MH3100 Real Analysis I

Midterm Practice 2 (Solutions)

AY 2026/2027, Semester 1 – QRS original

Quantitative Research Society @NTU

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TIME ALLOWED: 60 minutes.

Instructions to Students

1. This test contains **FOUR (4)** questions.
2. Answer **all** questions. The marks for each question are indicated below.
3. Write your answers in the box provided. Use the **facing page** for more space.
4. This is a **CLOSED BOOK** test.
5. You may use calculators. You may cite any results proved in the lectures and tutorials, **unless you are asked by the question to prove the result.**

Question	1	2	3	4	Total
Total Marks	6	6	6	6	24
Awarded Marks					

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Question 1 (6 Marks)

Is each of the following statement true or false? You do NOT need to justify your answer.

(a) (2 marks) If $\{a_n\}$ is a bounded sequence such that every convergent subsequence of $\{a_n\}$ converges to the same real number L, then $\{a_n\}$ converges to L.

(b) (2 marks) There is a monotone sequence that has a convergent subsequence but is itself divergent.

(c) (2 marks) If $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences with $b_n \neq 0$ for every $n \in \mathbb{N}$, then $\left\{\dfrac{a_n}{b_n}\right\}$ is a Cauchy sequence.

Solution

Part (a), Method 1: Contradiction via Bolzano–Weierstrass

Proof. Since $\{a_n\}$ is bounded, there is $M > 0$ such that $|a_n| \leq M$ for every $n \in \mathbb{N}$. Suppose, for contradiction, that $\{a_n\}$ does not converge to L. By the negation of the definition of $L = \lim a_n$, there is $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there is an index $n \geq N$ with $|a_n - L| \geq \varepsilon$. Fix this ε and let

$$E := \{n \in \mathbb{N} : |a_n - L| \geq \varepsilon\}.$$

For every $N \in \mathbb{N}$ the set E contains some $n \geq N$, so E is not bounded above. Every finite subset of $\mathbb{N}$ is bounded above by its largest element; therefore E is infinite.

For $n \in E$, $|a_n - L| \geq \varepsilon$ means that $a_n - L \geq \varepsilon$ or $L - a_n \geq \varepsilon$. Hence $E = E^+ \cup E^-$, where

$$E^+ := \{n \in \mathbb{N} : a_n \geq L + \varepsilon\}, \qquad E^- := \{n \in \mathbb{N} : a_n \leq L - \varepsilon\}.$$

If E^+ and E^- were both finite, then their union E would be finite. Therefore E^+ is infinite or E^- is infinite.

Case 1: E^+ is infinite. List the elements of E^+ in increasing order, $m_1 < m_2 < m_3 < \cdots$, and let $b_k := a_{m_k}$ for $k \in \mathbb{N}$. Then $\{b_k\}$ is a subsequence of $\{a_n\}$, and

$$b_k \geq L + \varepsilon \quad \text{and} \quad |b_k| \leq M \qquad \text{for every } k \in \mathbb{N}.$$

Hence $\{b_k\}$ is bounded, so by (BW) it has a convergent subsequence $\{b_{k_j}\}$; let $L' := \lim_j b_{k_j}$. Since $j \mapsto k_j$ and $k \mapsto m_k$ are strictly increasing, $j \mapsto m_{k_j}$ is strictly increasing. Therefore $\{b_{k_j}\} = \{a_{m_{k_j}}\}$ is a convergent subsequence of $\{a_n\}$, and the hypothesis gives $L' = L$. On the other hand, $b_{k_j} \geq L + \varepsilon$ for every $j \in \mathbb{N}$, and the constant sequence with every term $L + \varepsilon$ converges to $L + \varepsilon$. Hence the Order Limit Theorem (part 2) gives

$$L' = \lim_j b_{k_j} \;\geq\; L + \varepsilon.$$

Therefore $L = L' \geq L + \varepsilon$, so $\varepsilon \leq 0$, which contradicts $\varepsilon > 0$.

Case 2: E^- is infinite. List the elements of E^- in increasing order, $m_1 < m_2 < m_3 < \cdots$, and let $b_k := a_{m_k}$ for $k \in \mathbb{N}$. Then

$$b_k \leq L - \varepsilon \quad \text{and} \quad |b_k| \leq M \qquad \text{for every } k \in \mathbb{N}.$$

Hence $\{b_k\}$ is bounded, so by (BW) it has a convergent subsequence $\{b_{k_j}\}$; let $L' := \lim_j b_{k_j}$. As in Case 1, $j \mapsto m_{k_j}$ is strictly increasing, so $\{b_{k_j}\}$ is a convergent subsequence of $\{a_n\}$, and

the hypothesis gives $L' = L$. Since $b_{k_j} \leq L - \varepsilon$ for every $j \in \mathbb{N}$, the Order Limit Theorem (part 3) gives

$$L' = \lim_j b_{k_j} \ \leq \ L - \varepsilon.$$

Therefore $L = L' \leq L - \varepsilon$, so $\varepsilon \leq 0$, which contradicts $\varepsilon > 0$.

Both cases lead to a contradiction. Therefore $\{a_n\}$ converges to L. □

True

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Part (a), Method 2: Divergence Criterion

Proof. **Step 1: $\{a_n\}$ converges.** Suppose, for contradiction, that $\{a_n\}$ diverges. Since $\{a_n\}$ is bounded and divergent, the Divergence Criterion gives two subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ of $\{a_n\}$ such that

$$P := \lim_k a_{n_k}, \qquad Q := \lim_k a_{m_k}, \qquad P \neq Q.$$

Both $\{a_{n_k}\}$ and $\{a_{m_k}\}$ are convergent subsequences of $\{a_n\}$, so the hypothesis gives $P = L$ and $Q = L$. Hence $P = Q$, which contradicts $P \neq Q$. Therefore $\{a_n\}$ converges; let $a := \lim a_n$.

