Nanyang Technological University
SPMS/Division of Mathematical Sciences

2021/2022 Semester 1 **MH2500 Probability & Introduction to Statistics** **Test 2**

Name & Matriculation Number		HINTS

1. (10 Points)
   Let $X \sim \text{Exp}(\lambda)$.

   (a) Find $\mathbb{E}(X^n)$, where $n \geq 1$.

   (b) Find $\text{Var}(X)$.

   Solution: For $n \geq 1$,

$$\begin{aligned}\mathbb{E}(X^n) &= \int_0^\infty x^n \lambda e^{-\lambda x} dx \\ &= -e^{-\lambda x} x^n \Big|_{x=0}^{\infty} + \int_0^\infty e^{-\lambda x} n x^{n-1} dx \\ &= \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x} dx \\ &= \frac{n}{\lambda} \mathbb{E}(X^{n-1}).\end{aligned}$$

   This gives $\mathbb{E}(X^n) = \dfrac{n!}{\lambda^n} \mathbb{E}(1) = \dfrac{n!}{\lambda^n} \mathbb{E}(1) = \dfrac{n!}{\lambda^n}$.

   In particular, $\mathbb{E}(X) = \dfrac{1}{\lambda}$, $\mathbb{E}(X^2) = \dfrac{2!}{\lambda^2}$ and hence

$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{2!}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

2. (15 Points)

Suppose that of all the clouds that are seeded with silver iodide, 58% show splendid growth. If 60 clouds are seeded with silver iodide, find the probability that exactly 35 show splendid growth, by using normal distribution to approximate it.

Solution: Let X be the number of clouds that show splendid growth. Then

- $\mathbb{E}(X) = (60)(0.58) = 34.80$,
- $\sigma_X = \sqrt{(60)(0.58)(1-0.58)} = 3.82$.

Thus

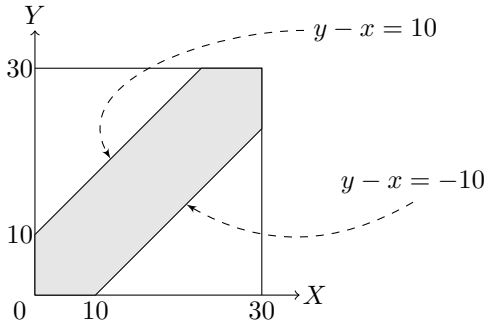
$$
\begin{aligned}
& P\{X = 35\} \approx P\{34.5 < X < 35.5\} \\
&= P\left\{\frac{34.5 - 34.8}{3.82} < \frac{X - 34.8}{3.82} < \frac{35.5 - 34.8}{3.82}\right\} \\
&= P\{-0.08 < Z < 0.18\} = \Phi(0.18) - \Phi(-0.08) \\
&= \Phi(0.18) - (1 - \Phi(0.08)) = 0.5714 - 0.4681 \\
&= 0.1033.
\end{aligned}
$$

3. (15 Points)

Jack invites his friend to Canteen B for a coffee break, and they agreed to meet at SPMS lobby between 3:30pm to 4:00pm. If they arrive at random times during this period, what is the probability that they will meet within 10 minutes?

Solution: Let X, Y be the minutes past 3:30pm, that Jack and his friend arrive at the lobby. Then

$$P\{|X-Y| \le 10\} = \frac{\text{area between } y-x=10 \text{ and } y-x=-10 \text{ (shaded)}}{\text{area of the square}}$$
$$= \frac{30^2-(30-10)^2}{30^2} = \frac{5}{9}.$$



4. (20 Points)

The joint probability density function of random variables X and Y is given by

$$f(x,y) = \begin{cases} cxy^2 & \text{if } 0 < x < y < 1; \\ \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Find the marginal probability density function of X and Y.

(c) Calculate $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

Solution: (a) By $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\ dydx = 1$, we have

$$\int_0^1 \int_x^1 (cxy^2)\ dydx = 1,$$

and $c = 10$.

(b)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\ dy = \int_x^1 10xy^2\ dy = \left[\frac{10}{3}xy^3\right]_x^1 = \frac{10}{3}(x - x^4)$$

for $0 < x < 1$;

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\ dx = \int_0^y 10xy^2\ dx = \left[5x^2y^2\right]_0^y = 5y^4$$

for $0 < y < 1$.

