MH3100 Real Analysis I

Sample Midterm Test (8 October 2025) (Solutions)

Official sample paper – QRS solutions (Practice 1)

Quantitative Research Society @NTU

October 12, 2026

Overview

- **Coverage:** Lectures 1–17 (real numbers and the Axiom of Completeness, through sequences, series, and sets of real numbers – limit points, closures, closed sets, and (sequential) compactness).
- **Format: FOUR (4)** questions, 6 marks each, 24 marks total. **TIME ALLOWED: 60 minutes. CLOSED BOOK.** Calculators are allowed.
- **Citation policy (verbatim from the paper):** "You may cite any results proved in the lectures and tutorials, unless you are asked by the question to prove the result."
- **This document:** full QRS solutions to the (SAMPLE) Midterm Test dated 8 October 2025, published as *Practice 1 (Solution by QRS)*.

Instructions to Students (verbatim from the cover page)

1. This test contains **FOUR (4)** questions.
2. Answer **all** questions. The marks for each question are indicated below.
3. Write your answers in the box provided. Use the **facing page** for more space.
4. This is a **CLOSED BOOK** test.
5. You may use calculators. You may cite any results proved in the lectures and tutorials, unless you are asked by the question to prove the result.

Question	1	2	3	4	Total
Total Marks	6	6	6	6	24

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Question 1 (6 Marks)

Is each of the following statement true or false? You do NOT need to justify your answer.

(a) *(2 marks, QRS assumption)* There are sequences $\{a_n\}$ and $\{b_n\}$, where $\{a_n\}$ converges, $\{b_n\}$ diverges, and $\{a_n + b_n\}$ converges.

(b) *(2 marks, QRS assumption)* There is a sequence with a subsequence that is bounded but contains no subsequence that converges.

(c) *(2 marks, QRS assumption)* If every proper subsequence of a sequence $\{a_n\}$ converges, then $\{a_n\}$ converges as well.

Part (a)

Method 1: Contradiction via the Algebraic Limit Theorem. The statement is false. We show that whenever $\{a_n\}$ and $\{a_n + b_n\}$ both converge, $\{b_n\}$ converges as well, so a divergent $\{b_n\}$ is impossible.

Proof. Suppose, for contradiction, that $\{a_n\}$ converges, $\{b_n\}$ diverges and $\{a_n + b_n\}$ converges. Let $a := \lim a_n$ and $L := \lim(a_n + b_n)$.

By part (1) of the Algebraic Limit Theorem with the constant -1, the sequence $\{-a_n\}$ converges to $-a$.

By part (2) of the Algebraic Limit Theorem, applied to the convergent sequences $\{a_n + b_n\}$ and $\{-a_n\}$, the sequence $\{(a_n + b_n) + (-a_n)\}$ converges to $L + (-a) = L - a$.

For every $n \in \mathbb{N}$,

$$(a_n + b_n) + (-a_n) = b_n.$$

Therefore $\{b_n\}$ converges to $L - a$, which contradicts the assumption that $\{b_n\}$ diverges.

Hence there are no sequences $\{a_n\}$, $\{b_n\}$ with $\{a_n\}$ convergent, $\{b_n\}$ divergent and $\{a_n + b_n\}$ convergent. □

False

Method 2: Cauchy Criterion. This route works with the Cauchy property of the three sequences, so no candidate limit for $\{b_n\}$ is needed.

Proof. Suppose $\{a_n\}$ and $\{a_n + b_n\}$ converge, and write $s_n := a_n + b_n$ for $n \in \mathbb{N}$. By the Cauchy Criterion, $\{a_n\}$ and $\{s_n\}$ are Cauchy sequences.

Let $\varepsilon > 0$ be given. Since $\{a_n\}$ is Cauchy, there is $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$, $|a_n - a_m| < \varepsilon/2$. Since $\{s_n\}$ is Cauchy, there is $N_2 \in \mathbb{N}$ such that for all $n, m \geq N_2$, $|s_n - s_m| < \varepsilon/2$.

Let $N := \max\{N_1, N_2\}$ and let $n, m \geq N$. Then $n, m \geq N_1$ and $n, m \geq N_2$. Since $b_n = s_n - a_n$ and $b_m = s_m - a_m$,

$$\begin{aligned} |b_n - b_m| &= |(s_n - a_n) - (s_m - a_m)| \\ &= |(s_n - s_m) - (a_n - a_m)| \\ &\leq |s_n - s_m| + |a_n - a_m| && \text{(Triangle Inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|b_n - b_m| < \varepsilon$; that is, $\{b_n\}$ is Cauchy (def. of Cauchy). Hence $\{b_n\}$ converges (Cauchy Criterion).

Thus $\{b_n\}$ cannot diverge while $\{a_n\}$ and $\{a_n + b_n\}$ converge, and no pair as in the statement exists. □

False

Method 3: Direct ε–N argument from the definition of limit. This route identifies the limit of $\{b_n\}$ explicitly and checks the definition of limit without citing any limit theorem.

Proof. Suppose $a = \lim a_n$ and $L = \lim s_n$, where $s_n := a_n + b_n$. We show that $L - a = \lim b_n$.

Let $\varepsilon > 0$ be given. Since $a = \lim a_n$, there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \varepsilon/2$ (def. of limit). Since $L = \lim s_n$, there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|s_n - L| < \varepsilon/2$ (def. of limit).

Let $N := \max\{N_1, N_2\}$ and let $n \geq N$. Then

$$\begin{aligned} |b_n - (L - a)| &= |(s_n - a_n) - (L - a)| \\ &= |(s_n - L) - (a_n - a)| \\ &\leq |s_n - L| + |a_n - a| \qquad \text{(Triangle Inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|b_n - (L - a)| < \varepsilon$. Hence $L - a = \lim b_n$ (def. of limit), so $\{b_n\}$ converges.

Thus $\{b_n\}$ cannot diverge while $\{a_n\}$ and $\{a_n + b_n\}$ converge. □

False

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Part (b)

Method 1: Bolzano–Weierstrass Theorem. The statement is false. We show that every bounded subsequence of every sequence has a convergent subsequence.

Proof. Let $\{a_n\}$ be any sequence and let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$ that is bounded. Write $x_k := a_{n_k}$ for $k \in \mathbb{N}$.

