\pi}{4},\frac{5\pi}{4}$), and $xy=-\frac{1}{4}$ exactly when $r=1$ and $\sin 2\theta=-1$ ($\theta=\frac{3\pi}{4},\frac{7\pi}{4}$). As $u\mapsto\mathrm{e}^{-u}$ is strictly decreasing,

$$\mathrm{e}^{-1/4}\le\mathrm{e}^{-xy}\le\mathrm{e}^{1/4} \quad \text{on } C,$$

with the bounds attained at the points listed in (c). Therefore

$$\boxed{\min_C f=\mathrm{e}^{-1/4} \text{ at } \left(\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}}\right),\left(-\frac{1}{\sqrt{2}},-\frac{1}{2\sqrt{2}}\right),}$$

$$\boxed{\max_C f=\mathrm{e}^{1/4} \text{ at } \left(\frac{1}{\sqrt{2}},-\frac{1}{2\sqrt{2}}\right),\left(-\frac{1}{\sqrt{2}},\frac{1}{2\sqrt{2}}\right).}$$

**Method 3: Completing the square**

(d) For all $(x, y) \in \mathbb{R}^2$,

$$0 \le (x-2y)^2 = x^2 - 4xy + 4y^2 \implies 4xy \le x^2 + 4y^2,$$
$$0 \le (x+2y)^2 = x^2 + 4xy + 4y^2 \implies -4xy \le x^2 + 4y^2.$$

On $C$, $x^2 + 4y^2 \le 1$, so

$$-\tfrac{1}{4} \le xy \le \tfrac{1}{4} \quad \text{on } C.$$

$xy = \frac{1}{4}$ requires $x^2+4y^2 = 1$ and $x-2y = 0$; then $8y^2 = 1$, so $y = \pm\frac{1}{2\sqrt{2}}$ and $x = 2y = \pm\frac{1}{\sqrt{2}}$ (matched signs). $xy = -\frac{1}{4}$ requires $x^2 + 4y^2 = 1$ and $x + 2y = 0$; then $y = \pm\frac{1}{2\sqrt{2}}$ and $x = -2y = \mp\frac{1}{\sqrt{2}}$. As $u \mapsto \mathrm{e}^{-u}$ is strictly decreasing,

$$\mathrm{e}^{-1/4} \le f(x, y) \le \mathrm{e}^{1/4} \quad \text{on } C,$$

and both bounds are values of $f$ at points of $C$. This exhibits the extremal points directly, without the Extreme Value Theorem:

$\boxed{f|_C \text{ attains both an absolute minimum and an absolute maximum.}}$

(e) From (d), the minimum $\mathrm{e}^{-1/4}$ is attained exactly where $x = 2y$ on $\mathrm{Bd}(C)$, and the maximum $\mathrm{e}^{1/4}$ exactly where $x = -2y$ on $\mathrm{Bd}(C)$:

$$\boxed{\min_C f = \mathrm{e}^{-1/4} \text{ at } \left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right),}$$

$$\boxed{\max_C f = \mathrm{e}^{1/4} \text{ at } \left(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right).}$$

**Marking scheme (25 marks)**

(a) **(5 marks)**

- 3 marks: differentiates and solves $\operatorname{grad} f = 0$.
- 2 marks: identifies $(0, 0)$ as the sole interior critical point.

(b) **(5 marks)**

- 3 marks: computes the required second derivatives at $(0, 0)$.
- 1 mark: obtains discriminant $-1$.
- 1 mark: concludes that the point is a saddle point.

(c) **(10 marks)**

- 3 marks: states the Lagrange system and boundary constraint.
- 3 marks: derives $x^2 = 4y^2$.
- 3 marks: finds all four boundary points.
- 1 mark: gives the associated function values or multipliers correctly.

(d) **(3 marks)**

- 1 mark: states continuity of $f$.
- 1 mark: states compactness of $C$.
- 1 mark: invokes the Extreme Value Theorem.

(e) **(2 marks)**

- 1 mark: identifies the absolute minimum and its points.
- 1 mark: identifies the absolute maximum and its points.