(c)

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^1 \left[x\left(\frac{10}{3}(x - x^4)\right)\right] dx = \frac{5}{9}.$$

$$\mathbb{E}(Y) = \int_{-\infty}^{\infty} yf_Y(y)dy = \int_0^1 \left[y\left(5y^4\right)\right] dx = \frac{5}{6}.$$

5. (20 Points)

Let the conditional probability density function of X, given that $Y = y$, be

$$f_{X|Y}(x|y) = \frac{x+y}{1+y}e^{-x}, \qquad 0 < x < \infty, \;\; 0 < y < \infty.$$

(a) Find $P\{X < 1|Y = 2\}$.

(b) Find $\mathbb{E}(X|Y = 2)$.

Solution: (a) The pdf of X given $Y = 2$ is

$$f_{X|Y}(x|2) = \frac{x+2}{3}e^{-x},$$

for $0 < x < \infty$. Therefore,

$$P\{X < 1|Y = 2\} = \int_0^1 \frac{x+2}{3}e^{-x}dx = \frac{1}{3}\left[\int_0^1 xe^{-x}dx + \int_0^1 2e^{-x}dx\right]$$

$$= \frac{1}{3}\Big[-xe^{-x} - e^{-x}\Big]_0^1 - \frac{2}{3}\Big[e^{-x}\Big]_0^1 = 1 - \frac{4}{3}e^{-1} \approx 0.509.$$

(b)

$$\mathbb{E}(X|Y = 2) = \int_{-\infty}^{\infty} xf_{X|Y}(x|2)dx = \int_0^{\infty} x\left(\frac{x+2}{3}e^{-x}\right)dx = \frac{1}{3}\int_0^{\infty}(x^2+2x)e^{-x}\,dx$$

$$= \frac{1}{3}\Big[-(x^2+2x)e^{-x}\Big]_0^{\infty} - \frac{1}{3}\int_0^{\infty} e^{-x}(-2x-2)dx = \frac{2}{3}\int_0^{\infty}(x+1)e^{-x}dx$$

$$= \frac{2}{3}\Big[e^{-x}(-x-1)\Big]_0^{\infty} - \frac{2}{3}\int_0^{\infty} e^{-x}(-1)dx = \frac{2}{3} + \frac{2}{3}\int_0^{\infty} e^{-x}dx = \frac{4}{3}.$$

6. (20 Points)

Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common cdf F and density f, and let U, V denote the *maximum* and the *minimum* of the X_i's, respectively. That is, $U = \max\{X_1, X_2, \ldots, X_n\}$ and $V = \min\{X_1, X_2, \ldots, X_n\}$.

(a) Find the cumulative distributive function F_U and the probability density function f_U for U.

(b) Find the cumulative distributive function F_V and the probability density function f_V for V.

(c) Suppose that $Y_1, Y_2, \ldots, Y_n \sim \text{Exp}(\lambda)$ are independent. Find the probability density function for the variable $\min\{Y_1, Y_2, \ldots, Y_n\}$.

Solution: (a) The cdf of U is given by

$$F_U(u) = P(U \le u) = P(X_1 \le u)P(X_2 \le u)\cdots P(X_n \le u) = (F(u))^n.$$

and, by differentiating it, we have the density function

$$f_U(u) = n(F(u))^{n-1}(F(u))' = nf(u)(F(u))^{n-1}.$$

(b) Let V denote the minimum of the X_i's. Then

$$1 - F_V(v) = P(V \ge v) = P(X_1 \ge v)P(X_2 \ge v)\cdots P(X_n \ge v) = (1 - F(v))^n.$$

Hence

$$F_V(v) = 1 - (1 - F(v))^n$$

and

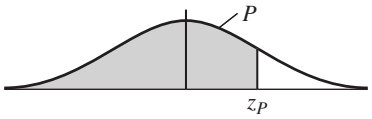
$$f_V(v) = (1 - (1 - F(v))^n)' = -n(1-F(v))^{n-1}(-F(v))' = -n(1-F(v))^{n-1}(-f(v)) = nf(v)(1-F(v))^{n-1}.$$

(c) Let $W = \min\{Y_1, Y_2, \ldots, Y_n\}$. By (b), we have

$$f_W(w) = n(\lambda e^{-\lambda w})(e^{-\lambda w})^{n-1} = n\lambda\left(e^{-\lambda w}(e^{-\lambda w})^{n-1}\right) = (n\lambda)e^{-(n\lambda)w},$$

$W \sim \text{Exp}(n\lambda)$.

TABLE Cumulative Normal Distribution—Values of P Corresponding to z_p for the Normal Curve



***z* is the standard normal variable.**

z_p	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998