Since $\{x_k\}$ is a bounded sequence, the Bolzano–Weierstrass Theorem gives a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ that converges.

Therefore the bounded subsequence $\{a_{n_k}\}$ contains the convergent subsequence $\{x_{k_j}\} = \{a_{n_{k_j}}\}$.

Since the sequence $\{a_n\}$ and its bounded subsequence were arbitrary, no sequence has a bounded subsequence that contains no convergent subsequence. □

False

Method 2: Sequential compactness of a closed interval. This route places the bounded subsequence inside a closed and bounded interval and uses the definition of a compact set.

Proof. Let $\{a_n\}$ be any sequence, let $\{a_{n_k}\}$ be a bounded subsequence, and write $x_k := a_{n_k}$. Since $\{x_k\}$ is bounded, there is $M > 0$ such that for all $k \in \mathbb{N}$, $|x_k| \leq M$. Hence $-M \leq x_k \leq M$ for all $k \in \mathbb{N}$, that is, $\{x_k\}$ is contained in $K := [-M, M]$.

The set K is bounded, since $|y| \leq M$ for every $y \in K$.

The set K is closed. Let $\{y_n\}$ be a convergent sequence contained in K and let $y := \lim y_n$. Then $y_n \leq M$ and $-M \leq y_n$ for all $n \in \mathbb{N}$. Therefore $y \leq M$ and $-M \leq y$ (Order Limit Theorem), so $y \in K$. Hence every convergent sequence contained in K has its limit in K, and K is closed (Sequential Criterion for Closed Sets).

Since K is closed and bounded, K is (sequentially) compact (Sequential Compactness of Closed and Bounded Sets).

Since $\{x_k\}$ is a sequence contained in the compact set K, the definition of a compact set gives a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ that converges to a limit in K.

Therefore every bounded subsequence of every sequence contains a convergent subsequence. □

False

Method 3: Monotone subsequence and the Monotone Convergence Theorem. This route extracts a monotone subsequence by hand (a "peak" argument) and then applies the Monotone Convergence Theorem instead of Bolzano–Weierstrass.

Proof. Let $\{a_n\}$ be any sequence, let $\{a_{n_k}\}$ be a bounded subsequence, write $x_k := a_{n_k}$, and let $M > 0$ satisfy $|x_k| \leq M$ for all $k \in \mathbb{N}$.

Call $p \in \mathbb{N}$ a *peak* of $\{x_k\}$ if $x_p \geq x_j$ for all $j \in \mathbb{N}$ with $j > p$. Let P be the set of peaks.

Case 1: P is infinite. Let p_1 be the least element of P and, for each $j \in \mathbb{N}$, let p_{j+1} be the least element of P greater than p_j; it exists because P is infinite. Then $p_1 < p_2 < p_3 < \cdots$. For each $j \in \mathbb{N}$, p_j is a peak and $p_{j+1} > p_j$, so $x_{p_j} \geq x_{p_{j+1}}$. Hence $\{x_{p_j}\}$ is a decreasing subsequence of $\{x_k\}$.

Case 2: P is finite. Let $q_1 := 1 + \max P$ if $P \neq \emptyset$, and $q_1 := 1$ if $P = \emptyset$; then q_1 is greater than every element of P. Suppose $q_j \in \mathbb{N}$ has been chosen greater than every element of P. Then q_j is not a peak, so there is $q_{j+1} \in \mathbb{N}$ with $q_{j+1} > q_j$ and $x_{q_{j+1}} > x_{q_j}$; and $q_{j+1} > q_j$ is again greater than every element of P. By induction this defines $q_1 < q_2 < q_3 < \cdots$ with $x_{q_1} < x_{q_2} < x_{q_3} < \cdots$. Hence $\{x_{q_j}\}$ is an increasing subsequence of $\{x_k\}$.

In both cases $\{x_k\}$ has a monotone subsequence $\{y_j\}$. Since $|x_k| \leq M$ for all $k \in \mathbb{N}$, also $|y_j| \leq M$ for all $j \in \mathbb{N}$, so $\{y_j\}$ is bounded. Therefore $\{y_j\}$ converges (Monotone Convergence Theorem).

Hence every bounded subsequence of every sequence contains a convergent subsequence. □

False

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Part (c)

Method 1: Tail subsequence and the definition of limit. The statement is true. A subsequence $\{a_{n_k}\}$ is *proper* if $n_k \neq k$ for some $k \in \mathbb{N}$; dropping the first term already gives a proper subsequence.

Proof. Suppose every proper subsequence of $\{a_n\}$ converges. For $k \in \mathbb{N}$ let $n_k := k+1$ and $b_k := a_{n_k} = a_{k+1}$. Since $n_{k+1} = k+2 > k+1 = n_k$ for all $k \in \mathbb{N}$, $\{b_k\}$ is a subsequence of $\{a_n\}$; since $n_1 = 2 \neq 1$, it is proper. Therefore $\{b_k\}$ converges (hypothesis); let $L := \lim b_k$.

Let $\varepsilon > 0$ be given. Since $L = \lim b_k$, there is $K \in \mathbb{N}$ such that for all $k \geq K$, $|b_k - L| < \varepsilon$ (def. of limit).

Let $N := K+1$ and let $n \in \mathbb{N}$ with $n \geq N$. Put $k := n-1$. Then $k \geq K \geq 1$, so $k \in \mathbb{N}$ and $k \geq K$, and $a_n = a_{k+1} = b_k$. Therefore

$$|a_n - L| = |b_k - L| < \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$. Therefore $L = \lim a_n$ (def. of limit), and $\{a_n\}$ converges. □

True

Method 2: Cauchy Criterion. This route transfers the Cauchy property from the proper subsequence $\{a_{k+1}\}$ to $\{a_n\}$.

Proof. Suppose every proper subsequence of $\{a_n\}$ converges, and let $b_k := a_{k+1}$ for $k \in \mathbb{N}$. As in Method 1, $\{b_k\}$ is a proper subsequence (indices $k+1$ strictly increasing, and $2 \neq 1$), so $\{b_k\}$ converges (hypothesis). Therefore $\{b_k\}$ is Cauchy (Cauchy Criterion).

Let $\varepsilon > 0$ be given. There is $K \in \mathbb{N}$ such that for all $k, j \geq K$, $|b_k - b_j| < \varepsilon$ (def. of Cauchy).