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# Question 4 (25 Marks)

**PROBLEM 4.**

(a) Evaluate

$$\int_0^1 \left[ \int_0^{\sqrt{(1-x)/3}} \mathrm{e}^{y-y^3}\,\mathrm{d}y \right] \mathrm{d}x. \qquad \textbf{(10 points)}$$

**Hint:** Perform a change of order of integration.

(b) Evaluate

$$\int_0^2 \left[ \int_0^{\sqrt{2y-y^2}} \sqrt{x^2+y^2}\,\mathrm{d}x \right] \mathrm{d}y. \qquad \textbf{(15 points)}$$

**Hint:** Perform a transformation from Cartesian to polar coordinates.

## Solution

**Method 1: Reversing the order (Fubini) with substitution; polar coordinates, $r$ first**

(a) *Region in both orders.* The given order describes

$$R = \left\{ (x,y) \;\middle|\; 0 \le x \le 1,\ 0 \le y \le \sqrt{\tfrac{1-x}{3}} \right\}.$$

For $y \ge 0$,

$$y \le \sqrt{\tfrac{1-x}{3}} \iff y^2 \le \tfrac{1-x}{3} \iff x \le 1 - 3y^2.$$

The largest $y$ occurs at $x = 0$, where $y = \frac{1}{\sqrt{3}}$; and $0 \le x \le 1 - 3y^2$ is nonempty exactly when $0 \le y \le \frac{1}{\sqrt{3}}$ (the condition $x \le 1$ then holds automatically). Hence

$$R = \left\{ (x,y) \;\middle|\; 0 \le y \le \tfrac{1}{\sqrt{3}},\ 0 \le x \le 1 - 3y^2 \right\}.$$

*Fubini's Theorem.* The integrand $\mathrm{e}^{y-y^3}$ is continuous on $R$, and $R$ is bounded by graphs of continuous functions in both descriptions, so both iterated integrals equal $\iint_R \mathrm{e}^{y-y^3}\,\mathrm{d}A$:

$$\int_0^1 \int_0^{\sqrt{(1-x)/3}} \mathrm{e}^{y-y^3}\,\mathrm{d}y\,\mathrm{d}x = \int_0^{1/\sqrt{3}} \int_0^{1-3y^2} \mathrm{e}^{y-y^3}\,\mathrm{d}x\,\mathrm{d}y.$$

*Inner integral* ($\mathrm{e}^{y-y^3}$ is constant in $x$):

$$\int_0^{1-3y^2} \mathrm{e}^{y-y^3}\,\mathrm{d}x = \mathrm{e}^{y-y^3}\big[x\big]_0^{1-3y^2} = (1-3y^2)\,\mathrm{e}^{y-y^3}.$$

*Outer integral.* Substitute $u = y - y^3$, so $\mathrm{d}u = (1-3y^2)\,\mathrm{d}y$. New limits: $y = 0 \mapsto u = 0$, and

$$y = \tfrac{1}{\sqrt{3}} \mapsto u = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{3-1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}.$$

Hence

$$\int_0^{1/\sqrt{3}} (1-3y^2)\,\mathrm{e}^{y-y^3}\,\mathrm{d}y = \int_0^{2/(3\sqrt{3})} \mathrm{e}^u\,\mathrm{d}u = \Big[\mathrm{e}^u\Big]_0^{2/(3\sqrt{3})} = \mathrm{e}^{2/(3\sqrt{3})} - \mathrm{e}^0.$$

Therefore

$$\boxed{\mathrm{e}^{2/(3\sqrt{3})} - 1.}$$

(b) *Region.* The given order describes

$$S = \Big\{(x,y) \;\Big|\; 0 \le y \le 2,\ 0 \le x \le \sqrt{2y - y^2}\Big\}.$$

For $x \ge 0$,

$$x \le \sqrt{2y-y^2} \iff x^2 \le 2y - y^2 \iff x^2 + y^2 - 2y \le 0 \iff x^2 + (y-1)^2 \le 1,$$

and $2y - y^2 \ge 0$ exactly for $0 \le y \le 2$. So $S$ is the right half ($x \ge 0$) of the disc with centre $(0,1)$ and radius 1.