Step 2: $a = L$. Since $\{a_n\}$ is bounded, (BW) gives a convergent subsequence $\{a_{p_k}\}$ of $\{a_n\}$, and the hypothesis gives $\lim_k a_{p_k} = L$. Since $a = \lim a_n$, the theorem on Subsequential Limits gives $\lim_k a_{p_k} = a$. By Uniqueness of Limits, $a = L$. Therefore $L = \lim a_n$. □

True

Part (b), Method 1: Monotone Convergence Lemma

Proof. Let $\{a_n\}$ be any monotone sequence that has a convergent subsequence. A monotone sequence is eventually monotone (take $N_0 = 1$ in the definition). Hence $\{a_n\}$ is eventually monotone and has a convergent subsequence, so the Monotone Convergence Lemma gives that $\{a_n\}$ is convergent. Therefore every monotone sequence with a convergent subsequence is convergent, so there is no monotone sequence that has a convergent subsequence and is divergent. Hence the statement, which asserts that such a sequence exists, is false. □

False

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Part (b), Method 2: Boundedness and the Monotone Convergence Theorem

Proof. Let $\{a_n\}$ be a monotone sequence with a convergent subsequence $\{a_{n_k}\}$. Every convergent sequence is bounded (Boundedness of Convergent Sequences), so there is $M > 0$ such that $|a_{n_k}| \leq M$ for every $k \in \mathbb{N}$. For every $n \in \mathbb{N}$, the Subsequence Index Bound applied at $k = n$ gives $n_n \geq n$.

Case 1: $\{a_n\}$ is increasing. Then $a_p \leq a_q$ whenever $p \leq q$. For every $n \in \mathbb{N}$, $1 \leq n \leq n_n$, so

$$a_1 \;\leq\; a_n \;\leq\; a_{n_n} \;\leq\; M.$$

If $a_n \geq 0$, then $|a_n| = a_n \leq M$; if $a_n < 0$, then $|a_n| = -a_n \leq -a_1 \leq |a_1|$. Hence $|a_n| \leq \max\{|a_1|, M\}$ for every $n \in \mathbb{N}$, so $\{a_n\}$ is bounded.

Case 2: $\{a_n\}$ is decreasing. Then $a_p \geq a_q$ whenever $p \leq q$. For every $n \in \mathbb{N}$, $1 \leq n \leq n_n$, so

$$a_1 \;\geq\; a_n \;\geq\; a_{n_n} \;\geq\; -M.$$

If $a_n \leq 0$, then $|a_n| = -a_n \leq M$; if $a_n > 0$, then $|a_n| = a_n \leq a_1 \leq |a_1|$. Hence $|a_n| \leq \max\{|a_1|, M\}$ for every $n \in \mathbb{N}$, so $\{a_n\}$ is bounded.

In both cases $\{a_n\}$ is monotone and bounded, so (MCT) gives that $\{a_n\}$ converges. Therefore no monotone sequence with a convergent subsequence is divergent, and the statement is false. □

False

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Part (b), Method 3: Direct proof from the definition of limit

Proof. Let $\{a_n\}$ be a monotone sequence with a convergent subsequence $\{a_{n_k}\}$, and let $a := \lim_k a_{n_k}$.

Case 1: $\{a_n\}$ is increasing. First we show $a_n \le a$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. For every $k \ge n$, the Subsequence Index Bound gives $n_k \ge k \ge n$, so $a_{n_k} \ge a_n$ because $\{a_n\}$ is increasing. Hence $\{a_{n_k}\}_k$ eventually satisfies $a_{n_k} \ge a_n$, and the Order Limit Theorem (part 2), comparing with the constant sequence whose every term is a_n, gives $a = \lim_k a_{n_k} \ge a_n$.

Now let $\varepsilon > 0$. Since $a = \lim_k a_{n_k}$, there is $K \in \mathbb{N}$ such that for all $k \ge K$, $|a_{n_k} - a| < \varepsilon$; in particular $a_{n_K} > a - \varepsilon$. Let $N := n_K$. For every $n \ge N$, $\{a_n\}$ increasing gives $a_n \ge a_{n_K}$, and the first step gives $a_n \le a$. Therefore

$$a - \varepsilon \;<\; a_{n_K} \;\le\; a_n \;\le\; a \;<\; a + \varepsilon,$$

so $|a_n - a| < \varepsilon$ for all $n \ge N$.

Case 2: $\{a_n\}$ is decreasing. First we show $a_n \ge a$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. For every $k \ge n$, $n_k \ge k \ge n$, so $a_{n_k} \le a_n$ because $\{a_n\}$ is decreasing. Hence the Order Limit Theorem (part 3) gives $a = \lim_k a_{n_k} \le a_n$.

Now let $\varepsilon > 0$. There is $K \in \mathbb{N}$ such that for all $k \ge K$, $|a_{n_k} - a| < \varepsilon$; in particular $a_{n_K} < a + \varepsilon$. Let $N := n_K$. For every $n \ge N$, $\{a_n\}$ decreasing gives $a_n \le a_{n_K}$, and the first step gives $a_n \ge a$. Therefore

$$a - \varepsilon \;<\; a \;\le\; a_n \;\le\; a_{n_K} \;<\; a + \varepsilon,$$

so $|a_n - a| < \varepsilon$ for all $n \ge N$.

In both cases, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \ge N$, $|a_n - a| < \varepsilon$. Hence $a = \lim a_n$ (def. of limit). Therefore every monotone sequence with a convergent subsequence converges, and the statement is false. □

False

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Part (c), Method 1: Counterexample checked against the definition of a Cauchy sequence

Recall the definition: $\{x_n\}$ is Cauchy if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - x_N| < \varepsilon$.

Proof. Let $a_n := 1$ and $b_n := \dfrac{1}{n}$ for $n \in \mathbb{N}$. Then $b_n > 0$, so $b_n \neq 0$ for every $n \in \mathbb{N}$.

$\{a_n\}$ **is Cauchy.** Let $\varepsilon > 0$ and let $N := 1$. For all $n \geq 1$, $|a_n - a_1| = |1 - 1| = 0 < \varepsilon$.

$\{b_n\}$ **is Cauchy.** Let $\varepsilon > 0$. By (Archimedean) there is $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$. For all $n \geq N$, $0 < \frac{1}{n} \leq \frac{1}{N}$, so

$$|b_n - b_N| = \frac{1}{N} - \frac{1}{n} < \frac{1}{N} < \varepsilon.$$

$\{a_n/b_n\}$ **is not Cauchy.** Let

$$c_n := \frac{a_n}{b_n} = \frac{1}{1/n} = n.$$

Take $\varepsilon := 1$. Let $N \in \mathbb{N}$ be arbitrary and take $n := N + 1$, so $n \geq N$. Then

$$|c_{N+1} - c_N| = |(N + 1) - N| = 1 \geq \varepsilon.$$

Hence for $\varepsilon = 1$ no $N \in \mathbb{N}$ satisfies $|c_n - c_N| < \varepsilon$ for all $n \geq N$, so $\{c_n\}$ is not Cauchy.