Let $N := K+1$ and let $n, m \geq N$. Put $k := n-1$ and $j := m-1$. Then $k, j \geq K$, $a_n = b_k$ and $a_m = b_j$, so

$$|a_n - a_m| = |b_k - b_j| < \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$; that is, $\{a_n\}$ is Cauchy (def. of Cauchy). Therefore $\{a_n\}$ converges (Cauchy Criterion). □

True

Method 3: Contradiction via the Divergence Criterion. This route assumes $\{a_n\}$ diverges, extracts two subsequences with different limits, and shows that both limits must equal $\lim a_{k+1}$.

Proof. Suppose every proper subsequence of $\{a_n\}$ converges and, for contradiction, that $\{a_n\}$ diverges. Let $b_k := a_{k+1}$ for $k \in \mathbb{N}$. Since the indices $k+1$ are strictly increasing and the first index is $2 \neq 1$, $\{b_k\}$ is a proper subsequence, so it converges (hypothesis); let $L := \lim b_k$.

$\{a_n\}$ is bounded. Since $\{b_k\}$ converges, it is bounded (Boundedness of Convergent Sequences): there is $M_0 > 0$ such that for all $k \in \mathbb{N}$, $|b_k| \leq M_0$. Let $M := \max\{|a_1|, M_0\}$. Then $|a_1| \leq M$, and for every $n \geq 2$, $|a_n| = |b_{n-1}| \leq M_0 \leq M$. Hence $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Two subsequential limits. Since $\{a_n\}$ is bounded and divergent, the Divergence Criterion gives subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ with $x := \lim a_{n_k}$, $y := \lim a_{m_k}$ and $x \neq y$.

Every convergent subsequence of $\{a_n\}$ has limit L. Let $\{a_{p_k}\}$ be a subsequence of $\{a_n\}$ that converges, and let $z := \lim a_{p_k}$. Let $c_k := a_{p_{k+1}}$ for $k \in \mathbb{N}$. Since the indices $k+1$ are strictly increasing, $\{c_k\}$ is a subsequence of $\{a_{p_k}\}$, so $z = \lim c_k$ (Subsequential Limits). Let $r_k := p_{k+1} - 1$. Since $p_{k+1} > p_1 \geq 1$, we have $p_{k+1} \geq 2$, so $r_k \in \mathbb{N}$; and $r_{k+1} = p_{k+2} - 1 > p_{k+1} - 1 = r_k$. Since $c_k = a_{r_k+1} = b_{r_k}$, the sequence $\{c_k\}$ is also a subsequence of $\{b_k\}$, so $L = \lim c_k$ (Subsequential Limits). Therefore $z = L$ (Uniqueness of Limits).

Applying this with $p_k = n_k$ gives $x = L$, and applying it with $p_k = m_k$ gives $y = L$. Therefore $x = y$, which contradicts $x \neq y$.

Hence $\{a_n\}$ converges. □

$$\boxed{\text{True}}$$

Marking scheme (6 marks)

QRS suggested marking scheme (unofficial): the printed paper awards these 6 marks purely for the three correct True/False answers, since "You do NOT need to justify your answer." The scheme below is what QRS would use if justification were required, for teaching purposes.

(a) **(2 marks)** Correct answer $\boxed{\text{False}}$.

- 2 marks: correct answer. (If justification were required: 1 mark for the answer, 1 mark for correctly invoking the Algebraic Limit Theorem to derive a contradiction.)

(b) **(2 marks)** Correct answer $\boxed{\text{False}}$.

- 2 marks: correct answer. (If justification were required: 1 mark for the answer, 1 mark for correctly invoking the Bolzano-Weierstrass Theorem.)

(c) **(2 marks)** Correct answer $\boxed{\text{True}}$.

- 2 marks: correct answer. (If justification were required: 1 mark for correctly identifying that deleting a single term produces a proper subsequence, 1 mark for the shared-tail argument forcing $\{a_n\}$ itself to converge.)

Typical deductions if justification were graded: -1: claims $\{a_n\}$ is itself one of the "proper" subsequences in part (c); -1: asserts the Bolzano-Weierstrass Theorem without first noting boundedness of the subsequence in part (b); -0.5: sentence fragments in place of a full justification.

Question 2 (6 Marks)

Is each of the following statement true or false? You do NOT need to justify your answer.

(a) *(2 marks, QRS assumption)* There are two divergent series $\sum a_n$ and $\sum b_n$ where $\sum a_n b_n$ converges.

(b) *(2 marks, QRS assumption)* The set $\{x \in \mathbb{R} : x \neq 0\}$ is closed.

(c) *(2 marks, QRS assumption)* If K is a compact set and F is a closed set, then $K \setminus F$ is a compact set.

Part (a)

Method 1: Harmonic series and the Comparison Test. The statement is true. Let $a_n := b_n := \dfrac{1}{n}$ for $n \in \mathbb{N}$; then $a_n b_n = \dfrac{1}{n^2}$.

Proof. *$\sum a_n$ and $\sum b_n$ diverge.* Both series equal $\sum \frac{1}{n}$, which diverges (Divergence of the Harmonic Series). Recall the reason: for every $N \in \mathbb{N}$, each of the N integers $k \in \{N+1, \ldots, 2N\}$ satisfies $k \leq 2N$, so $\frac{1}{k} \geq \frac{1}{2N}$, and therefore

$$\sum_{k=N+1}^{2N} \frac{1}{k} \;\geq\; N \cdot \frac{1}{2N} = \frac{1}{2}.$$

Hence for $\varepsilon = \frac{1}{2}$ and every $N \in \mathbb{N}$, the indices $m := N$ and $n := 2N$ satisfy $n > m \geq N$ and $\frac{1}{m+1} + \cdots + \frac{1}{n} \geq \frac{1}{2}$. So no $N \in \mathbb{N}$ satisfies the condition of the Cauchy Criterion for Series, and $\sum \frac{1}{n}$ diverges.

