NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER I EXAMINATION 2018-2019

MH2500 – Probability and Introduction to Statistics

December 2018 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

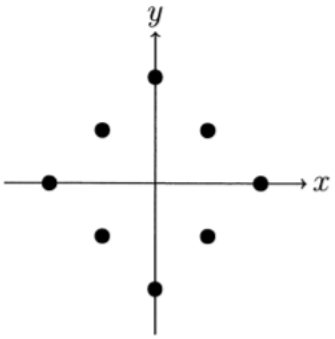
1. This examination paper contains **FOUR (4)** questions and comprises **FIVE (5)** printed pages including appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This is a **RESTRICTED OPEN BOOK** exam. Candidates are allowed **TWO** double-sided A4 size help sheet.
5. Candidates may use calculators. However, they should write down systematically the steps in the workings.
6. The table with the values of the standard normal distribution is included at the end of this examination paper.
7. All questions are the property of Nanyang Technological University.

QUESTION 1 **(25 marks)**

Suppose that (X, Y) is a point drawn uniformly at random from the set

$$S = \{(2,0), (1,1), (0,2), (-1,1), (-2,0), (-1,-1), (0,-2), (1,-1)\}.$$

Pictorially, (X, Y) is a uniform random point among the eight points illustrated in the picture below.



(a) (5 marks) Find $\Pr(X^2 + Y^2 \leq 3)$.

(b) (5 marks) Find the conditional probability $\Pr(Y \leq X^2 | Y \geq X^2)$.

(c) (5 marks) Find the marginal distribution of $|X|$ (the absolute value of X).

(d) (5 marks) Find in the range of Y the value of k that minimizes the conditional variance $\mathrm{Var}(X^7 | Y = k)$.

(e) (5 marks) Are X and Y independent? State your reason.

QUESTION 2 **(15 marks)**

Suppose that X and Y are (independent) uniformly distributed on $(0, 1]$. Let $Z = X/Y$.

(a) (10 marks) Find the probability density function of Z and the median of Z.

(b) (5 marks) Find the expected value and the variance of $\sqrt{Z}$. In the case of non-existence, answer 'do not exist' and give reasons.

QUESTION 3 **(15 marks)**

Let $n \geq 2$ and $X_1, \cdots, X_n$ be i.i.d. random variables taking the values in $\{-2, 1\}$ with probabilities $\Pr(X_1 = 1) = 2/3$ and $\Pr(X_1 = -2) = 1/3$. Denote $S_n = \sum_{i=1}^{n} X_i$.

(a) (5 marks) Find $\mathbb{E}(S_n), \mathrm{Var}(S_n)$ and $\lim_{n\to\infty} \Pr(S_n \geq \sqrt{n \ln n})$.

(b) (5 marks) Find $\lim_{n\to\infty} \Pr(S_n \geq 0)$.

(c) (5 marks) Show that there exists a constant C (which does not depend on n) such that

$$\Pr(S_n \geq \sqrt{n \ln n}) \leq \frac{C}{(\ln n)^{3/2}}$$

No points will be given for the limit in part (a) if you invoke part (c) without proving it.

QUESTION 4 **(20 marks)**

Let $X_1, \cdots X_n$ be independent uniform random variables over $[0, 1]$ and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ be their order statistics.

(a) (8 marks) Find the expectations $\mathbb{E}(X_{(1)})$ and $\mathbb{E}(X_{(n)})$.

(b) (12 marks) Find the covariance $\mathrm{Cov}(X_{(1)}, X_{(n)})$.

Note: Question 5 (Hypothesis Testing) is not tested for the Final Exam.

END OF PAPER