1. Curves, vectors, tangents

Parametric curve: $\boldsymbol{r} : I \to \mathbb{R}^n$, $\boldsymbol{r}(t) = (x_1(t), \dots, x_n(t))$; $\boldsymbol{r}' = (x_1', \dots, x_n')$, $\boldsymbol{v} = \boldsymbol{r}'$, speed $= \|\boldsymbol{r}'\|$, $\boldsymbol{a} = \boldsymbol{r}''$. Same image may have several parameters/directions.

Standard forms: Line $P + t\boldsymbol{v}$; circle $(a + R\cos t, b + R\sin t)$; ellipse $(a + A\cos t, b + B\sin t)$, $0 \le t < 2\pi$. State the parameter interval: it controls the traced image.

Dot / angle: $\boldsymbol{a} \cdot \boldsymbol{b} = \sum a_i b_i$; $\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$; for nonzero vectors, $\boldsymbol{a} \cdot \boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \cos\theta$. Perpendicular $\iff \boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Derivative rules: Differentiate vector functions componentwise; $(f\boldsymbol{u})' = f'\boldsymbol{u} + f\boldsymbol{u}'$. For planar $x = x(t), y = y(t)$ with $x' \neq 0$, $\mathrm{d}y/\mathrm{d}x = y'/x'$.

Acceleration data: $\boldsymbol{v} = \boldsymbol{r}'$, $\boldsymbol{a} = \boldsymbol{r}''$; speed is $\|\boldsymbol{v}\|$ and $\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\| = \boldsymbol{v} \cdot \boldsymbol{a}/\|\boldsymbol{v}\|$ when $\boldsymbol{v} \neq 0$.

Point vs parameter: A parametrization need not be one-to-one: a self-intersection has one geometric point but multiple t values. Tangent questions are about each parameter value, unless the question specifies a branch.

Secant direction: At t_0, use the normalized displacement $[\boldsymbol{r}(t) - \boldsymbol{r}(t_0)]/\|\boldsymbol{r}(t) - \boldsymbol{r}(t_0)\|$ when defined; one-sided limiting *line* directions can diagnose a singular tangent.

Regular tangent: $P = \boldsymbol{r}(t_0)$ and $\boldsymbol{r}'(t_0) \neq \boldsymbol{0}$: $\ell(s) = P + s\boldsymbol{r}'(t_0)$, $s \in \mathbb{R}$; unit tangent $\boldsymbol{T} = \boldsymbol{r}'/\|\boldsymbol{r}'\|$. Solve $\boldsymbol{r}(t) = P$ for every t_0 first.

Singular parameter: $\boldsymbol{r}'(t_0) = \boldsymbol{0}$ (or absent): regular formula unavailable. Inspect one-sided secants / local implicit form; cusp or self-intersection can give distinct tangents.

2. Limits, continuity, partials

Relative limit: a a limit point of D: $\lim_{x \to a,\, x \in D} f(x) = L \iff \forall \varepsilon > 0\, \exists \delta > 0$: $0 < \|x - a\| < \delta$, $x \in D \Rightarrow |f(x) - L| < \varepsilon$. Limit ignores $f(a)$; paths must lie in D.

Limit algebra: If $f \to L$, $g \to M$: $f \pm g \to L \pm M$, $fg \to LM$, $f/g \to L/M$ if $M \neq 0$; continuous outer ϕ gives $\phi(f) \to \phi(L)$. These need the expressions defined on the relative domain near the point.

Path test: Two admissible paths with different limits $\Rightarrow$ no limit; matching paths do *not* prove one. Prove with an algebraic/radial bound and squeeze.

Vector limits: $F = (f_1, \dots, f_m) \to (L_1, \dots, L_m) \iff f_i \to L_i$ for each i. Norm/distance are continuous: $x \to a \Rightarrow \|x\| \to \|a\|$.

Infinity: For unbounded D, $\lim_{\|x\| \to \infty,\, x \in D} f(x) = L$ means: $\forall \varepsilon > 0\, \exists R > 0$, $x \in D$, $\|x\| > R \Rightarrow |f(x) - L| < \varepsilon$. Compare dominant powers only after controlling direction/domain.

Domains: For quotients exclude zero denominators; for even real roots require nonnegative radicands; for logs require positive arguments. A limit or continuity assertion is always relative to this domain.

Level sets: $\{(x, y) : f(x, y) = c\}$ is a level curve. It describes inputs, not the graph $z = f(x, y)$; a regular level curve needs $\nabla f \neq 0$.

Continuity: a a limit point: f continuous at $a \iff \lim_{x \to a,\, x \in D} f(x) = f(a)$. At an isolated domain point, continuity is automatic. Sums/products/compositions preserve it; quotient needs nonzero limiting denominator.