Therefore $\{a_n\}$ and $\{b_n\}$ satisfy the hypotheses while $\{a_n/b_n\}$ is not Cauchy, so the statement is false. □

False

Part (c), Method 2: Cauchy Criterion and unboundedness

Proof. Let $a_n := 1$ and $b_n := \dfrac{1}{n} \neq 0$ for $n \in \mathbb{N}$. For every $\varepsilon > 0$ and every $n \in \mathbb{N}$, $|a_n - 1| = 0 < \varepsilon$; hence $1 = \lim a_n$. Let $\varepsilon > 0$. By (Archimedean) there is $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$, and for all $n \geq N$,

$$|b_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence $0 = \lim b_n$. Both sequences converge, so both are Cauchy (Cauchy Criterion).

Let $c_n := \dfrac{a_n}{b_n} = n$. Let $M > 0$. By (Archimedean) there is $n \in \mathbb{N}$ with $n > M$, so $|c_n| = n > M$. Hence no $M > 0$ satisfies $|c_n| \leq M$ for every $n \in \mathbb{N}$, so $\{c_n\}$ is unbounded. Every convergent sequence is bounded (Boundedness of Convergent Sequences), so $\{c_n\}$ is not convergent. By the Cauchy Criterion, a sequence is Cauchy if and only if it is convergent; therefore $\{c_n\}$ is not Cauchy, and the statement is false. □

False

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Part (c), Method 3: Bounded oscillating quotient and the Divergence Criterion

Proof. Let $a_n := \dfrac{1}{n}$ and $b_n := \dfrac{(-1)^n}{n}$ for $n \in \mathbb{N}$. Then $|b_n| = \frac{1}{n} > 0$, so $b_n \neq 0$ for every $n \in \mathbb{N}$. Let $\varepsilon > 0$. By (Archimedean) there is $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$, and for all $n \geq N$,

$$|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon, \qquad |b_n - 0| = \frac{|(-1)^n|}{n} = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence $0 = \lim a_n$ and $0 = \lim b_n$, so both sequences are Cauchy (Cauchy Criterion).

Let

$$c_n := \frac{a_n}{b_n} = \frac{1/n}{(-1)^n/n} = \frac{1}{(-1)^n} = (-1)^n.$$

Then $c_{2k} = 1$ and $c_{2k-1} = -1$ for every $k \in \mathbb{N}$. The index maps $k \mapsto 2k$ and $k \mapsto 2k-1$ are strictly increasing, so $\{c_{2k}\}$ and $\{c_{2k-1}\}$ are subsequences of $\{c_n\}$; as constant sequences they converge to 1 and to -1 respectively. Since $1 \neq -1$, the Divergence Criterion ("if" direction) gives that $\{c_n\}$ is divergent. By the Cauchy Criterion, $\{c_n\}$ is therefore not Cauchy, and the statement is false. □

$$\boxed{\text{False}}$$

Marking scheme (6 marks)

QRS suggested marking scheme.

(a) **(2 marks)** 2 marks for $\boxed{\text{True}}$; 0 marks for $\boxed{\text{False}}$. No justification is required on the actual exam.

(b) **(2 marks)** 2 marks for $\boxed{\text{False}}$; 0 marks for $\boxed{\text{True}}$.

(c) **(2 marks)** 2 marks for $\boxed{\text{False}}$; 0 marks for $\boxed{\text{True}}$.

Typical deductions (if justification is examined, e.g. in a tutorial recap): in (a), asserting the result without reference to a compactness-type tool (Bolzano-Weierstrass); in (c), attempting a "proof" instead of noticing a quotient can be unbounded; confusing Cauchy with merely bounded.

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Question 2 (6 Marks)

Is each of the following statement true or false? You do NOT need to justify your answer.

(a) (2 marks) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

(b) (2 marks) Every finite subset of $\mathbb{R}$ is a closed set.

(c) (2 marks) $\overline{\mathbb{Q} \cap [0,1]} = [0,1]$.

Solution

Part (a), Method 1: Counterexample via the Alternating Series Test and the harmonic series

For $s, t \geq 0$ with $s < t$ we have $s^2 \leq st < t^2$. Hence, for all $x, y \geq 0$,

$$x < y \iff \sqrt{x} < \sqrt{y}.$$

Indeed, ($\Leftarrow$) is the previous sentence with $s = \sqrt{x}$, $t = \sqrt{y}$; for ($\Rightarrow$), if $\sqrt{x} \geq \sqrt{y}$, then either $\sqrt{x} = \sqrt{y}$, so $x = y$, or $\sqrt{y} < \sqrt{x}$, so $y < x$; in both cases $x \geq y$.

Proof. Let $b_n := \dfrac{1}{\sqrt{n}}$ and $a_n := (-1)^n b_n = \dfrac{(-1)^n}{\sqrt{n}}$ for $n \in \mathbb{N}$.

Step 1: $\{b_n\}$ is decreasing. For every $n \in \mathbb{N}$, $n < n+1$, so $\sqrt{n} < \sqrt{n+1}$. Therefore

$$b_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = b_n.$$

Step 2: $0 = \lim b_n$. Let $\varepsilon > 0$. By (Archimedean) there is $N \in \mathbb{N}$ with $N > \dfrac{1}{\varepsilon^2}$. For every $n \geq N$, $n > \dfrac{1}{\varepsilon^2}$, so $\sqrt{n} > \sqrt{1/\varepsilon^2} = \dfrac{1}{\varepsilon}$, and therefore

$$|b_n - 0| = \frac{1}{\sqrt{n}} < \varepsilon.$$

Step 3: $\sum_{n=1}^{\infty} a_n$ converges. By Steps 1 and 2 and the Alternating Series Test, the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n$$

converges; let s be its sum. Since $a_n = (-1) \cdot (-1)^{n-1} b_n$ for every n, the Algebraic Limit Theorem for Series (constant multiple -1) gives that $\sum_{n=1}^{\infty} a_n$ converges, with sum $-s$.