*Polar coordinates.* Let $x = r\cos\theta$, $y = r\sin\theta$. The Jacobian determinant is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r\cos^2\theta + r\sin^2\theta = r,$$

so $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$ for $r \ge 0$. The disc condition becomes

$$x^2 + y^2 \le 2y \iff r^2 \le 2r\sin\theta \iff r \le 2\sin\theta \quad (r > 0).$$

The condition $x \ge 0$ is $\cos\theta \ge 0$, and $r \le 2\sin\theta$ with $r > 0$ needs $\sin\theta > 0$. Hence

$$S = \big\{(r,\theta) \;\big|\; 0 \le \theta \le \tfrac{\pi}{2},\ 0 \le r \le 2\sin\theta\big\},$$

and the integrand is $\sqrt{x^2+y^2} = \sqrt{r^2} = r$. Thus

$$\iint_S \sqrt{x^2+y^2}\,\mathrm{d}A = \int_0^{\pi/2}\int_0^{2\sin\theta} r \cdot r\,\mathrm{d}r\,\mathrm{d}\theta.$$

*Inner integral:*

$$\int_0^{2\sin\theta} r^2\,\mathrm{d}r = \Big[\frac{r^3}{3}\Big]_0^{2\sin\theta} = \frac{8\sin^3\theta}{3}.$$

*Outer integral.* Write $\sin^3\theta = (1-\cos^2\theta)\sin\theta$ and substitute $w = \cos\theta$, $\mathrm{d}w = -\sin\theta\,\mathrm{d}\theta$; new limits $\theta = 0 \mapsto w = 1$, $\theta = \frac{\pi}{2} \mapsto w = 0$:

$$\int_0^{\pi/2} \sin^3\theta\,\mathrm{d}\theta = \int_1^0 (1-w^2)(-\mathrm{d}w) = \int_0^1 (1-w^2)\,\mathrm{d}w = \Big[w - \frac{w^3}{3}\Big]_0^1 = \frac{2}{3}.$$

Hence

$$\iint_S \sqrt{x^2+y^2}\,\mathrm{d}A = \frac{8}{3}\cdot\frac{2}{3} = \frac{16}{9}.$$

Therefore

$$\boxed{\frac{16}{9}.}$$

**Method 2: Integration by parts in the given order; polar coordinates, $\theta$ first**

(a) The hint asks for a change of order; this route keeps the given order and should be used only if the course allows this interpretation.

Let $\varphi(x) = \sqrt{\frac{1-x}{3}}$ and $G(x) = \int_0^{\varphi(x)} \mathrm{e}^{y-y^3}\,\mathrm{d}y$, so the integral is $\int_0^1 G(x)\,\mathrm{d}x$. $G$ is continuous on $[0,1]$ with $G(1) = \int_0^0 = 0$. On $[0,1)$,

$$\varphi'(x) = -\frac{1}{2\sqrt{3}\,\sqrt{1-x}}, \qquad G'(x) = \mathrm{e}^{\varphi(x)-\varphi(x)^3}\,\varphi'(x)$$

by the Fundamental Theorem of Calculus and the Chain Rule. For $0 < b < 1$, integrate by parts (with $\frac{\mathrm{d}}{\mathrm{d}x}x = 1$):

$$\int_0^b G(x)\,\mathrm{d}x = \Big[x\,G(x)\Big]_0^b - \int_0^b x\,G'(x)\,\mathrm{d}x = b\,G(b) - \int_0^b x\,\mathrm{e}^{\varphi(x)-\varphi(x)^3}\varphi'(x)\,\mathrm{d}x.$$

Substitute $y = \varphi(x)$: then $x = 1-3y^2$, $\mathrm{d}y = \varphi'(x)\,\mathrm{d}x$, and $x = 0 \mapsto y = \frac{1}{\sqrt{3}}$, $x = b \mapsto y = \varphi(b)$:

$$\int_0^b x\,\mathrm{e}^{\varphi(x)-\varphi(x)^3}\varphi'(x)\,\mathrm{d}x = \int_{1/\sqrt{3}}^{\varphi(b)} (1-3y^2)\,\mathrm{e}^{y-y^3}\,\mathrm{d}y.$$