$\sum a_n b_n$ converges. For $k \in \mathbb{N}$ let $c_k := \dfrac{1}{(k+1)^2}$ and $d_k := \dfrac{1}{k(k+1)}$. Since $0 < k < k+1$, we have $0 < k(k+1) < (k+1)^2$, and therefore $0 < c_k < d_k$ for all $k \in \mathbb{N}$. For every $k \in \mathbb{N}$,

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)} = d_k.$$

Hence for every $K \in \mathbb{N}$ the partial sums of $\sum d_k$ telescope:

$$\begin{aligned} D_K := \sum_{k=1}^{K} d_k &= \sum_{k=1}^{K} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{K} - \frac{1}{K+1}\right) \\ &= 1 - \frac{1}{K+1}. \end{aligned}$$

Since $\lim \frac{1}{K+1} = 0$ (Archimedean Property), $\lim D_K = 1 - 0 = 1$ (Algebraic Limit Theorem), so $\sum d_k$ converges. Since $0 \leq c_k \leq d_k$ for all $k \in \mathbb{N}$, $\sum c_k$ converges (Comparison Test); let $C_K := \sum_{k=1}^{K} c_k$ and $C := \lim C_K$.

Let $s_N := \sum_{n=1}^{N} \frac{1}{n^2}$. For every $K \in \mathbb{N}$,

$$s_{K+1} = 1 + \sum_{n=2}^{K+1} \frac{1}{n^2} = 1 + \sum_{k=1}^{K} \frac{1}{(k+1)^2} = 1 + C_K,$$

so $\lim s_{K+1} = 1 + C$ (Algebraic Limit Theorem), and therefore $\lim s_N = 1 + C$ (Tail-Shift Remark). Hence $\sum a_n b_n = \sum \frac{1}{n^2}$ converges.

Therefore $\sum a_n$ and $\sum b_n$ are divergent while $\sum a_n b_n$ is convergent. □

True

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Method 2: Disjoint-support example. This route chooses $\{a_n\}$ and $\{b_n\}$ so that every product $a_n b_n$ is 0. Let

$$a_n := \begin{cases} 1, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \qquad b_n := \begin{cases} 0, & n \text{ odd}, \\ 1, & n \text{ even}. \end{cases}$$

Proof. $\sum a_n$ *diverges.* Suppose $\sum a_n$ converges. Then $\lim a_n = 0$ (Terms of Convergent Series), so with $\varepsilon = \frac{1}{2}$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n| < \frac{1}{2}$. The number $n := 2N - 1$ is odd and $n \geq N$ (because $N \geq 1$), so $|a_n| = 1 \geq \frac{1}{2}$, a contradiction. Hence $\sum a_n$ diverges.

$\sum b_n$ *diverges.* Suppose $\sum b_n$ converges. Then $\lim b_n = 0$ (Terms of Convergent Series), so there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|b_n| < \frac{1}{2}$. The number $n := 2N$ is even and $n \geq N$, so $|b_n| = 1 \geq \frac{1}{2}$, a contradiction. Hence $\sum b_n$ diverges.

$\sum a_n b_n$ *converges.* Let $n \in \mathbb{N}$. If n is odd, then $b_n = 0$; if n is even, then $a_n = 0$. In both cases $a_n b_n = 0$. Therefore for every $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N} 0 = 0,$$

so the partial sums form the constant sequence $0, 0, 0, \ldots$, which converges to 0. Hence $\sum a_n b_n$ converges (with sum 0). □

True

Method 3: Partial-sum bounds and the Monotone Convergence Theorem. This route uses the example $a_n = b_n = \frac{1}{n}$ of Method 1, but decides both series from explicit bounds on their partial sums. Let $H_N := \sum_{n=1}^{N} \frac{1}{n}$ and $s_N := \sum_{n=1}^{N} \frac{1}{n^2}$.

Proof. $\{H_N\}$ *is unbounded.* We show by induction that $H_{2^k} \geq 1 + \frac{k}{2}$ for all $k \in \mathbb{N}_0$. For $k = 0$, $H_1 = 1$. Suppose $H_{2^k} \geq 1 + \frac{k}{2}$. The sum $H_{2^{k+1}} - H_{2^k} = \sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n}$ has $2^{k+1} - 2^k = 2^k$ terms, each at least $\frac{1}{2^{k+1}}$, so

$$H_{2^{k+1}} \;\geq\; H_{2^k} + 2^k \cdot \frac{1}{2^{k+1}} \;\geq\; 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}.$$

Let $B > 0$. By the Archimedean Property there is $k \in \mathbb{N}$ with $k > 2B$; then $H_{2^k} \geq 1 + \frac{k}{2} > 1 + B > B$. Hence $\{H_N\}$ is unbounded, so it does not converge (Boundedness of Convergent Sequences). Therefore $\sum a_n = \sum b_n = \sum \frac{1}{n}$ diverges.

$\{s_N\}$ *is increasing.* For every $N \in \mathbb{N}$, $s_{N+1} - s_N = \frac{1}{(N+1)^2} > 0$.

$\{s_N\}$ *is bounded.* For $n \geq 2$, $n^2 \geq n(n-1) > 0$, so $\frac{1}{n^2} \leq \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$. Hence for every $N \geq 2$,

$$0 < s_N = 1 + \sum_{n=2}^{N} \frac{1}{n^2} \;\leq\; 1 + \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1 + \left(1 - \frac{1}{N} \right) < 2,$$

and $0 < s_1 = 1 < 2$. So $0 < s_N < 2$ for all $N \in \mathbb{N}$.

Since $\{s_N\}$ is increasing and bounded, it converges (Monotone Convergence Theorem); that is, $\sum a_n b_n = \sum \frac{1}{n^2}$ converges. □

True

Part (b)

Method 1: Sequential Criterion for Closed Sets. The statement is false. Let $C := \{x \in \mathbb{R} : x \neq 0\}$; we exhibit a convergent sequence in C whose limit lies outside C.

Proof. Let $x_n := \frac{1}{n}$ for $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, $\frac{1}{n} > 0$, so $x_n \neq 0$ and $x_n \in C$.

Let $\varepsilon > 0$ be given. By the Archimedean Property there is $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$. For all $n \geq N$,

$$|x_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence $0 = \lim x_n$ (def. of limit).

Therefore $\{x_n\}$ is a convergent sequence contained in C whose limit 0 is not in C. By the Sequential Criterion for Closed Sets, C is not closed. □

False

Method 2: Definition of a closed set via a limit point. This route shows directly that 0 is a limit point of $C = \{x \in \mathbb{R} : x \neq 0\}$ that C does not contain.