Step 4: $\sum_{n=1}^{\infty} a_n^2$ diverges. For every $n \in \mathbb{N}$,

$$a_n^2 = \frac{(-1)^{2n}}{(\sqrt{n})^2} = \frac{1}{n}.$$

Take $\varepsilon := \frac{1}{2}$ in the Cauchy Criterion for Series. Let $N \in \mathbb{N}$ be arbitrary and let $m := N$, $n := 2N$, so $n > m \geq N$. Each $k \in \{N+1, \ldots, 2N\}$ satisfies $k \leq 2N$, so $\frac{1}{k} \geq \frac{1}{2N}$, and there are exactly N such k. Therefore

$$a_{N+1}^2 + \cdots + a_{2N}^2 = \sum_{k=N+1}^{2N} \frac{1}{k} \;\geq\; N \cdot \frac{1}{2N} = \frac{1}{2} = \varepsilon.$$

Hence the Cauchy condition fails for $\varepsilon = \frac{1}{2}$, and $\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Cauchy Criterion for Series; this is the Divergence of the Harmonic Series).

By Steps 3 and 4, $\sum_{n=1}^{\infty} a_n$ converges while $\sum_{n=1}^{\infty} a_n^2$ diverges, so the statement is false. □

$$\boxed{\text{False}}$$

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Part (a), Method 2: Block counterexample without square roots

Proof. For $k \in \mathbb{N}$ let the k-th block be the $2k$ consecutive indices

$$P_k := \{k(k-1)+1,\ k(k-1)+2,\ \ldots,\ k(k+1)\}.$$

The last index of P_k is $k(k+1)$ and the first index of P_{k+1} is $(k+1)k+1$, so the blocks follow one another without gaps or overlaps, starting from $P_1 = \{1, 2\}$. Since $k(k+1) \geq k$, every $n \in \mathbb{N}$ satisfies $n \leq k(k+1)$ for some k; for the least such k, $n > (k-1)k$, so $n \in P_k$. Hence every $n \in \mathbb{N}$ lies in exactly one block. For $n = k(k-1)+j \in P_k$ with $1 \leq j \leq 2k$, define

$$a_n := \frac{(-1)^{j-1}}{k},$$

so block k reads $\frac{1}{k}, -\frac{1}{k}, \frac{1}{k}, -\frac{1}{k}, \ldots, \frac{1}{k}, -\frac{1}{k}$ (k pairs).

Step 1: $\sum_{n=1}^{\infty} a_n$ converges (to 0). Let $s_n := a_1 + \cdots + a_n$ and $s_0 := 0$. Each block is k pairs $\frac{1}{k} - \frac{1}{k} = 0$, so the sum over block k is 0, and by induction on k,

$$s_{k(k+1)} = s_{(k-1)k} + 0 = 0 \qquad \text{for every } k \in \mathbb{N}.$$

For $n = k(k-1)+j \in P_k$, the partial sum adds the first j terms of block k to $s_{k(k-1)} = 0$:

$$s_n = \sum_{i=1}^{j} \frac{(-1)^{i-1}}{k} = \begin{cases} \frac{1}{k}, & j \text{ odd}, \\ 0, & j \text{ even}. \end{cases}$$

Hence $0 \leq s_n \leq \frac{1}{k}$ whenever $n \in P_k$. Let $\varepsilon > 0$. By (Archimedean) there is $K \in \mathbb{N}$ with $\frac{1}{K} < \varepsilon$. Let $N := K(K-1)+1$. If $n \geq N$ and $n \in P_k$, then $k \geq K$, since $k \leq K-1$ would give $n \leq (K-1)K < N$. Therefore, for all $n \geq N$,

$$|s_n - 0| \leq \frac{1}{k} \leq \frac{1}{K} < \varepsilon.$$

Hence $0 = \lim s_n$, so $\sum_{n=1}^{\infty} a_n$ converges.

Step 2: $\sum_{n=1}^{\infty} a_n^2$ diverges. For $n \in P_k$, $a_n^2 = \frac{1}{k^2}$, so the sum of a_n^2 over block k is $2k \cdot \frac{1}{k^2} = \frac{2}{k}$. Take $\varepsilon := 1$ in the Cauchy Criterion for Series. Let $N \in \mathbb{N}$ be arbitrary, let $K := N$, $m := K(K+1)$ and $n := 2K(2K+1)$; then $n > m \geq K = N$. The indices $m+1, \ldots, n$ are exactly the blocks $P_{K+1}, \ldots, P_{2K}$, and each $k \in \{K+1, \ldots, 2K\}$ has $\frac{2}{k} \geq \frac{2}{2K}$. Therefore

$$a_{m+1}^2 + \cdots + a_n^2 = \sum_{k=K+1}^{2K} \frac{2}{k} \ \geq\ K \cdot \frac{2}{2K} = 1 = \varepsilon.$$

Hence the Cauchy condition fails for $\varepsilon = 1$, and $\sum_{n=1}^{\infty} a_n^2$ diverges (Cauchy Criterion for Series).

By Steps 1 and 2, $\sum_{n=1}^{\infty} a_n$ converges while $\sum_{n=1}^{\infty} a_n^2$ diverges, so the statement is false. □

False

Part (b), Method 1: Definition of limit point (minimum distance)

Proof. Let $F \subseteq \mathbb{R}$ be finite and let $a \in \mathbb{R}$. We show that a is not a limit point of F.

Case 1: $F \setminus \{a\} = \emptyset$. Take $\varepsilon := 1$. There is no $x \in F \setminus \{a\}$ at all, in particular none with $|x - a| < 1$. Hence a is not a limit point of F (def. of limit point).

Case 2: $F \setminus \{a\} \neq \emptyset$. Let

$$D := \{|x - a| : x \in F \setminus \{a\}\}.$$

D is a nonempty finite set of real numbers, and each element of D is positive because $x \neq a$. Hence $d := \min D$ exists and $d > 0$. For every $x \in F \setminus \{a\}$, $|x - a| \in D$, so $|x - a| \geq d$. Therefore, for $\varepsilon := d$ there is no $x \in F \setminus \{a\}$ with $|x - a| < \varepsilon$, so a is not a limit point of F (def. of limit point).

