2021 Test 2

Saturday, October 30, 2021 2:03 PM

Nanyang Technological University
SPMS/Division of Mathematical Sciences

2021/2022 Semester 1 **MH2500 Probability & Introduction to Statistics** **Test 2**

Name & Matriculation Number	Solutions	**Score:**	

Tutorial Group: **9:00-10:20, 29/10/2021**

1. (10 Points)
   Let $X \sim \text{Exp}(\lambda)$.
   (a) Find $\mathbb{E}(X^n)$, where $n \geq 1$.
   (b) Find $\text{Var}(X)$.

(a) pdf of X: $\lambda e^{-\lambda x}$, $x \geq 0$

$$E[X^n] = \int_0^\infty \underbrace{x^n}_{u} \underbrace{\lambda e^{-\lambda x}}_{v'} dx$$ Apply Integration-By-Parts.

$$= \left[\frac{x^n \lambda e^{-\lambda x}}{-\lambda}\right]_0^\infty - \int_0^\infty \frac{n x^{n-1} \lambda e^{-\lambda x}}{-\lambda} dx$$

$$= 0 + \frac{n}{\lambda} \int_0^\infty \underbrace{x^{n-1}}_{x^{n-1}} \underbrace{\lambda e^{-\lambda x}}_{\text{pdf of } X} dx$$

$$= \frac{n}{\lambda} E[X^{n-1}]$$

$$\therefore E[X^n] = \frac{n}{\lambda} \times \frac{n-1}{\lambda} \times \frac{n-2}{\lambda} \times \cdots \times \frac{2}{\lambda} \times \frac{1}{\lambda} = \frac{n!}{\lambda^n}$$

(b) $\text{Var}(X) = E[X^2] - \{E[X]\}^2 = \frac{2!}{\lambda^2} - \left(\frac{1!}{\lambda}\right)^2 = \frac{1}{\lambda^2}$ //

2. (15 Points)

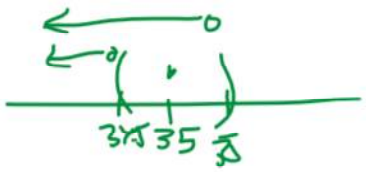
Suppose that of all the clouds that are seeded with silver iodide, 58% show splendid growth. If 60 clouds are seeded with silver iodide, find the probability that exactly 35 show splendid growth, by using normal distribution to approximate it.

Let X be the r.v. denoting the number of clouds that show splendid growth.

$X \sim Bin(60, 0.58)$ $\quad E(X) = 60(0.58) = 34.8$

$Var(X) = 60(0.58)[1-0.58] = 14.616$

Normal Approximation:



$$P(X=35) = P(34.5 < X < 35.5)$$

$$= P\left(\frac{34.5-34.8}{\sqrt{14.616}} < Z < \frac{35.5-34.8}{\sqrt{14.616}}\right)$$

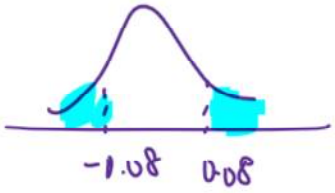
$$= P(-0.08 < Z < 0.18)$$

$$= \Phi(0.18) - \Phi(-0.08)$$

$$= 0.5714 - (1 - \Phi(0.08))$$

$$= 0.5714 - (1 - 0.5319)$$

$$\approx 0.1033$$

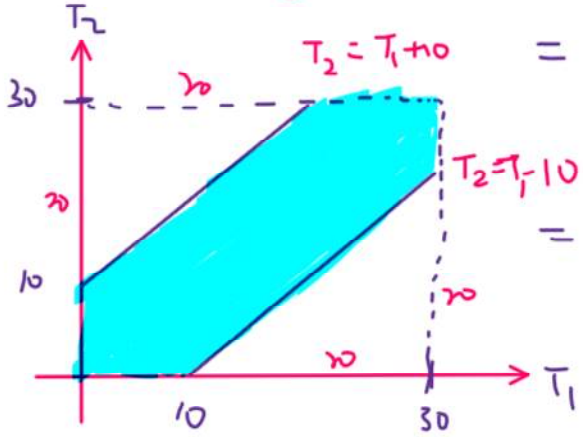


3. (15 Points)

Jack invites his friend to Canteen B for a coffee break, and they agreed to meet at SPMS lobby between 3:30pm to 4:00pm. If they arrive at random times during this period, what is the probability that they will meet within 10 minutes?