Partials: $f_x(a, b) = \lim_{h \to 0}[f(a + h, b) - f(a, b)]/h$ and similarly f_y; hold the other coordinate fixed. Partials existing at one point do not imply continuity or differentiability.

Gradient: $\nabla f = (f_x, f_y)$; for $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f = (f_{x_1}, \dots, f_{x_n})$. Derivative notation at a point: $Df(p)h = \nabla f(p) \cdot h$ only when differentiable.

Mixed partials: $f \in C^2$ near (a, b) (in particular f_{xy}, f_{yx} continuous) $\Rightarrow f_{xy} = f_{yx}$.

3. Extrema

Candidates: Interior local extremum + existing first partials $\Rightarrow \nabla f = \boldsymbol{0}$. Include interior points where a first partial fails; include all relevant boundary pieces and endpoints.

Definitions: Local max/min compares only nearby domain points; absolute max/min compares every point of the stated domain. Boundary points can be absolute extrema but are not covered by the interior stationary condition.

2D Hessian: $f \in C^2$ near stationary (a, b); $D = f_{xx}f_{yy} - f_{xy}^2$. $D > 0, f_{xx} > 0$: strict min; $D > 0, f_{xx} < 0$: strict max; $D < 0$: saddle; $D = 0$: inconclusive.

Quadratic view: $Q(h, k) = f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2$: positive definite gives min, negative definite max, indefinite saddle (under the C^2 stationary setup).

Boundary method: Restrict f to each boundary curve/segment as a one-variable function; include its stationary and endpoint candidates. A compact region may have several boundary pieces.

Necessary only: $\nabla f = 0$ identifies candidates, not extrema: evaluate the Hessian or compare nearby values. An extremum can occur at an interior nondifferentiable point.

Hessian caveat: D test is for two variables at a stationary point and needs C^2 nearby. Do not apply it at a boundary point without first reducing to the boundary.

Absolute extrema: If D is nonempty, compact and f continuous, EVT gives attained absolute max/min. List all interior, boundary and singular candidates, then compare values.

4. Planes, differentiability, approximation

Ordinary plane: Through $P_0 = (x_0, y_0, z_0)$ with $\boldsymbol{n} = (A, B, C) \neq \boldsymbol{0}$: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$. Through directions $\boldsymbol{u}, \boldsymbol{v}$: $P_0 + s\boldsymbol{u} + t\boldsymbol{v}$ if independent; normal $\boldsymbol{u} \times \boldsymbol{v}$.

Plane relations: Planes with normals $\boldsymbol{n}_1, \boldsymbol{n}_2$ are perpendicular if $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = 0$, parallel if normals are parallel. A line of direction $\boldsymbol{v}$ is perpendicular to a plane with normal $\boldsymbol{n}$ iff $\boldsymbol{v} \parallel \boldsymbol{n}$.

Plane from points: Given noncollinear P, Q, R, use directions $Q - P$, $R - P$, normal $(Q - P) \times (R - P)$, then point-normal form. Collinear points do not determine a unique plane.

Cross product: $\boldsymbol{u} \times \boldsymbol{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$; it is normal to both; $\|\boldsymbol{u} \times \boldsymbol{v}\| = \|\boldsymbol{u}\|\|\boldsymbol{v}\| \sin\theta$.

Line / plane: Line $P + t\boldsymbol{v}$ meets plane $\boldsymbol{n} \cdot (X - P_0) = 0$ by substituting $X = P + t\boldsymbol{v}$. If $\boldsymbol{n} \cdot \boldsymbol{v} = 0$, it is parallel (or lies in the plane).

Graph tangent plane: $z = f(x, y)$ at $(a, b, f(a, b))$: $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$. It is a first-order approximation when f is differentiable.

Differential form: At (a, b), $\mathrm{d}z = f_x(a, b)\,\mathrm{d}x + f_y(a, b)\,\mathrm{d}y$ is the linear change. Substitute actual increments only after evaluating derivatives at the base point.

Differentiability: $f(p+h) = f(p) + Df(p)h + o(\|h\|)$; scalar case $Df(p)h = \nabla f(p) \cdot h$. Differentiable $\Rightarrow$ continuous + all partials. C^1 on an open neighbourhood $\Rightarrow$ differentiable; partials merely existing at p are insufficient.

Remainder test: f differentiable at p iff $[f(p+h) - f(p) - \nabla f(p) \cdot h]/\|h\| \to 0$ (when the displayed partials exist). Test along the full two-variable displacement, not only coordinate axes.

Linear / error: $L_p(p + h) = f(p) + \nabla f(p) \cdot h$; $\Delta f \approx \mathrm{d}f = \nabla f(p) \cdot \Delta\boldsymbol{x}$. For input bounds $|\Delta x_i| \le e_i$: $|\Delta f| \lesssim \sum_i |f_{x_i}(p)| e_i$.