In both cases a is not a limit point of F. Since $a \in \mathbb{R}$ was arbitrary, the set of limit points of F is $\emptyset$, and $\emptyset \subseteq F$. Hence F contains all its limit points, i.e. F is closed (def. of closed set). □

True

Part (b), Method 2: Sequential Criterion for Closed Sets with a constant subsequence

Proof. Let $F \subseteq \mathbb{R}$ be finite. If $F = \emptyset$, there is no sequence contained in F, so every convergent sequence contained in F has its limit in F (vacuously). Otherwise write $F = \{f_1, \ldots, f_m\}$ with $m \in \mathbb{N}$. Let $\{x_n\}$ be a sequence contained in F with $x = \lim x_n$. For $i = 1, \ldots, m$ let $E_i := \{n \in \mathbb{N} : x_n = f_i\}$. Each x_n equals some f_i, so

$$\mathbb{N} = E_1 \cup E_2 \cup \cdots \cup E_m.$$

$\mathbb{N}$ is infinite and a finite union of finite sets is finite, so some E_i is infinite. List its elements in increasing order, $n_1 < n_2 < \cdots$. Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ with $x_{n_k} = f_i$ for every $k \in \mathbb{N}$. For every $\varepsilon > 0$ and every k, $|x_{n_k} - f_i| = 0 < \varepsilon$, so $f_i = \lim_k x_{n_k}$. By Subsequential Limits, $x = \lim_k x_{n_k}$. By Uniqueness of Limits, $x = f_i \in F$. Hence every convergent sequence contained in F has its limit in F, so F is closed (Sequential Criterion for Closed Sets). □

True

Part (b), Method 3: Induction on the number of elements

Proof. Singletons are closed. Let $c \in \mathbb{R}$ and $a \in \mathbb{R}$. If $a = c$, then $\{c\} \setminus \{a\} = \emptyset$, so for $\varepsilon := 1$ no $x \in \{c\} \setminus \{a\}$ has $|x - a| < \varepsilon$. If $a \neq c$, take $\varepsilon := |c - a| > 0$; the only $x \in \{c\} \setminus \{a\}$ is $x = c$, and $|c - a| = \varepsilon$ is not $< \varepsilon$. In both cases a is not a limit point of $\{c\}$. Hence $\{c\}$ has no limit points, so $\{c\}$ is closed (def. of closed set).

Induction. For $m \in \mathbb{N}_0$ let $P(m)$ be the statement: every subset of $\mathbb{R}$ with exactly m elements is closed. $P(0)$: the only such set is $\emptyset$; for every $a \in \mathbb{R}$ and $\varepsilon := 1$ there is no $x \in \emptyset \setminus \{a\}$, so $\emptyset$ has no limit points and is closed. Suppose $P(m)$ holds, and let $F \subseteq \mathbb{R}$ have exactly $m + 1$ elements. Choose $c \in F$. Then

$$F = (F \setminus \{c\}) \cup \{c\},$$

where $F \setminus \{c\}$ has exactly m elements, so it is closed by $P(m)$, and $\{c\}$ is closed by the first paragraph. The union of two closed sets is closed (Tutorial 5, Q4(a)); therefore F is closed, and $P(m + 1)$ holds. By induction, $P(m)$ holds for every $m \in \mathbb{N}_0$, so every finite subset of $\mathbb{R}$ is closed. □

True

Part (c), Method 1: Sequential Criterion for Closures, both inclusions

Let $B := \mathbb{Q} \cap [0, 1]$.

Proof. ($\subseteq$) Let $a \in \overline{B}$. By the Sequential Criterion for Closures there is a sequence $\{y_n\}$ contained in B with $a = \lim y_n$. Every $y_n \in B \subseteq [0, 1]$, so $0 \leq y_n$ and $y_n \leq 1$ for every $n \in \mathbb{N}$. Hence the Order Limit Theorem (parts 1 and 3) gives $0 \leq a$ and $a \leq 1$. Therefore $a \in [0, 1]$, and $\overline{B} \subseteq [0, 1]$.

($\supseteq$) Let $x \in [0, 1]$ and fix $n \in \mathbb{N}$. Let

$$S_n := \{k \in \mathbb{N}_0 : k \leq nx\}.$$

Since $nx \geq 0$, $0 \in S_n$. Since $nx \leq n$, every $k \in S_n$ satisfies $k \leq n$, so $S_n \subseteq \{0, 1, \ldots, n\}$. Hence S_n is a nonempty finite set, so $k_n := \max S_n$ exists and $0 \leq k_n \leq n$. Since $k_n + 1 \in \mathbb{N}_0$ and $k_n + 1 > \max S_n$, $k_n + 1 \notin S_n$, i.e. $k_n + 1 > nx$. Together with $k_n \in S_n$ this gives

$$k_n \leq nx < k_n + 1 \quad \Longrightarrow \quad 0 \leq nx - k_n < 1.$$

Let $q_n := \dfrac{k_n}{n}$. Then $q_n \in \mathbb{Q}$ (a quotient of integers) and $0 \leq q_n \leq 1$ (since $0 \leq k_n \leq n$), so $q_n \in B$. Moreover

$$|q_n - x| = \frac{|k_n - nx|}{n} = \frac{nx - k_n}{n} < \frac{1}{n}.$$

Let $\varepsilon > 0$. By (Archimedean) there is $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$. For all $n \geq N$,

$$|q_n - x| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence $x = \lim q_n$ (def. of limit), with $\{q_n\}$ contained in B, so $x \in \overline{B}$ by the Sequential Criterion for Closures. Therefore $[0, 1] \subseteq \overline{B}$.

By the two inclusions, $\overline{\mathbb{Q} \cap [0, 1]} = [0, 1]$. □

True

Part (c), Method 2: ε-Criterion for Closures, both inclusions

Let $B := \mathbb{Q} \cap [0,1]$.

Proof. ($\subseteq$) We prove the contrapositive: if $a \notin [0,1]$, then $a \notin \overline{B}$. *Case* $a < 0$. Let $\varepsilon := -a > 0$. Every $y \in B$ has $y \geq 0$, so

$$|y - a| = y - a \geq -a = \varepsilon.$$

Case $a > 1$. Let $\varepsilon := a - 1 > 0$. Every $y \in B$ has $y \leq 1$, so

$$|y - a| = a - y \geq a - 1 = \varepsilon.$$

In both cases no $y \in B$ satisfies $|y - a| < \varepsilon$, so $a \notin \overline{B}$ (ε-Criterion for Closures). Therefore $\overline{B} \subseteq [0,1]$.