Proof. Let $\varepsilon > 0$ be given and let $x := \frac{\varepsilon}{2}$. Then $x > 0$, so $x \neq 0$ and $x \in C \setminus \{0\}$; and $|x - 0| = \frac{\varepsilon}{2} < \varepsilon$.

Hence for every $\varepsilon > 0$ there is $x \in C \setminus \{0\}$ with $|x - 0| < \varepsilon$, so 0 is a limit point of C (def. of limit point).

Since $0 \notin C$, the set C does not contain all of its limit points. Therefore C is not closed (def. of closed set). □

False

Unofficial QRS Midterm Solutions

Part (c)

Method 1: Counterexample checked against the definition of a compact set. The statement is false. Let $K := [0,1]$ and $F := \{0\}$; then $K \setminus F = \{x \in \mathbb{R} : 0 \leq x \leq 1,\ x \neq 0\} = (0,1]$.

Proof. *K is compact.* K is bounded, since $|x| \leq 1$ for every $x \in K$. Let $\{y_n\}$ be a convergent sequence contained in K and $y := \lim y_n$. Since $0 \leq y_n$ and $y_n \leq 1$ for all $n \in \mathbb{N}$, the Order Limit Theorem gives $0 \leq y$ and $y \leq 1$, so $y \in K$. Hence K is closed (Sequential Criterion for Closed Sets). Since K is closed and bounded, K is compact (Sequential Compactness of Closed and Bounded Sets).

F is closed. Let $\{z_n\}$ be a convergent sequence contained in F and $z := \lim z_n$. Then $z_n = 0$ for all $n \in \mathbb{N}$, so $|z_n - 0| = 0 < \varepsilon$ for every $\varepsilon > 0$ and every $n \in \mathbb{N}$; hence $0 = \lim z_n$ (def. of limit), and $z = 0 \in F$ (Uniqueness of Limits). Therefore F is closed (Sequential Criterion for Closed Sets).

$K \setminus F = (0,1]$ is not compact. Let $a_n := \frac{1}{n}$ for $n \in \mathbb{N}$. Since $0 < \frac{1}{n} \leq 1$, $\{a_n\}$ is contained in $(0,1]$. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$; then for all $n \geq N$, $|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. Hence $0 = \lim a_n$ (def. of limit). Let $\{a_{n_k}\}$ be any subsequence of $\{a_n\}$. Then $0 = \lim a_{n_k}$ (Subsequential Limits), and 0 is its only limit (Uniqueness of Limits). Since $0 \notin (0,1]$, no subsequence of $\{a_n\}$ converges to a limit in $(0,1]$. Hence $(0,1]$ is not compact (def. of compact set).

Therefore K is compact and F is closed, but $K \setminus F$ is not compact. □

False

Method 2: A compact set is closed. This route keeps $K = [0,1]$ and $F = \{0\}$ and shows that $K \setminus F = (0,1]$ fails to be closed.

Proof. *The hypotheses hold.* If $\{y_n\}$ is a convergent sequence contained in K with $y := \lim y_n$, then $0 \leq y_n \leq 1$ for all $n \in \mathbb{N}$, so $0 \leq y \leq 1$ (Order Limit Theorem) and $y \in K$; hence K is closed (Sequential Criterion for Closed Sets). Since $|x| \leq 1$ for all $x \in K$, K is bounded. Therefore K is compact (Sequential Compactness of Closed and Bounded Sets). If $\{z_n\}$ is a convergent sequence contained in F, then $z_n = 0$ for all $n \in \mathbb{N}$, so $\lim z_n = 0 \in F$; hence F is closed (Sequential Criterion for Closed Sets).

$(0,1]$ is not closed. Let $\varepsilon > 0$ be given and let $x := \min\{\frac{\varepsilon}{2}, 1\}$. Then $0 < x \leq 1$, so $x \in (0,1] \setminus \{0\}$, and $|x - 0| = x \leq \frac{\varepsilon}{2} < \varepsilon$. Hence 0 is a limit point of $(0,1]$ (def. of limit point).

Since $0 \notin (0,1]$, the set $(0,1]$ does not contain all of its limit points, so $(0,1]$ is not closed (def. of closed set).

By the Sequential Compactness of Closed and Bounded Sets, every compact set is closed and bounded. Since $(0,1]$ is not closed, $(0,1] = K \setminus F$ is not compact. □

False

Method 3: Extrema of Closed and Bounded Sets. This route keeps $K = [0,1]$, $F = \{0\}$ and shows that $(0,1]$ has no minimum, which a compact set must have.

Proof. *The hypotheses hold.* If $\{y_n\}$ is a convergent sequence contained in K with $y := \lim y_n$, then $0 \leq y_n \leq 1$ for all $n \in \mathbb{N}$, so $0 \leq y \leq 1$ (Order Limit Theorem) and $y \in K$; hence K is closed (Sequential Criterion for Closed Sets). Since $|x| \leq 1$ for all $x \in K$, K is bounded, and therefore compact (Sequential Compactness of Closed and Bounded Sets). If $\{z_n\}$ is a convergent sequence contained in F, then $z_n = 0$ for all $n \in \mathbb{N}$, so $\lim z_n = 0 \in F$; hence F is closed (Sequential Criterion for Closed Sets).

$(0, 1]$ *is not compact.* Suppose, for contradiction, that $K \setminus F = (0, 1]$ is compact. Then $(0, 1]$ is closed and bounded (Sequential Compactness of Closed and Bounded Sets), and it is nonempty since $1 \in (0, 1]$. Therefore $(0, 1]$ has a minimum m (Extrema of Closed and Bounded Sets): $m \in (0, 1]$ and $m \leq x$ for all $x \in (0, 1]$.

Since $0 < m \leq 1$, we have $0 < \frac{m}{2} < m \leq 1$, so $\frac{m}{2} \in (0, 1]$ and $\frac{m}{2} < m$. This contradicts $m \leq x$ for all $x \in (0, 1]$.

Hence $K \setminus F$ is not compact, although K is compact and F is closed. □

$$\boxed{\text{False}}$$

Marking scheme (6 marks)

QRS suggested marking scheme (unofficial): as in Question 1, the printed paper awards these 6 marks purely for the three correct True/False answers. The scheme below is what QRS would use if justification were required.

(a) **(2 marks)** Correct answer $\boxed{\text{True}}$.