5. Chain rule, IFT, directions

Chain rule: $z = f(x,y), x = x(t), y = y(t)$: $\mathrm{d}z/\mathrm{d}t = f_x x' + f_y y'$ (evaluate f_x, f_y at $(x(t), y(t))$). $x = x(s,t), y = y(s,t)$: $z_s = f_x x_s + f_y y_s$, $z_t = f_x x_t + f_y y_t$.

General chain: $F : \mathbb{R}^m \to \mathbb{R}^q$, $G : \mathbb{R}^n \to \mathbb{R}^m$ differentiable: $D(F \circ G)(p) = DF(G(p))DG(p)$. Product order matters; evaluate DF at $G(p)$.

Three variables: $w = f(x,y,z)$, each depending on t: $\mathrm{d}w/\mathrm{d}t = f_x x' + f_y y' + f_z z'$. With x, y, z depending on s: $w_s = f_x x_s + f_y y_s + f_z z_s$.

Second mixed: C^2 data: $z_{st} = f_{xx} x_s x_t + f_{xy}(x_s y_t + y_s x_t) + f_{yy} y_s y_t + f_x x_{st} + f_y y_{st}$.

Evaluation trap: Write the transformed point beside the formula: f_x, f_y, f_z are functions until evaluated. Do not differentiate f_x as a constant through a composite.

Implicit function: $F \in C^1$ near p, $F(p) = 0$, and $F_y(p) \neq 0 \Rightarrow y = y(x)$ locally with $y' = -F_x/F_y$. In $F(x,y,z) = 0$, $F_z \neq 0 \Rightarrow z_x = -F_x/F_z$, $z_y = -F_y/F_z$. Choose only a variable with nonzero partial.

IFT scope: IFT gives a local graph and derivative formula, not necessarily a global function. If the selected partial is zero, the conclusion/formula cannot be invoked from that coordinate choice.

Directional derivative: $D_{\boldsymbol{u}} f(p) = \lim_{t\to 0}[f(p + t\boldsymbol{u}) - f(p)]/t$. If f differentiable and $\|\boldsymbol{u}\| = 1$: $D_{\boldsymbol{u}} f = \nabla f \cdot \boldsymbol{u}$. Given nonzero $\boldsymbol{v}$, use $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$.

Existence caveat: A directional derivative may exist without differentiability. The dot-product formula and steepest-direction conclusion require differentiability at the point.

Steepest / level: If $\nabla f(p) \neq \boldsymbol{0}$, max/min unit rates $\pm\|\nabla f(p)\|$ occur in $\pm\nabla f/\|\nabla f\|$. On regular $F = c$, $\nabla F(p) \neq 0$ is normal and tangents $\boldsymbol{v}$ obey $\nabla F \cdot \boldsymbol{v} = 0$; $\nabla F = 0$ needs separate analysis.

Level tangent: For $F(x,y) = c$ with $F_y \neq 0$, tangent line at (a,b): $F_x(a,b)(x-a) + F_y(a,b)(y-b) = 0$; its slope is $-F_x/F_y$.

6. Lagrange multipliers

One equality: $f, g \in C^1$ near feasible local extremum p, $g(p) = k$, $\nabla g(p) \neq \boldsymbol{0}$: $\nabla f(p) = \lambda \nabla g(p)$ together with $g(p) = k$. Necessary candidates only; compare objective values.

Geometry: At a regular constraint, feasible tangent directions are perpendicular to ∇g; at an extremum ∇f is also normal, hence parallel to ∇g.

Several equalities: $f, g_1, \ldots, g_m \in C^1$ near p and $\nabla g_1(p), \ldots, \nabla g_m(p)$ linearly independent: $\nabla f = \sum_{i=1}^m \lambda_i \nabla g_i$, $g_i = k_i$.

Two constraints in $\mathbb{R}^3$: $g_1 = k_1, g_2 = k_2$: if $\nabla g_1, \nabla g_2$ independent, solve $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ plus both constraints. Their cross product gives a tangent direction.

Regularity trap: Where constraint gradient(s) vanish or are dependent, multiplier theorem is unavailable: include feasible singular points by direct analysis. For absolute problems also include feasible boundary candidates; compact feasible set + continuous f gives existence.

Candidate audit: For each solution verify feasibility, regularity status and domain. Lagrange equations neither prove existence nor classify local max/min without comparison or a further argument.

7. Double integrals and polar coordinates

Meaning: $\iint_R f \,\mathrm{d}A = \lim_{\|P\|\to 0} \sum_{i,j} f(x_{ij}^*, y_{ij}^*)\Delta x_i \Delta y_j$ when the common tagged limit exists. $f \geq 0$: volume; density ρ: mass $M = \iint_D \rho \,\mathrm{d}A$; if $\mathrm{Area}(D) > 0$, $f_{\mathrm{avg}} = \frac{1}{\mathrm{Area}(D)} \iint_D f \,\mathrm{d}A$.