($\supseteq$) Let $x \in [0,1]$ and $\varepsilon > 0$. By (Archimedean) there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Let $S := \{k \in \mathbb{N}_0 : k \leq nx\}$. Since $0 \leq nx \leq n$, $0 \in S$ and $S \subseteq \{0, 1, \ldots, n\}$, so $k := \max S$ exists with $0 \leq k \leq n$. Since $k + 1 \in \mathbb{N}_0$ and $k + 1 > \max S$, $k + 1 \notin S$, so

$$k \leq nx < k + 1 \quad \Longrightarrow \quad 0 \leq nx - k < 1.$$

Let $q := \dfrac{k}{n}$. Then $q \in \mathbb{Q}$ and $0 \leq q \leq 1$, so $q \in B$, and

$$|q - x| = \frac{nx - k}{n} < \frac{1}{n} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, for every $\varepsilon > 0$ there is $q \in B$ with $|q - x| < \varepsilon$, so $x \in \overline{B}$ (ε-Criterion for Closures). Therefore $[0,1] \subseteq \overline{B}$.

By the two inclusions, $\overline{\mathbb{Q} \cap [0,1]} = [0,1]$. □

True

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Part (c), Method 3: Closedness of $[0,1]$ and dyadic bisection

Let $B := \mathbb{Q} \cap [0,1]$.

Proof. **$[0,1]$ is closed.** Let $\{x_n\}$ be contained in $[0,1]$ with $a = \lim x_n$. Then $0 \le x_n \le 1$ for every n, so the Order Limit Theorem (parts 1 and 3) gives $0 \le a \le 1$. Hence every convergent sequence contained in $[0,1]$ has its limit in $[0,1]$, so $[0,1]$ is closed (Sequential Criterion for Closed Sets), and $\overline{[0,1]} = [0,1]$ (Closure Criterion for Closed Sets).

($\subseteq$) Let $a \in \overline{B}$. By the Sequential Criterion for Closures there is $\{y_n\}$ contained in $B \subseteq [0,1]$ with $a = \lim y_n$. Hence $\{y_n\}$ is a sequence contained in $[0,1]$ converging to a, so $a \in \overline{[0,1]} = [0,1]$ (Sequential Criterion for Closures). Therefore $\overline{B} \subseteq [0,1]$.

($\supseteq$) Let $x \in [0,1]$. Define $l_0 := 0$ and $r_0 := 1$. Then $l_0, r_0 \in B$, $l_0 \le x \le r_0$ and $r_0 - l_0 = 1 = 2^{-0}$. Now let $n \in \mathbb{N}_0$ and suppose $l_n, r_n \in B$ satisfy

$$l_n \le x \le r_n, \qquad r_n - l_n = 2^{-n}.$$

Let $c_n := \frac{l_n + r_n}{2}$, which is rational and lies in $[l_n, r_n] \subseteq [0,1]$. If $x < c_n$, let $l_{n+1} := l_n$, $r_{n+1} := c_n$; if $x \ge c_n$, let $l_{n+1} := c_n$, $r_{n+1} := r_n$. In both cases $l_{n+1}, r_{n+1} \in B$, $l_{n+1} \le x \le r_{n+1}$ and $r_{n+1} - l_{n+1} = \frac{1}{2}(r_n - l_n) = 2^{-(n+1)}$. By induction, $l_n, r_n \in B$, $l_n \le x \le r_n$ and $r_n - l_n = 2^{-n}$ for every $n \in \mathbb{N}_0$.

Next, $2^n \ge n+1$ for every $n \in \mathbb{N}_0$: $2^0 = 1$, and if $2^n \ge n+1$ then $2^{n+1} = 2 \cdot 2^n \ge 2n+2 \ge n+2$. Hence, for every $n \in \mathbb{N}$,

$$|l_n - x| = x - l_n \le r_n - l_n = \frac{1}{2^n} \le \frac{1}{n+1} < \frac{1}{n}.$$

Let $\varepsilon > 0$. By (Archimedean) there is $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$. For all $n \ge N$, $|l_n - x| < \frac{1}{n} \le \frac{1}{N} < \varepsilon$. Hence $x = \lim l_n$ with $\{l_n\}_{n \ge 1}$ contained in B, so $x \in \overline{B}$ (Sequential Criterion for Closures). Therefore $[0,1] \subseteq \overline{B}$.

By the two inclusions, $\overline{\mathbb{Q} \cap [0,1]} = [0,1]$. □

$$\boxed{\text{True}}$$

Marking scheme (6 marks)

QRS suggested marking scheme.

(a) **(2 marks)** 2 marks for $\boxed{\text{False}}$; 0 marks for $\boxed{\text{True}}$.

(b) **(2 marks)** 2 marks for $\boxed{\text{True}}$; 0 marks for $\boxed{\text{False}}$.

(c) **(2 marks)** 2 marks for $\boxed{\text{True}}$; 0 marks for $\boxed{\text{False}}$.

Typical deductions (if justification is examined): in (a), assuming convergence of $\sum_{n=1}^{\infty} a_n$ is absolute; in (b), confusing "closed" with "bounded" or "compact"; in (c), forgetting to prove the reverse inclusion (that the closure cannot exceed $[0,1]$).

Question 3 (6 Marks)

In this question, A is a nonempty set of real numbers that is bounded below, and $c > 0$. Write down the definition of $s = \inf A$, and use the definition to show that

$$\inf(cA) = c \inf A, \qquad \text{where } cA := \{ca : a \in A\}.$$

Do not use the ε-Criterion for Suprema and Infima.

Solution

Method 1: Direct verification of the two defining properties

Definition. Let $A \subseteq \mathbb{R}$. A number $s \in \mathbb{R}$ is the infimum of A, written $s = \inf A$, if (1) s is a lower bound for A, that is, $s \leq a$ for every $a \in A$; and (2) s is not less than any lower bound for A, that is, $L \leq s$ for every lower bound L of A.

Proof. Let $s := \inf A$. We show that the number cs satisfies (1) and (2) for the set cA.