Let T_1 and T_2 be the times that Jack and his friend arrives at the lobby. $T_1 \sim \text{Unif}[0,30]$, $T_2 \sim \text{Unif}[0,30]$

$P\{\text{They meet within 10 mins}\} = P(|T_1 - T_2| \leq 10)$



$= \dfrac{\text{Area of shaded Region}}{\text{Area of Square}}$

$= \dfrac{30^2 - 2\left(\frac{1}{2}\right)(20)(20)}{30^2}$

$= \dfrac{5}{9}$ //

4. (20 Points)

The joint probability density function of random variables X and Y is given by

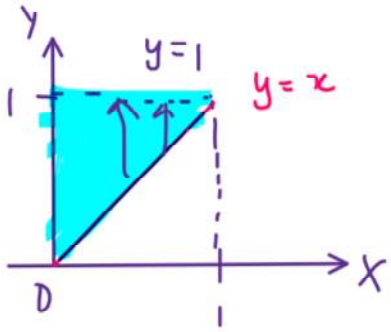
$$f(x,y) = \begin{cases} cxy^2 & \text{if } 0 < x < y < 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Find the marginal probability density function of X and Y.

(c) Calculate $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

(a)



$$\int_0^1 \int_x^1 cxy^2 \, dy \, dx = 1$$

$$c\int_0^1 x \left[\frac{y^3}{3}\right]_x^1 dx = 1$$

$$\frac{c}{3}\int_0^1 x - x^4 \, dx = 1$$

$$\frac{c}{3}\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 1$$

$$c = 10 //$$

(b) $f_X(x) = \int_x^1 10xy^2 \, dy$

$= 10x\left[\frac{y^3}{3}\right]_x^1$

$= \frac{10}{3}x(1-x^3)$ if $0<x<1$ //

$f_Y(y) = \int_0^y 10xy^2 \, dx = 10y^2\int_0^y x \, dx = 10y^2\left[\frac{x^2}{2}\right]_0^y$

$= 5y^4$ $0<y<1$ //

(c) $E(X) = \int_0^1 x f_X(x) \, dx$

$= \int_0^1 \frac{10}{3}x^2(1-x^3) \, dx$

$= \frac{10}{3}\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1$

$= \frac{5}{9}$ //

$E(Y) = \int_0^1 y f_Y(y) \, dy$

$= \int_0^1 5y^5 \, dy$

$= \left[\frac{5y^6}{6}\right]_0^1$

$= \frac{5}{6}$ //

5. (20 Points)

Let the conditional probability density function of X, given that $Y = y$, be

$$f_{X|Y}(x|y) = \frac{x+y}{1+y}e^{-x}, \qquad 0 < x < \infty, \quad 0 < y < \infty.$$

(a) Find $P\{X < 1|Y = 2\}$.

(b) Find $\mathbb{E}(X|Y = 2)$.

(a) $P\{X<1 \mid Y=2\} = \int_0^1 \frac{x+2}{1+2} e^{-x}\,dx$

$= \frac{1}{3}\int_0^1 \underbrace{x}_{u}\underbrace{e^{-x}}_{v'} + 2e^{-x}\,dx$

$= \frac{1}{3}\left[\left[-xe^{-x}\right]_0^1 - \int_0^1 -e^{-x}\,dx + \left[-2e^{-x}\right]_0^1\right]$

$= \frac{1}{3}\left[-e^{-1} + \left[-e^{-x}\right]_0^1 + \left[-2e^{-x}\right]_0^1\right]$

$= \frac{1}{3}\left[-e^{-1} - 3\left(e^{-1} - 1\right)\right]$

$= \frac{1}{3}\left(3 - 4e^{-1}\right)$ //

(b) $E[X|Y=2] = \int_0^\infty x\left(\frac{x+2}{4+2}e^{-x}\right)dx$

$= \frac{1}{3}\left[\int_0^\infty \underbrace{x^2}_{u}\underbrace{e^{-x}}_{v'}\,dx + \int_0^\infty \underbrace{2x}_{u}\underbrace{e^{-x}}_{v'}\,dx\right]$

$\vdots$ ← exercise...

$= \frac{1}{3}\left[2! + 2(1!)\right]$

$= \frac{4}{3}$ //

6. (20 Points)

Let $X_1, X_2, \ldots, X_n$ be independent random variables with a common cdf F and density f, and let U, V denote the *maximum* and the *minimum* of the X_i's, respectively. That is, $U = \max\{X_1, X_2, \ldots, X_n\}$ and $V = \min\{X_1, X_2, \ldots, X_n\}$.

(a) Find the cumulative distributive function F_U and the probability density function f_U for U.

(b) Find the cumulative distributive function F_V and the probability density function f_V for V.

