NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2017-2018

MH2500 – Probability and Introduction to Statistics

December 2017 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FIVE (5)** printed pages including appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **TWO** double-sided A4 size help sheet.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.
6. The table with the values of the standard normal distribution is included at the end of this examination paper.
7. All questions are the property of Nanyang Technological University.

QUESTION 1 **(20 marks)**

Suppose that X is a random variable taking values in $\{0, 1, 2\}$ and Y is a random variable taking values in $\{0, 1, 2, 3\}$. A partial joint probability table of X and Y is given below.

	$Y = 0$	$Y = 1$	$Y = 2$	$Y = 3$
$X = 0$	0.10		0.16	
$X = 1$	0.05	0.04	0.08	0.04
$X = 2$	0.10	0.08		0.08

(a) (5 marks) Find the probability $\Pr(X > Y)$.

(b) (5 marks) Find the conditional mass function of X given that $Y = 0$.

(c) (5 marks) If the marginal distribution of Y is uniform, what is the probability $\Pr(X = 0, Y = 1)$?

(d) (5 marks) Find the probability $\Pr(X = 1)$ and determine whether X and Y are independent when the marginal distribution of Y is uniform.

QUESTION 2 **(15 marks)**

Suppose you have in your pocket a fair coin and a two-headed coin. You select one of the coins at random (with equal probability). All questions below are about this one coin; it is not replaced, and no other coin is drawn.

(a) (5 marks) When you flip it, it shows head. What is the probability that it is the fair coin?

(b) (5 marks) What is the probability that it will show a head on a second flip, given that it shows a head on the first flip?

(c) (5 marks) When you flip the coin a second time, it shows head again. What is the probability that it is the fair coin?

QUESTION 3 **(20 marks)**

A random rectangle is formed in the following way: The base X is chosen to be uniformly random on $[0, 1]$, and after having generated the base, the height H is chosen to be uniform on $[0, X]$.

(a) (5 marks) Find the conditional expectation $\mathbb{E}(H|X)$.

(b) (5 marks) Find the expected area of the rectangle.

(c) (5 marks) Find the variance of the area of the rectangle.

(d) (5 marks) Find the correlation between the base and the area of the rectangle.

Leave your answers as fractions, or numerical values that are accurate up to 0.001.

QUESTION 4 **(25 marks)**

Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables of the following density function:

$$f(x) = \begin{cases} Cx^{-(1+\alpha)}, & \text{if } x \geq 1; \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ is a constant and C is the normalization constant. Let Z_n be the geometric mean of $X_1, \cdots, X_n$, that is, $Z_n = (X_1 X_2 \cdots X_n)^{1/n}$.

(a) (5 marks) Determine the value of C. (The value of C may depend on α, in which case your answer should be expressed in terms of α).

(b) (10 marks) Find the density function of $Y_i = \ln X_i$.

(c) (5 marks) Determine whether Z_n converges to some constant L in probability as $n \longrightarrow \infty$. If yes, find the constant L. (Hint: you may find part (b) useful).

(d) (5 marks) If your answer to part (c) is 'yes', show that

$$\Pr(Z_n \leq e^{\frac{1}{\alpha}} L) \geq 1 - \frac{1}{n}.$$

If your answer to part (c) is 'no', show that

$$\Pr(Z_n \geq e^{\frac{1}{\alpha}} L) \geq 1 - \frac{1}{n}.$$

Note: Question 5 (Hypothesis Testing) is not tested for the Final Exam.

END OF PAPER