Step 1: cs is a lower bound for cA. Let $y \in cA$. By the definition of cA, $y = ca$ for some $a \in A$. By (1) for $s = \inf A$, $s \leq a$. Multiplying by $c > 0$ preserves the inequality, so

$$cs \leq ca = y.$$

Since $y \in cA$ was arbitrary, $cs \leq y$ for every $y \in cA$. Therefore cs is a lower bound for cA.

Step 2: every lower bound L of cA satisfies $L \leq cs$. Let L be a lower bound for cA, and let $a \in A$. Then $ca \in cA$, so $L \leq ca$. Dividing by $c > 0$ preserves the inequality, so

$$\frac{L}{c} \leq a.$$

Since $a \in A$ was arbitrary, $\frac{L}{c}$ is a lower bound for A. Hence (2) for $s = \inf A$ gives $\frac{L}{c} \leq s$, and multiplying by $c > 0$,

$$L \leq cs.$$

Since L was an arbitrary lower bound for cA, every lower bound L of cA satisfies $L \leq cs$.

By Steps 1 and 2, cs satisfies (1) and (2) for the set cA. Therefore $\inf(cA) = cs = c \inf A$. □

$$\boxed{\inf(cA) = c \inf A}$$

Method 2: Contradiction for the greatest-lower-bound property

Definition. Let $A \subseteq \mathbb{R}$. A number $s \in \mathbb{R}$ is the infimum of A, written $s = \inf A$, if (1) $s \leq a$ for every $a \in A$; and (2) $L \leq s$ for every lower bound L of A.

Proof. Let $s := \inf A$.

Step 1: cs is a lower bound for cA. Let $y \in cA$, so $y = ca$ for some $a \in A$. By (1), $s \leq a$, and since $c > 0$, $cs \leq ca = y$. Since $y \in cA$ was arbitrary, cs is a lower bound for cA.

Step 2: no lower bound of cA exceeds cs. Suppose, for contradiction, that some lower bound L of cA satisfies $L > cs$. Dividing by $c > 0$,

$$\frac{L}{c} > s.$$

By (2) for $s = \inf A$, every lower bound of A is $\leq s$. Since $\frac{L}{c} > s$, the number $\frac{L}{c}$ is therefore not a lower bound for A, so there is $a \in A$ with $a < \frac{L}{c}$. Multiplying by $c > 0$ gives

$$ca < L.$$

On the other hand, $ca \in cA$ and L is a lower bound for cA, so $L \leq ca$. This contradicts $ca < L$. Therefore every lower bound L of cA satisfies $L \leq cs$.

By Steps 1 and 2, cs satisfies (1) and (2) for the set cA. Therefore $\inf(cA) = cs = c \inf A$. □

$$\boxed{\inf(cA) = c \inf A}$$

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Method 3: Two inequalities by the scaling symmetry $c \leftrightarrow 1/c$

Definition. Let $A \subseteq \mathbb{R}$. A number $s \in \mathbb{R}$ is the infimum of A, written $s = \inf A$, if (1) $s \leq a$ for every $a \in A$; and (2) $L \leq s$ for every lower bound L of A.

Proof. **Lemma.** Let $B \subseteq \mathbb{R}$ be nonempty with $t = \inf B$, and let $d > 0$. Then $\inf(dB)$ exists and $dt \leq \inf(dB)$.

Proof of the Lemma. Fix $b_0 \in B$; then $db_0 \in dB$, so dB is nonempty. Let $y \in dB$, so $y = db$ for some $b \in B$. By (1) for $t = \inf B$, $t \leq b$, and since $d > 0$, $dt \leq db = y$. Since $y \in dB$ was arbitrary, dt is a lower bound for dB. Hence dB is nonempty and bounded below, so $\inf(dB)$ exists (Completeness, infimum form). Applying (2) for $\inf(dB)$ to the lower bound dt gives $dt \leq \inf(dB)$.

First inequality. Let $s := \inf A$. The Lemma with $B = A$, $t = s$, $d = c$ gives that $u := \inf(cA)$ exists and

$$cs \leq u.$$

Second inequality. The Lemma with $B = cA$, $t = u$, $d = \frac{1}{c} > 0$ gives

$$\frac{1}{c}\, u \leq \inf\left(\tfrac{1}{c}(cA)\right).$$

By the definition of cA,

$$\tfrac{1}{c}(cA) = \left\{\tfrac{1}{c}(ca) : a \in A\right\} = \{a : a \in A\} = A,$$

so $\frac{1}{c}\, u \leq \inf A = s$. Multiplying by $c > 0$ gives

$$u \leq cs.$$

Combining the two inequalities, $cs \leq u \leq cs$, so $u = cs$. Therefore $\inf(cA) = c \inf A$. □

$$\boxed{\inf(cA) = c \inf A}$$

Marking scheme (6 marks)

QRS suggested marking scheme.

- **2 marks**: definition of $s = \inf A$ stated correctly, with both clauses (lower bound; not less than any lower bound) and no quantifier-order errors.
- **3 marks**: complete proof of both Step 1 (1 mark: cs is a lower bound for cA) and Step 2 (2 marks: cs is the greatest lower bound of cA, including the division step and the correct use of $s = \inf A$).
- **1 mark**: the three writing expectations are met throughout (complete sentences; related statements joined by connectives; inference marked with "therefore/hence/so").

Typical deductions: omitting Step 2 entirely; using the ε-Criterion for Suprema and Infima despite the prohibition; silently assuming $c > 0$ preserves inequality direction without stating it; sign errors when dividing by c.

Question 4 (6 Marks)

In this question, $\{a_n\}$ is a sequence of nonnegative real numbers. Write down the definition of $a = \lim a_n$, and use the definition to show that if $a = \lim a_n$, then $\sqrt{a} = \lim \sqrt{a_n}$.

Do not use the Algebraic Limit Theorem.

Solution

Method 1: Rationalisation with cases $a = 0$ and $a > 0$

Definition. Let $\{a_n\}$ be a sequence and $a \in \mathbb{R}$. Then $a = \lim a_n$ means: for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $|a_n - a| < \varepsilon$.