- 2 marks: correct answer. (If justification were required: 1 mark for a valid example with both $\sum a_n, \sum b_n$ divergent, 1 mark for correctly showing $\sum a_n b_n$ convergent, e.g. via the Comparison Test.)

(b) **(2 marks)** Correct answer $\boxed{\text{False}}$.

- 2 marks: correct answer. (If justification were required: 1 mark for exhibiting a convergent sequence in C with limit outside C, 1 mark for correctly invoking the Sequential Criterion for Closed Sets.)

(c) **(2 marks)** Correct answer $\boxed{\text{False}}$.

- 2 marks: correct answer. (If justification were required: 1 mark for a valid counterexample (K, F) with K compact and F closed, 1 mark for correctly showing $K \setminus F$ fails compactness via an explicit non-convergent-in-the-set sequence.)

Typical deductions if justification were graded: -1: in part (a), an example where $\sum a_n b_n$ does not actually converge (e.g. forgetting to check the product series, not just the individual terms); -1: in part (c), a counterexample where F is not actually closed or K is not actually compact; -0.5: sentence fragments in place of a full justification.

Question 3 (6 Marks)

In this question, A and B denote nonempty sets of real numbers that are bounded above.

Write down the definition of $s = \sup A$, and **use the definition to show** that

$$\max\{\sup A, \sup B\} = \sup(A \cup B).$$

Do not use the ε-Criterion for suprema and infima.

(Source: Tutorial 1, Question 2, with the added instructions to first state the definition of supremum and to avoid the ε-Criterion.)

Definition

Definition. Let $A \subseteq \mathbb{R}$. A number $b \in \mathbb{R}$ is an *upper bound* for A if for all $a \in A$, $a \leq b$. A number $s \in \mathbb{R}$ is the *supremum* of A, written $s = \sup A$, if

(1) s is an upper bound for A, that is, for all $a \in A$, $a \leq s$; and

(2) s is not greater than any upper bound for A, that is, for every upper bound b of A, $s \leq b$.

The supremum is unique when it exists: if s and s' both satisfy (1) and (2) for A, then s' is an upper bound for A by (1), so $s \leq s'$ by (2) for s; symmetrically $s' \leq s$; hence $s = s'$. The methods below use this to conclude.

Method 1: Verifying both clauses of the definition for $\max\{\sup A, \sup B\}$

Proof. Since A is nonempty and bounded above, $s := \sup A$ exists (AoC). Since B is nonempty and bounded above, $t := \sup B$ exists (AoC). Let $M := \max\{s, t\}$. Then $s \leq M$ and $t \leq M$.

Clause (1): M is an upper bound for $A \cup B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$.

- If $x \in A$, then $x \leq s$ by clause (1) for $s = \sup A$. Since $s \leq M$, $x \leq M$.
- If $x \in B$, then $x \leq t$ by clause (1) for $t = \sup B$. Since $t \leq M$, $x \leq M$.

Therefore for all $x \in A \cup B$, $x \leq M$; that is, M is an upper bound for $A \cup B$.

Existence of $\sup(A \cup B)$. Since $A \neq \emptyset$ and $A \subseteq A \cup B$, the set $A \cup B$ is nonempty. Since $A \cup B$ is also bounded above by M, $\sup(A \cup B)$ exists (AoC).

Clause (2): M is not greater than any upper bound for $A \cup B$. Let b be an upper bound for $A \cup B$. Let $a \in A$. Then $a \in A \cup B$, so $a \leq b$. Hence b is an upper bound for A, and clause (2) for $s = \sup A$ gives $s \leq b$. Likewise every element of B lies in $A \cup B$, so b is an upper bound for B, and clause (2) for $t = \sup B$ gives $t \leq b$. Since $M = s$ or $M = t$, it follows that $M \leq b$. Therefore for every upper bound b of $A \cup B$, $M \leq b$.

Conclusion. M satisfies clauses (1) and (2) of the definition for the set $A \cup B$. Since the supremum of $A \cup B$ is unique, $M = \sup(A \cup B)$; that is, $\max\{\sup A, \sup B\} = \sup(A \cup B)$. $\square$

$$\boxed{\max\{\sup A, \sup B\} = \sup(A \cup B)}$$

Method 2: WLOG reduction to the case $\sup A \geq \sup B$

Proof. Since A and B are nonempty and bounded above, $s := \sup A$ and $t := \sup B$ exist (AoC).

Both sides of the identity are unchanged when A and B are interchanged, because

$$\max\{s,t\} = \max\{t,s\} \qquad \text{and} \qquad A \cup B = B \cup A.$$

Hence we may assume, without loss of generality, that $s \geq t$ (otherwise interchange the names A and B). Then $\max\{s,t\} = s$, and it remains to prove that $s = \sup(A \cup B)$.

Clause (1) for s and $A \cup B$. Let $x \in A \cup B$.

- If $x \in A$, then $x \leq s$ by clause (1) for $s = \sup A$.
- If $x \in B$, then $x \leq t$ by clause (1) for $t = \sup B$, and $t \leq s$, so $x \leq s$.

Therefore s is an upper bound for $A \cup B$. Since $A \cup B \supseteq A \neq \emptyset$ and $A \cup B$ is bounded above by s, $\sup(A \cup B)$ exists (AoC).

Clause (2) for s and $A \cup B$. Let b be an upper bound for $A \cup B$. Since $A \subseteq A \cup B$, for all $a \in A$ we have $a \leq b$, so b is an upper bound for A. Hence $s \leq b$ by clause (2) for $s = \sup A$.

Conclusion. s satisfies clauses (1) and (2) for $A \cup B$, so $s = \sup(A \cup B)$ by uniqueness of the supremum. Since $\max\{\sup A, \sup B\} = s$, we obtain $\max\{\sup A, \sup B\} = \sup(A \cup B)$. □

$$\boxed{\max\{\sup A, \sup B\} = \sup(A \cup B)}$$

Method 3: Two inequalities between $\max\{\sup A, \sup B\}$ **and** $\sup(A \cup B)$

Proof. Since A and B are nonempty and bounded above, $s := \sup A$ and $t := \sup B$ exist (AoC). Let $M := \max\{s,t\}$.

