2020 Test2

Wednesday, October 27, 2021 12:44 AM

Nanyang Technological University
SPMS/Division of Mathematical Sciences

AY20/21 MH2500 PROBABILITY AND INTRODUCTION TO STATISTICS Test 2

Name & Matriculation Number	Suggested Solutions	Score	

Venue: Time: 9:35-10:15, 27 October, 2020

1. (30 Points)

A process yields 10% defective items. If 100 items are randomly selected from the process, use Normal approximation to find the probability that the number of defectives

(a) exceeds 13; and $X > 13$
(b) is less than 8. $X < 8$

Please refer to page 4 for the table of standard normal variable.

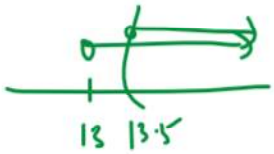
(a) Let X be the r.v. denoting the # of defectives in 100 items

$X \sim Bin(100, 0.1)$ $\quad np(1-p) = 9$

$E(X) = 100(0.1) = 10$ $\quad Var(X) = 100(0.1)(1-0.1) = 9$

Apply normal approximation,

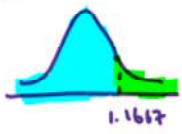
$P(X > 13) \approx P(X \geq 13.5)$ ← correction



$= P\left(Z \geq \frac{13.5-10}{\sqrt{9}}\right)$

$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

$= P(Z \geq 1.1667)$



$= 1 - \Phi(1.1667)$

$\approx 1 - 0.8790$

≈ 0.121 //

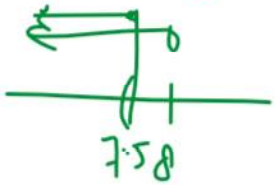
(b) $P(X < 8) \approx P(X \leq 7.5) = P\left(Z \leq \frac{7.5-10}{\sqrt{9}}\right)$

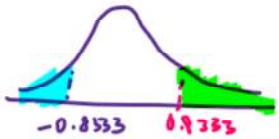
$= P(Z \leq -0.833)$

$= 1 - \Phi(0.8333)$

$= 1 - 0.7967$

≈ 0.2033 //





· $f(x,y) \geq 0$

· $\int\int_{\mathbb{R}^2} \frac{12}{5} x(2-x-y)\,dx\,dy.$

2. (40 Points) Let

$$f(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & \text{if } 0<x<1, 0<y<1; \\ 0 & \text{otherwise.} \end{cases}$$

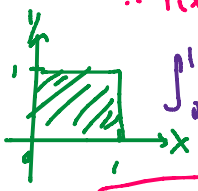
(a) Prove that $f(x,y)$ is a joint density function of two random variables, X and Y say.

(b) Compute the conditional density of X given that $Y = y$, where $0 < y < 1$.

$f_{X|Y}(x|y)$

(a) It is clear that for $0<x<1$, $0<y<1$, $x>0$ and $2-x-y>0$

$\therefore f(x,y) \geq 0$ for all x,y.



$$\int_0^1 \int_0^1 \frac{12}{5} x(2-x-y)\,dy\,dx = \frac{12}{5}\int_0^1 x\left[\left(2y - xy - \tfrac{1}{2}y^2\right)\right]_{y=0}^{y=1} dx$$

$$= \frac{12}{5}\int_0^1 x\left[2 - x - \tfrac{1}{2}\right] dx$$

$$= \frac{12}{5}\int_0^1 \tfrac{3}{2}x - x^2\,dx \quad \checkmark$$

$$= \frac{12}{5}\left[\tfrac{3}{4}x^2 - \tfrac{1}{3}x^3\right]_0^1$$

$$= \frac{12}{5}\left[\tfrac{3}{4} - \tfrac{1}{3}\right]$$

$$= \frac{12}{5}\left(\frac{5}{12}\right)$$

$$= 1 \quad \checkmark$$

$\therefore f(x,y)$ is a pdf.

$$\int_0^\infty y^n e^{-y}\,dy = n!$$

$n \in \mathbb{Z}^+$

(b) $f_{X|Y}(x|Y=y) = \dfrac{f_{XY}(x,y)}{f_Y(y)}$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$

$$= \frac{\frac{12}{5}x(2-x-y)}{\int_0^1 \frac{12}{5}x(2-x-y)\,dx}$$

$$= \frac{\frac{12}{5}x(2-x-y)}{\frac{12}{5}\int_0^1 2x - x^2 - xy\,dx} = \frac{x(2-x-y)}{\left[x^2 - \frac{1}{3}x^3 - \frac{yx^2}{2}\right]_0^1}$$

$$= \frac{x(2-x-y)}{\frac{2}{3} - \frac{1}{2}y}$$

$$= \frac{6x(2-x-y)}{4-3y}$$

if $0<x<1, 0<y<1$

$$\therefore f_{X|Y}(x|y) = \begin{cases} \frac{6x(2-x-y)}{4-3y} & \text{if } 0<x<1, 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

3. (30 Points)

Evidence concerning the guilt or innocence of a defendant in a criminal investigation can be summarized by the value of an exponential random variable X whose mean μ depends on whether the defendant is guilty. If innocent, $\mu = 1$; if guilty, $\mu = 2$. The deciding judge will rule the defendant guilty if $X > c$ for some suitably chosen value of c.

(given)

(a) If the judge wants to be 95 percent certain that an innocent man will not be convicted, what should be the value of c?

(b) Using the value of c found in part (a), what is the probability that a guilty defendant will be convicted?

(a) 95% certainty that innocent man is not convicted

$\Rightarrow P\{X > c\} = 0.05$

$X \sim \text{Expo}(\lambda_1)$, $E(X) = 1 = \mu$

$\frac{1}{\lambda_1} = 1 \Rightarrow \lambda_1 = 1.$

$F(x) = 1 - e^{-\lambda x} = 1 - e^{-x}$

$1 - P\{X \le c\} = 0.05$

$P\{X \le c\} = 0.95$

$1 - e^{-c} = 0.95$

$e^{-c} = 0.05$

$c = -\ln(0.05)$

$c = \ln 20 \approx 2.996$

$e^{-c} = \frac{1}{20}$

$-c = \ln\frac{1}{20}$

$-c = \ln 1 - \ln 20$

$c = \ln 20$

(b) $X \sim \text{Expo}(\lambda_2)$, where $E(Y) = \frac{1}{\lambda_2} = 2 \Rightarrow \lambda_2 = \frac{1}{2}$

$P\{X > c\} = 1 - P\{X \le c\}$

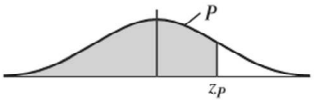
$= 1 - P\{X \le \ln 20\}$

$= 1 - (1 - e^{-\frac{1}{2}\ln 20})$

$= e^{-\frac{1}{2}\ln 20} = e^{\ln 20^{-\frac{1}{2}}}$

$= \frac{1}{\sqrt{20}} \approx 0.2236$

TABLE Cumulative Normal Distribution—Values of P Corresponding to z_p for the Normal Curve



z is the standard normal variable.

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998