(c) Suppose that $Y_1, Y_2, \ldots, Y_n \sim \text{Exp}(\lambda)$ are independent. Find the probability density function for the variable $\min\{Y_1, Y_2, \ldots, Y_n\}$.

(a) $F_U(u) = P(U \le u)$

$= P(X_1 \le u, X_2 \le u, \ldots, X_n \le u)$

independence → $= P\{X_1 \le u\} P\{X_2 \le u\} \cdots P\{X_n \le u\}$

$= [F(u)]^n$ //

$f_U(u) = \frac{d}{du}\{[F(u)]^n\} = n[F(u)]^{n-1} f(u)$

(b) $F_V(v) = P(V \le v)$

$= 1 - P(V > v)$

$= 1 - P(X_1 > v) P(X_2 > v) \cdots P(X_n > v)$

$= 1 - (1 - F(v))^n$ //

$f_V(v) = \frac{d}{dv}\{1 - (1 - F(v))^n\} = -n(1 - F(v))^{n-1}\{-f(v)\}$

$= n(1 - F(v))^{n-1} f(v)$ //

(c) $Y_i \sim \text{Expo}(\lambda)$, $f_Y(y) = \lambda e^{-\lambda y}$ if $y \ge 0$

$F(y) = 1 - e^{-\lambda y}$

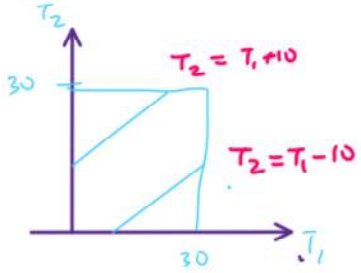
$$F_Y(y) = 1 - e^{-\lambda y}$$

From (5), $f_V(v) = n\left[1-(1-e^{-\lambda y})\right]^{n-1} \lambda e^{-\lambda y}$

$= n\lambda e^{-\lambda y(n-1)} e^{-\lambda y}$

$= n\lambda e^{-n\lambda}$, if $y \geq 0$. $\therefore V \sim \text{Expo}(n\lambda)$.

EXTRA: Calculus(!) Approach for Q3. (Unnecessary, but u can try this approach if u like)



Consider $0 \leq T_2 \leq 10$, $10 \leq T_2 \leq 20$, $T_2 \geq 20$.

$$\int_0^{10}\int_0^{T_2+10} \frac{1}{900}\, dT_1\, dT_2$$

$= \frac{1}{900}\int_0^{10}\left[T_1\right]_0^{T_2+10} dT_2$

$= \frac{1}{900}\int_0^{10} T_2 + 10\, dT_2$

$= \frac{1}{900}\left[\frac{1}{2}T_2^2 + 10T_2\right]_0^{10}$

$= \frac{1}{900}\left[\frac{1}{2}(10)^2 + 100\right]$

$= \frac{150}{900}$

$$\int_{10}^{20}\int_{T_2-10}^{T_2+10} \frac{1}{900}\, dT_1\, dT_2$$

$= \frac{1}{900}\int_{10}^{20} 20\, dT_2$

$= \frac{1}{900}(20)(10)$

$= \frac{200}{900}$

Complement:

$$1 - \int_0^{20}\int_{T_1+10}^{30} \frac{1}{900}\, dT_2\, dT_1 - \int_{10}^{30}\int_0^{T_1-10} \frac{1}{900}\, dT_2\, dT_1$$

$= 1 - \frac{1}{900}\int_0^{20} 20 - T_1\, dT_1 - \frac{1}{900}\int_{10}^{30} T_1 - 10\, dT_1$

$= 1 - \frac{1}{900}\left[20T_1 - \frac{1}{2}T_1^2\right]_0^{20} - \frac{1}{900}\left[\frac{T_1^2}{2} - 10T_1\right]_{10}^{30}$

$= 1 - \frac{1}{900}(200) - \frac{1}{900}\left((450-300) - (50-100)\right)$

$= 1 - \frac{2}{9} - \frac{2}{9} = \frac{5}{9}$.

$$\int_{10}^{30}\int_{T_2-10}^{30} \frac{1}{900}\, dT_1\, dT_2$$

$= \frac{1}{900}\int_{20}^{30} 40 - T_2\, dT_2$

$= \frac{1}{900}\left[40T_2 - \frac{1}{2}T_2^2\right]_{20}^{30}$

$= \frac{1}{900}\left[(1200 - 450) - (800 - 200)\right]$

$= \frac{1}{900}(150)$

$$\frac{150}{900} + \frac{200}{900} + \frac{150}{900} = \frac{5}{9}$$