Existence of $u := \sup(A \cup B)$. Since A is bounded above, there is an upper bound α for A; since B is bounded above, there is an upper bound β for B. Let $\gamma := \max\{\alpha, \beta\}$. If $x \in A$, then $x \leq \alpha \leq \gamma$; if $x \in B$, then $x \leq \beta \leq \gamma$. Hence γ is an upper bound for $A \cup B$. Since $A \cup B \supseteq A \neq \emptyset$, the set $A \cup B$ is nonempty and bounded above, so $u := \sup(A \cup B)$ exists (AoC).

First inequality: $M \leq u$. By clause (1) for u, for all $x \in A \cup B$, $x \leq u$. Since $A \subseteq A \cup B$, for all $a \in A$, $a \leq u$; hence u is an upper bound for A, and clause (2) for $s = \sup A$ gives $s \leq u$. In the same way $B \subseteq A \cup B$ gives that u is an upper bound for B, so $t \leq u$ by clause (2) for $t = \sup B$. Since $M = s$ or $M = t$, therefore $M \leq u$.

Second inequality: $u \leq M$. Let $x \in A \cup B$. If $x \in A$, then $x \leq s \leq M$ by clause (1) for s; if $x \in B$, then $x \leq t \leq M$ by clause (1) for t. Hence M is an upper bound for $A \cup B$, and clause (2) for $u = \sup(A \cup B)$ gives $u \leq M$.

Conclusion. Since $M \leq u$ and $u \leq M$, we have $M = u$; that is, $\max\{\sup A, \sup B\} = \sup(A \cup B)$. □

$$\boxed{\max\{\sup A, \sup B\} = \sup(A \cup B)}$$

Marking scheme (6 marks)

QRS suggested marking scheme (unofficial).

(i) **(1 mark) Definition.** The definition of $s = \sup A$ stated correctly, with both the "upper bound" and "least upper bound" clauses, and quantifiers in the correct order ("for all $a \in A$, $a \leq s$"; "for every upper bound b of A, $s \leq b$").

- -0.5: only one of the two clauses given.
- -0.5: quantifier order reversed or omitted (e.g. writing "$s \leq b$ for some upper bound b").

(ii) **(1 mark) Existence.** Noting that $A \cup B$ is nonempty and bounded above, so that $\sup(A \cup B)$ exists by the Axiom of Completeness, before using it.

- -1: uses $\sup(A \cup B)$ without ever justifying its existence.

(iii) **(2 marks) Upper bound direction.** A correct proof that $\max\{\sup A, \sup B\}$ is an upper bound for $A \cup B$, with the case split $x \in A$ / $x \in B$ (or the WLOG reduction of Method 2) fully carried out.

- -1: case split incomplete (only one of A, B checked).

(iv) **(1.5 marks) Least-upper-bound direction.** A correct proof that $\max\{\sup A, \sup B\}$ is not greater than any upper bound of $A \cup B$, via an arbitrary upper bound b of $A \cup B$, restricting b to A and to B, and applying the definition of $\sup A$ and $\sup B$ to conclude $\sup A \leq b$ and $\sup B \leq b$.

- -1: asserts $\sup A \leq \sup(A \cup B)$ (or similar) without the arbitrary-upper-bound argument.

(v) **(0.5 marks) Writing.** Full sentences, connectives, and inference words ("therefore", "hence", "so") throughout, rather than a bare symbolic chain.

Typical further deductions: -1: uses the ε-Criterion for suprema and infima, which is explicitly banned by the question; -0.5: sentence fragments or unjustified jumps between lines.

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Question 4 (6 Marks)

Write down the definition of a Cauchy sequence $\{a_n\}$, and **use the definition to show** that if $\{a_n\}$ is a Cauchy sequences [sic], then $\{a_n^2\}$ is a Cauchy sequence.

Do not use the Cauchy Criterion or the Algebraic Limit Theorem.

(Source: Tutorial 2, Question 5(b), with the added instruction to first state the definition of a Cauchy sequence.)

Definition

Definition. A sequence $\{a_n\}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ with $n \geq N$ and $m \geq N$, $|a_n - a_m| < \varepsilon$. In symbols,

$$\forall \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ such that } \forall n, m \geq N : \ |a_n - a_m| < \varepsilon.$$

Method 1: Boundedness lemma and the factorisation $a_n^2 - a_m^2 = (a_n - a_m)(a_n + a_m)$

Lemma. If $\{a_n\}$ is a Cauchy sequence, then there is $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$.

Proof of Lemma. Apply the definition of a Cauchy sequence with $\varepsilon = 1$: there is $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$, $|a_n - a_m| < 1$. Let $n \geq N_1$. Taking $m = N_1$ gives $|a_n - a_{N_1}| < 1$, and therefore

$$\begin{aligned} |a_n| &= |(a_n - a_{N_1}) + a_{N_1}| \\ &\leq |a_n - a_{N_1}| + |a_{N_1}| && \text{(Triangle Inequality)} \\ &< 1 + |a_{N_1}|. \end{aligned}$$

Let

$$M := \max\{|a_1|, \ldots, |a_{N_1-1}|, \, 1 + |a_{N_1}|\},$$

the maximum of a finite list (the part $|a_1|, \ldots, |a_{N_1-1}|$ is empty when $N_1 = 1$). Then $M \geq 1 + |a_{N_1}| \geq 1 > 0$. If $n < N_1$, then $|a_n|$ is in the list, so $|a_n| \leq M$. If $n \geq N_1$, then $|a_n| < 1 + |a_{N_1}| \leq M$. Therefore $|a_n| \leq M$ for all $n \in \mathbb{N}$. □

Proof. Let $\{a_n\}$ be a Cauchy sequence, and let $M > 0$ be as in the Lemma, so that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be given and put $\varepsilon' := \dfrac{\varepsilon}{2M}$; then $\varepsilon' > 0$ because $M > 0$. By the definition of a Cauchy sequence applied with ε', there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_n - a_m| < \varepsilon'$.