Hence

$$\int_0^b G(x)\,\mathrm{d}x = b\,G(b) + \int_{\varphi(b)}^{1/\sqrt{3}} (1-3y^2)\,\mathrm{e}^{y-y^3}\,\mathrm{d}y.$$

Let $b \to 1^-$: $G(b) \to G(1) = 0$ and $\varphi(b) \to 0$, and the left side tends to $\int_0^1 G$ ($G$ is continuous on $[0,1]$). So

$$\int_0^1 G(x)\,\mathrm{d}x = \int_0^{1/\sqrt{3}} (1-3y^2)\,\mathrm{e}^{y-y^3}\,\mathrm{d}y.$$

By the Chain Rule, $\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{e}^{y-y^3} = (1-3y^2)\mathrm{e}^{y-y^3}$, so by the Fundamental Theorem of Calculus

$$\int_0^{1/\sqrt{3}} (1-3y^2)\,\mathrm{e}^{y-y^3}\,\mathrm{d}y = \Big[\mathrm{e}^{y-y^3}\Big]_0^{1/\sqrt{3}} = \mathrm{e}^{\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}}} - \mathrm{e}^0 = \mathrm{e}^{2/(3\sqrt{3})} - 1.$$

Therefore

$$\boxed{\mathrm{e}^{2/(3\sqrt{3})} - 1.}$$

(b) As in Method 1, $S = \{0 \le \theta \le \frac{\pi}{2},\ 0 \le r \le 2\sin\theta\}$ in polar coordinates, with $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$ and integrand $r$. Describe $S$ in the other order: for fixed $r \in [0,2]$ and $\theta \in [0, \frac{\pi}{2}]$,

$$r \le 2\sin\theta \iff \sin\theta \ge \frac{r}{2} \iff \theta \ge \arcsin\frac{r}{2},$$

because sin is increasing on $[0, \frac{\pi}{2}]$. Hence

$$S = \Big\{(r,\theta) \;\Big|\; 0 \le r \le 2,\ \arcsin\tfrac{r}{2} \le \theta \le \tfrac{\pi}{2}\Big\}.$$

By Fubini's Theorem (continuous integrand $r^2$ on a region bounded by graphs),

$$\iint_S \sqrt{x^2+y^2}\,\mathrm{d}A = \int_0^2 \int_{\arcsin(r/2)}^{\pi/2} r^2\,\mathrm{d}\theta\,\mathrm{d}r = \int_0^2 r^2\Big(\frac{\pi}{2} - \arcsin\frac{r}{2}\Big)\,\mathrm{d}r = \int_0^2 r^2 \arccos\frac{r}{2}\,\mathrm{d}r,$$

using $\frac{\pi}{2} - \arcsin z = \arccos z$. Substitute $r = 2s,\ \mathrm{d}r = 2\,\mathrm{d}s$; new limits $r = 0 \mapsto s = 0$, $r = 2 \mapsto s = 1$:

$$\int_0^2 r^2 \arccos\frac{r}{2}\,\mathrm{d}r = \int_0^1 4s^2 \arccos s \cdot 2\,\mathrm{d}s = 8\int_0^1 s^2 \arccos s\,\mathrm{d}s.$$

Integrate by parts with $p = \arccos s,\ \mathrm{d}q = s^2\,\mathrm{d}s$, so $\mathrm{d}p = -\frac{\mathrm{d}s}{\sqrt{1-s^2}}$, $q = \frac{s^3}{3}$:

$$\int_0^1 s^2 \arccos s\,\mathrm{d}s = \Big[\frac{s^3}{3}\arccos s\Big]_0^1 + \frac{1}{3}\int_0^1 \frac{s^3}{\sqrt{1-s^2}}\,\mathrm{d}s = \Big(\frac{1}{3}\cdot 0 - 0\Big) + \frac{1}{3}J,$$

where $J$ is a convergent improper integral. For $J$, substitute $w = 1 - s^2,\ \mathrm{d}w = -2s\,\mathrm{d}s$, $s^2 = 1 - w$; new limits $s = 0 \mapsto w = 1$, $s = 1 \mapsto w = 0$:

$$J = \int_1^0 \frac{1-w}{\sqrt{w}}\Big(-\frac{\mathrm{d}w}{2}\Big) = \frac{1}{2}\int_0^1 \big(w^{-1/2} - w^{1/2}\big)\,\mathrm{d}w = \frac{1}{2}\Big[2w^{1/2} - \frac{2}{3}w^{3/2}\Big]_0^1 = \frac{1}{2}\Big(2 - \frac{2}{3}\Big) = \frac{2}{3}.$$

Hence $\int_0^1 s^2 \arccos s\,\mathrm{d}s = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$, and the integral is $8 \cdot \frac{2}{9} = \frac{16}{9}$:

$$\boxed{\frac{16}{9}.}$$

Unofficial QRS@NTU Question Bank

**Method 3: Cartesian evaluation with the antiderivative of $\sqrt{x^2+y^2}$**

This method applies to part (b). The hint asks for polar coordinates; this route should be used only if the course allows this interpretation.

For fixed $y>0$, an antiderivative in $x$ is

$$\Phi(x)=\frac{x}{2}\sqrt{x^2+y^2}+\frac{y^2}{2}\ln\Big(x+\sqrt{x^2+y^2}\Big).$$

Check by differentiation, writing $\rho=\sqrt{x^2+y^2}$ and $\frac{\mathrm{d}\rho}{\mathrm{d}x}=\frac{x}{\rho}$:

$$\Phi'(x)=\frac{\rho}{2}+\frac{x^2}{2\rho}+\frac{y^2}{2}\cdot\frac{1+\frac{x}{\rho}}{x+\rho}=\frac{\rho}{2}+\frac{x^2}{2\rho}+\frac{y^2}{2\rho}=\frac{\rho}{2}+\frac{\rho^2}{2\rho}=\rho.$$

*Inner integral.* At the upper limit $X=\sqrt{2y-y^2}$ we have $X^2+y^2=2y$, so $\sqrt{X^2+y^2}=\sqrt{2y}$; at the lower limit $x=0$, $\sqrt{0+y^2}=y$. Hence

$$\int_0^X\sqrt{x^2+y^2}\,\mathrm{d}x=\frac{X}{2}\sqrt{2y}+\frac{y^2}{2}\Big[\ln\big(X+\sqrt{2y}\big)-\ln y\Big].$$

Simplify both terms using $X=\sqrt{y}\,\sqrt{2-y}$:

$$X\sqrt{2y}=\sqrt{y(2-y)}\,\sqrt{2y}=y\sqrt{4-2y},\qquad X+\sqrt{2y}=\sqrt{y}\big(\sqrt{2-y}+\sqrt{2}\big),$$

so, with $L(y)=\ln\big(\sqrt{2-y}+\sqrt{2}\big)-\frac{1}{2}\ln y$,

$$\int_0^X\sqrt{x^2+y^2}\,\mathrm{d}x=\frac{y}{2}\sqrt{4-2y}+\frac{y^2}{2}L(y).$$

The required integral is $A+B$ with

$$A=\int_0^2\frac{y}{2}\sqrt{4-2y}\,\mathrm{d}y,\qquad B=\int_0^2\frac{y^2}{2}L(y)\,\mathrm{d}y.$$

*The integral $A$.* Substitute $w=4-2y$, $y=\frac{4-w}{2}$, $\mathrm{d}y=-\frac{\mathrm{d}w}{2}$; new limits $y=0\mapsto w=4$, $y=2\mapsto w=0$:

$$\begin{aligned}A&=\int_4^0\frac{4-w}{4}\sqrt{w}\Big(-\frac{\mathrm{d}w}{2}\Big)=\frac{1}{8}\int_0^4\big(4w^{1/2}-w^{3/2}\big)\,\mathrm{d}w=\frac{1}{8}\Big[\frac{8}{3}w^{3/2}-\frac{2}{5}w^{5/2}\Big]_0^4\\&=\frac{1}{8}\Big(\frac{8}{3}\cdot 8-\frac{2}{5}\cdot 32\Big)=\frac{1}{8}\Big(\frac{64}{3}-\frac{64}{5}\Big)=\frac{1}{8}\cdot\frac{128}{15}=\frac{16}{15}.\end{aligned}$$