Proof. Since $a_n \geq 0$ for every $n \in \mathbb{N}$ and $a = \lim a_n$, the Order Limit Theorem (part 1) gives $a \geq 0$, so $\sqrt{a}$ is defined. For $s, t \geq 0$ with $s < t$ we have $s^2 \leq st < t^2$. Hence, for all $x, y \geq 0$: if $\sqrt{x} < \sqrt{y}$, then $x < y$; and if $x < y$, then $\sqrt{x} < \sqrt{y}$, because $\sqrt{x} = \sqrt{y}$ would give $x = y$ and $\sqrt{y} < \sqrt{x}$ would give $y < x$.

Case $a = 0$. Let $\varepsilon > 0$. Then $\varepsilon^2 > 0$, so the definition of $0 = \lim a_n$ applied with ε^2 gives $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| < \varepsilon^2$. For every $n \geq N$, $0 \leq a_n < \varepsilon^2$, so $\sqrt{a_n} < \sqrt{\varepsilon^2} = \varepsilon$, and therefore

$$|\sqrt{a_n} - 0| = \sqrt{a_n} < \varepsilon.$$

Hence $0 = \lim \sqrt{a_n}$, i.e. $\sqrt{a} = \lim \sqrt{a_n}$.

Case $a > 0$. Then $\sqrt{a} > 0$. Let $\varepsilon > 0$. Since $\varepsilon\sqrt{a} > 0$, the definition of $a = \lim a_n$ applied with $\varepsilon\sqrt{a}$ gives $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon\sqrt{a}$. For every $n \in \mathbb{N}$, $\sqrt{a_n} + \sqrt{a} \geq \sqrt{a} > 0$, and

$$\left(\sqrt{a_n} - \sqrt{a}\right)\left(\sqrt{a_n} + \sqrt{a}\right) = \left(\sqrt{a_n}\right)^2 - \left(\sqrt{a}\right)^2 = a_n - a.$$

Therefore, for every $n \geq N$,

$$\begin{aligned} \left|\sqrt{a_n} - \sqrt{a}\right| &= \frac{\left|\sqrt{a_n} - \sqrt{a}\right|\left(\sqrt{a_n} + \sqrt{a}\right)}{\sqrt{a_n} + \sqrt{a}} = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \\ &\leq \frac{|a_n - a|}{\sqrt{a}} && \left(\sqrt{a_n} + \sqrt{a} \geq \sqrt{a} > 0\right) \\ &< \frac{\varepsilon\sqrt{a}}{\sqrt{a}} = \varepsilon. \end{aligned}$$

Hence $\sqrt{a} = \lim \sqrt{a_n}$.

In both cases, $\sqrt{a} = \lim \sqrt{a_n}$. □

$$\boxed{\sqrt{a} = \lim \sqrt{a_n}}$$

Method 2: Square-root difference inequality, no case split

Definition. Let $\{a_n\}$ be a sequence and $a \in \mathbb{R}$. Then $a = \lim a_n$ means: for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $|a_n - a| < \varepsilon$.

Proof. Since $a_n \geq 0$ for every $n \in \mathbb{N}$, the Order Limit Theorem (part 1) gives $a \geq 0$, so $\sqrt{a}$ is defined. For $s, t \geq 0$ with $s < t$ we have $s^2 \leq st < t^2$. Hence, for all $x, y \geq 0$: if $\sqrt{x} < \sqrt{y}$, then $x < y$; and if $x < y$, then $\sqrt{x} < \sqrt{y}$, because $\sqrt{x} = \sqrt{y}$ would give $x = y$ and $\sqrt{y} < \sqrt{x}$ would give $y < x$. In particular, $x \leq y$ implies $\sqrt{x} \leq \sqrt{y}$.

Step 1: $\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|}$ for all $x, y \geq 0$. First let $x \geq y \geq 0$. Then $x - y \geq 0$ and

$$\left(\sqrt{y} + \sqrt{x - y}\right)^2 = y + 2\sqrt{y}\sqrt{x - y} + (x - y) = x + 2\sqrt{y}\sqrt{x - y} \;\geq\; x = \left(\sqrt{x}\right)^2.$$

Both $\sqrt{y} + \sqrt{x - y}$ and $\sqrt{x}$ are nonnegative, so $\sqrt{y} + \sqrt{x - y} \geq \sqrt{x}$, i.e. $\sqrt{x} - \sqrt{y} \leq \sqrt{x - y}$. Also $x \geq y$ gives $\sqrt{x} \geq \sqrt{y}$. Hence

$$0 \leq \sqrt{x} - \sqrt{y} \leq \sqrt{x - y}, \qquad \text{so} \qquad \left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|}.$$

If instead $y > x \geq 0$, the same argument with x and y interchanged gives $\left|\sqrt{y} - \sqrt{x}\right| \leq \sqrt{|y - x|}$, which is the same inequality.

Step 2: the limit. Let $\varepsilon > 0$. Then $\varepsilon^2 > 0$, so the definition of $a = \lim a_n$ applied with ε^2 gives $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon^2$. For every $n \geq N$, $|a_n - a| < \varepsilon^2$ gives $\sqrt{|a_n - a|} < \sqrt{\varepsilon^2} = \varepsilon$, so by Step 1

$$\left|\sqrt{a_n} - \sqrt{a}\right| \;\leq\; \sqrt{|a_n - a|} \;<\; \varepsilon.$$

Hence, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $\left|\sqrt{a_n} - \sqrt{a}\right| < \varepsilon$. Therefore $\sqrt{a} = \lim \sqrt{a_n}$. □

$$\boxed{\sqrt{a} = \lim \sqrt{a_n}}$$

Marking scheme (6 marks)

QRS suggested marking scheme.

- **2 marks**: definition of $a = \lim a_n$ stated correctly, with the quantifiers in the correct order (N may depend on ε, not vice versa).
- **1 mark**: correctly establishes $a \geq 0$ via the Order Limit Theorem before defining $\sqrt{a}$.
- **2 marks**: correct treatment of both cases ($a = 0$ and $a > 0$), including the rationalisation identity and a valid choice of N for the given ε.
- **1 mark**: the three writing expectations are met throughout (complete sentences; related statements joined by connectives; inference marked with "therefore/hence/so").

Typical deductions: omitting the case $a = 0$; using the Algebraic Limit Theorem or citing "continuity of $\sqrt{\cdot}$" despite the prohibition; dividing by $\sqrt{a}$ without justifying $a > 0$ first; choosing N depending on n.