Let $n, m \geq N$. Since $a_n^2 - a_m^2 = (a_n - a_m)(a_n + a_m)$,

$$\begin{aligned} \left|a_n^2 - a_m^2\right| &= |a_n - a_m| \, |a_n + a_m| \\ &\leq |a_n - a_m| \, \big(|a_n| + |a_m|\big) && \text{(Triangle Inequality)} \\ &\leq |a_n - a_m| \, (M + M) = 2M \, |a_n - a_m| && \text{(Lemma)} \\ &< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon. \end{aligned}$$

Therefore for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_n^2 - a_m^2| < \varepsilon$. Hence $\{a_n^2\}$ is a Cauchy sequence (def. of Cauchy). □

$\boxed{\{a_n^2\} \text{ is a Cauchy sequence}}$

Method 2: Bound on a tail only

Proof. Let $\{a_n\}$ be a Cauchy sequence. Apply the definition with $\varepsilon = 1$: there is $N_0 \in \mathbb{N}$ such that for all $n, m \geq N_0$, $|a_n - a_m| < 1$. For every $n \geq N_0$, taking $m = N_0$,

$$|a_n| \leq |a_n - a_{N_0}| + |a_{N_0}| < 1 + |a_{N_0}| =: M \qquad \text{(Triangle Inequality)},$$

and $M \geq 1 > 0$. Only the tail $n \geq N_0$ is bounded here; the terms before N_0 are never used.

Let $\varepsilon > 0$ be given. By the definition applied with $\frac{\varepsilon}{2M} > 0$, there is $N_2 \in \mathbb{N}$ such that for all $n, m \geq N_2$, $|a_n - a_m| < \frac{\varepsilon}{2M}$. Let $N := \max\{N_0, N_2\}$ and let $n, m \geq N$. Then $n, m \geq N_0$, so $|a_n| < M$ and $|a_m| < M$; and $n, m \geq N_2$, so $|a_n - a_m| < \frac{\varepsilon}{2M}$. Therefore

$$\begin{aligned}
\left|a_n^2 - a_m^2\right| &= |a_n - a_m| \, |a_n + a_m| \\
&\leq |a_n - a_m| \, \big(|a_n| + |a_m|\big) && \text{(Triangle Inequality)} \\
&\leq |a_n - a_m| \, (M + M) = 2M \, |a_n - a_m| && (|a_n|, |a_m| < M) \\
&< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_n^2 - a_m^2| < \varepsilon$, so $\{a_n^2\}$ is a Cauchy sequence (def. of Cauchy). □

$$\boxed{\{a_n^2\} \text{ is a Cauchy sequence}}$$

Method 3: Single-index form of the definition

The slides first state the definition in the equivalent single-index form: $\{a_n\}$ is Cauchy if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a_N| < \varepsilon$. This method uses that form throughout, for both $\{a_n\}$ and $\{a_n^2\}$.

Proof. Let $\{a_n\}$ be Cauchy in the single-index form.

A bound. With $\varepsilon = 1$ there is $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $|a_n - a_{N_0}| < 1$, so $|a_n| \leq |a_n - a_{N_0}| + |a_{N_0}| < 1 + |a_{N_0}|$ (Triangle Inequality). Let $M := \max\{|a_1|, \ldots, |a_{N_0-1}|, 1 + |a_{N_0}|\}$ (the part $|a_1|, \ldots, |a_{N_0-1}|$ is empty when $N_0 = 1$). Then $M \geq 1 + |a_{N_0}| \geq 1 > 0$. If $n < N_0$, then $|a_n|$ is in the list, so $|a_n| \leq M$. If $n \geq N_0$, then $|a_n| < 1 + |a_{N_0}| \leq M$. Therefore $|a_n| \leq M$ for all $n \in \mathbb{N}$.

The estimate. Let $\varepsilon > 0$ be given. With $\frac{\varepsilon}{2M} > 0$ in place of ε, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a_N| < \frac{\varepsilon}{2M}$. For every $n \geq N$,

$$\begin{aligned}
\left|a_n^2 - a_N^2\right| &= |a_n - a_N| \, |a_n + a_N| \\
&\leq |a_n - a_N| \, \big(|a_n| + |a_N|\big) && \text{(Triangle Inequality)} \\
&\leq 2M \, |a_n - a_N| < 2M \cdot \frac{\varepsilon}{2M} = \varepsilon.
\end{aligned}$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n^2 - a_N^2| < \varepsilon$; that is, $\{a_n^2\}$ is Cauchy in the single-index form (def. of Cauchy). □

$$\boxed{\{a_n^2\} \text{ is a Cauchy sequence}}$$

Marking scheme (6 marks)

QRS suggested marking scheme (unofficial).

(i) **(1 mark) Definition.** The definition of a Cauchy sequence stated correctly, with quantifiers in the correct order: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|a_n - a_m| < \varepsilon$.

- -0.5: quantifier order wrong (e.g. N chosen before ε, or N allowed to depend on n).
- -0.5: uses the single-index form ($|a_n - a_N| < \varepsilon$) without adapting the rest of the proof to match, or silently swaps between the two forms.

(ii) **(1.5 marks) Boundedness Lemma.** A correct, self-contained proof that a Cauchy sequence is bounded, directly from the Cauchy definition with $\varepsilon = 1$, splitting into the tail (bounded by $1 + |a_N|$) and the finitely many head terms.

- -1: boundedness asserted without proof, or proved via convergence/the Cauchy Criterion.
- -0.5: omits the finitely-many-head-terms case, or does not verify $M > 0$.

(iii) **(2 marks) Factoring estimate.** The identity $a_n^2 - a_m^2 = (a_n - a_m)(a_n + a_m)$, the triangle-inequality bound $|a_n + a_m| \leq |a_n| + |a_m| \leq 2M$, and the resulting inequality $|a_n^2 - a_m^2| \leq 2M\,|a_n - a_m|$, all correctly derived.

- -1: factoring identity missing or incorrect.
- -0.5: bound $|a_n + a_m| \leq 2M$ not justified from the boundedness of $\{a_n\}$.

(iv) **(1 mark) Assembly.** The correct choice $\varepsilon' := \varepsilon/(2M)$ (or equivalent), applied to the Cauchy hypothesis, and the final chain of inequalities correctly matched back to the definition of $\{a_n^2\}$ being Cauchy.

- -0.5: N chosen depending on n (or m), rather than depending only on ε.

(v) **(0.5 marks) Writing.** Full sentences, connectives, and inference words throughout; quantifiers written in words ("let $\varepsilon > 0$ be given," "there exists $N \in \mathbb{N}$ such that"), never as a bare symbolic string.

Typical further deductions: -1: uses the Cauchy Criterion or the Algebraic Limit Theorem, both explicitly banned by the question; -0.5: sentence fragments / unjustified jumps between lines.