*The integral $B$.* Differentiate $L$:

$$L'(y)=\frac{-\frac{1}{2\sqrt{2-y}}}{\sqrt{2-y}+\sqrt{2}}-\frac{1}{2y}.$$

Since $\big(\sqrt{2}+\sqrt{2-y}\big)\big(\sqrt{2}-\sqrt{2-y}\big)=2-(2-y)=y$, we have $\frac{1}{\sqrt{2-y}+\sqrt{2}}=\frac{\sqrt{2}-\sqrt{2-y}}{y}$, so

$$L'(y)=-\frac{\sqrt{2}-\sqrt{2-y}}{2y\sqrt{2-y}}-\frac{1}{2y}=-\frac{\sqrt{2}}{2y\sqrt{2-y}}+\frac{1}{2y}-\frac{1}{2y}=-\frac{\sqrt{2}}{2y\sqrt{2-y}}.$$

Integrate $B$ by parts with $p = L(y)$, $\mathrm{d}q = \frac{y^2}{2}\,\mathrm{d}y$, $q = \frac{y^3}{6}$. The boundary term vanishes: $L(2) = \ln\sqrt{2} - \frac{1}{2}\ln 2 = 0$, and $y^3 L(y) \to 0$ as $y \to 0^+$ because $y^3 \ln y \to 0$. Hence

$$B = -\int_0^2 \frac{y^3}{6} L'(y)\,\mathrm{d}y = \int_0^2 \frac{y^3}{6} \cdot \frac{\sqrt{2}}{2y\sqrt{2-y}}\,\mathrm{d}y = \frac{\sqrt{2}}{12}\int_0^2 \frac{y^2}{\sqrt{2-y}}\,\mathrm{d}y,$$

a convergent improper integral at $y = 2$. Substitute $w = 2 - y$, $\mathrm{d}y = -\,\mathrm{d}w$; new limits $y = 0 \mapsto w = 2$, $y = 2 \mapsto w = 0$:

$$\begin{aligned}\int_0^2 \frac{y^2}{\sqrt{2-y}}\,\mathrm{d}y &= \int_0^2 \frac{(2-w)^2}{\sqrt{w}}\,\mathrm{d}w = \int_0^2 \left(4w^{-1/2} - 4w^{1/2} + w^{3/2}\right)\mathrm{d}w \\ &= \left[8w^{1/2} - \frac{8}{3}w^{3/2} + \frac{2}{5}w^{5/2}\right]_0^2 = 8\sqrt{2} - \frac{16\sqrt{2}}{3} + \frac{8\sqrt{2}}{5} = \frac{64\sqrt{2}}{15}.\end{aligned}$$

So $B = \frac{\sqrt{2}}{12} \cdot \frac{64\sqrt{2}}{15} = \frac{128}{180} = \frac{32}{45}$, and

$$A + B = \frac{16}{15} + \frac{32}{45} = \frac{48+32}{45} = \frac{80}{45} = \frac{16}{9}.$$

Therefore

$$\boxed{\frac{16}{9}.}$$

**Marking scheme (25 marks)**

(a) **(10 marks)**

- 3 marks: describes the region correctly after reversing the order.
- 2 marks: supplies the correct reversed bounds.
- 3 marks: integrates with respect to $x$ and recognizes the substitution $u = y - y^3$.
- 2 marks: evaluates the endpoints correctly.

(b) **(15 marks)**

- 3 marks: recognizes and states the Cartesian disk region.
- 4 marks: obtains the correct polar bounds $0 \le \theta \le \pi/2$, $0 \le r \le 2\sin\theta$.
- 3 marks: transforms both the integrand and area element correctly.
- 3 marks: evaluates the $r$-integral and $\int_0^{\pi/2} \sin^3\theta\,\mathrm{d}\theta$.
- 2 marks: obtains 16/9.