Darboux check: For bounded f on a rectangle: integrable $\iff$ for every $\varepsilon > 0$ some partition has $U(f,P) - L(f,P) < \varepsilon$. This is equivalent to the common tagged-sum limit.

Area / mass: $\mathrm{Area}(D) = \iint_D 1 \,\mathrm{d}A$. For density $\rho \geq 0$, $M = \iint_D \rho \,\mathrm{d}A$; for a probability density $p \geq 0$ with $\iint_D p \,\mathrm{d}A = 1$, $\Pr(A) = \iint_A p \,\mathrm{d}A$.

Integrability: Continuous on a rectangle $\Rightarrow$ Riemann integrable. On a Jordan domain, bounded f is integrable $\iff$ $\mathrm{Disc}(f)$ has Lebesgue area zero: it has countable rectangle covers of arbitrarily small total area. Bounded alone is insufficient.

Jordan domain: For bounded D with boundary of area zero, define $\iint_D f$ by zero extension to an enclosing rectangle when f is integrable. A piecewise C^1 boundary curve has area zero; a two-dimensional overlap need not.

Integral laws: Linearity: $\iint_D (\alpha f + \beta g) = \alpha \iint_D f + \beta \iint_D g$. If $f \leq g$, integrals preserve order; $|\iint_D f| \leq \iint_D |f|$.

Bounds: If $m \leq f \leq M$ on D, then $m\,\mathrm{Area}(D) \leq \iint_D f \,\mathrm{d}A \leq M\,\mathrm{Area}(D)$. Useful for sign/sanity checks.

Fubini: f continuous on $[a,b] \times [c,d]$: $\iint_R f \,\mathrm{d}A = \int_a^b \int_c^d f \,\mathrm{d}y \,\mathrm{d}x = \int_c^d \int_a^b f \,\mathrm{d}x \,\mathrm{d}y$. For continuous f and continuous bounds, Type I: $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$; Type II reverses variables. Sketch; split where slice bounds change.

Order reversal: The order changes bounds, never the region/value. Read the original inner bounds as a moving slice, solve the inequalities in the other orientation, and split at every switch of boundary.

Additivity: If $D = \bigcup_{k=1}^N D_k$, f is integrable on the pieces, and overlaps have area zero, then $\iint_D f \,\mathrm{d}A = \sum_k \iint_{D_k} f \,\mathrm{d}A$. Shared curves/points are harmless; shared area is not.

Polar: $x = r\cos\theta$, $y = r\sin\theta$, $\mathrm{d}A = r \,\mathrm{d}r \,\mathrm{d}\theta$. For continuous f, h_1, h_2 with $0 \leq h_1 \leq h_2$ on $\alpha \leq \theta \leq \beta$, use $\int_\alpha^\beta \int_{h_1}^{h_2} f(r\cos\theta, r\sin\theta) r \,\mathrm{d}r \,\mathrm{d}\theta$; choose one covering (apart from boundary rays), include r.

Polar translation: $x^2 + y^2 = r^2$; circle $x^2 + y^2 = R^2 \Rightarrow r = R$; ray angle θ; $x = r\cos\theta$, $y = r\sin\theta$. Check outer radius is nonnegative and an angle interval does not double-cover interior points.

Polar areas: Disk $0 \leq r \leq R$: area πR^2; annulus $0 \leq a \leq r \leq b$: $\pi(b^2 - a^2)$; sector angle $\beta - \alpha$: $\frac{1}{2}R^2(\beta - \alpha)$. All follow from integrating the factor r.

Triggers & Traps

"Tangent at a curve point": Solve $\boldsymbol{r}(t) = P$, then test $\boldsymbol{r}'(t) \neq 0$; singular means secants/local geometry.

"Does the limit exist?": State domain; two valid paths can disprove, a norm bound can prove.

"Classify critical point": Hessian only after C^2 near the point; $D = 0$ means use another argument.

"Linearize / estimate error": Use the graph tangent plane only with differentiability; C^1 nearby is a sufficient check.

"Rate in direction $\boldsymbol{v}$": Normalize: $\boldsymbol{u} = \boldsymbol{v}/\|\boldsymbol{v}\|$ before $\nabla f \cdot \boldsymbol{u}$.

"Implicitly defined": Verify $F(p) = 0$, $F \in C^1$ nearby, and the solved-against partial is nonzero.

"Constrained extremum": Solve constraints with Lagrange equations, then add singular/boundary candidates and compare.

"Change order / polar": Draw the same region first; update both bounds. In polar convert integrand and retain the factor r.

Scope: No general Jacobians, triple integrals, line integrals or surface integrals.