MH3100 Real Analysis I

Midterm Practice 3 (Solutions)

AY 2026/2027, Semester 1 – QRS original

Quantitative Research Society @NTU

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TIME ALLOWED: 60 minutes

Instructions to Students

1. This test contains **FOUR (4)** questions.
2. Answer **all** questions. The marks for each question are indicated below.
3. Write your answers in the space provided. If you need more space, continue on the facing page and clearly indicate that you have done so.
4. This is a **CLOSED BOOK** test.
5. You may use calculators. You may cite any results proved in the lectures and tutorials, **unless you are asked by the question to prove the result.**

Question	1	2	3	4	Total
Total Marks	6	6	6	6	24
Awarded Marks					

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Question 1 (6 Marks)

Is each of the following statement true or false? You do NOT need to justify your answer.

(a) (2 marks) If A and B are nonempty subsets of $\mathbb{R}$ with $A \subseteq B$ and B bounded above, then $\sup A \leq \sup B$.

(b) (2 marks) The set $A = \{q \in \mathbb{Q} : q > 0 \text{ and } q^2 < 2\}$ has a supremum in $\mathbb{Q}$.

(c) (2 marks) Every bounded monotone sequence of rational numbers converges to a rational number.

Solution

Part (a), Method 1: Least-upper-bound comparison

Proof. Let A and B be nonempty subsets of $\mathbb{R}$ with $A \subseteq B$ and B bounded above. Since B is nonempty and bounded above, $\sup B$ exists in $\mathbb{R}$ (AoC).

Let $a \in A$ be arbitrary. Since $A \subseteq B$, we have $a \in B$. Since $\sup B$ is an upper bound of B, it follows that $a \leq \sup B$. In one line:

$$a \in A \implies a \in B \implies a \leq \sup B.$$

Hence $a \leq \sup B$ for every $a \in A$; that is, $\sup B$ is an upper bound of A. In particular, A is bounded above.

Since A is nonempty and bounded above, $\sup A$ exists in $\mathbb{R}$ (AoC). By the definition of supremum, $\sup A \leq u$ for every upper bound u of A. Taking the upper bound $u = \sup B$ gives

$$\sup A \leq \sup B.$$

Since A and B were arbitrary sets satisfying the hypotheses, the statement holds. □

True

Part (a), Method 2: Contradiction via the ε-Criterion

Proof. Let A and B be nonempty subsets of $\mathbb{R}$ with $A \subseteq B$ and B bounded above. Since B is nonempty and bounded above, $\sup B$ exists in $\mathbb{R}$ (AoC). Let u be an upper bound of B. For every $a \in A$ we have $a \in B$, and therefore $a \leq u$. Hence A is nonempty and bounded above, so $\sup A$ exists in $\mathbb{R}$ (AoC).

Suppose, for a contradiction, that $\sup A > \sup B$. Let

$$\varepsilon := \sup A - \sup B > 0.$$

By the ε-Criterion for Suprema, there is $a \in A$ with $a > \sup A - \varepsilon$. Substituting the value of ε,

$$a > \sup A - (\sup A - \sup B) = \sup B.$$

But $a \in A \subseteq B$, and $\sup B$ is an upper bound of B, so $a \leq \sup B$. This contradicts $a > \sup B$. Therefore $\sup A > \sup B$ is impossible, and hence $\sup A \leq \sup B$. □

True

Part (b), Method 1: Contradiction via the Archimedean Property

Proof. Suppose, for a contradiction, that $s \in \mathbb{Q}$ is a supremum of A in $\mathbb{Q}$; that is, $q \le s$ for every $q \in A$, and $s \le u$ for every $u \in \mathbb{Q}$ that is an upper bound of A.

Since $1 \in \mathbb{Q}$, $1 > 0$ and $1^2 = 1 < 2$, we have $1 \in A$. Hence $s \ge 1 > 0$. No rational number squares to 2 (Lecture 1: $\sqrt{2} \notin \mathbb{Q}$). Since $s \in \mathbb{Q}$, therefore $s^2 \ne 2$, so either $s^2 < 2$ or $s^2 > 2$.

Case 1: $s^2 < 2$. Then $2 - s^2 > 0$ and $2s + 1 > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with

$$n > \frac{2s+1}{2-s^2}.$$

Let $h := \frac{1}{n}$. Then $h \in \mathbb{Q}$, $0 < h \le 1$, and $h(2s+1) < 2 - s^2$. Since $0 < h \le 1$, we have $h^2 \le h$. Therefore

$$\begin{aligned}(s+h)^2 &= s^2 + 2sh + h^2 \\ &\le s^2 + 2sh + h && (h^2 \le h) \\ &= s^2 + h(2s+1) \\ &< s^2 + (2 - s^2) = 2.\end{aligned}$$

Hence $s + h \in \mathbb{Q}$, $s + h > 0$ and $(s+h)^2 < 2$, so $s + h \in A$. But $s + h > s$, which contradicts $q \le s$ for every $q \in A$.

Case 2: $s^2 > 2$. Then $s^2 - 2 > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with

$$n > \max\left\{\frac{1}{s},\ \frac{2s}{s^2-2}\right\}.$$

Let $h := \frac{1}{n}$. Then $h \in \mathbb{Q}$, $0 < h < s$ and $2sh < s^2 - 2$. Therefore $s - h \in \mathbb{Q}$, $s - h > 0$, and

$$\begin{aligned}(s-h)^2 &= s^2 - 2sh + h^2 \\ &> s^2 - 2sh && (h^2 > 0) \\ &> s^2 - (s^2 - 2) = 2.\end{aligned}$$

Let $q \in A$, and suppose $q \ge s - h$. Since $q > 0$ and $s - h > 0$,

$$q^2 = q \cdot q \ge q(s-h) \ge (s-h)(s-h) > 2,$$

which contradicts $q^2 < 2$. Hence $q < s - h$ for every $q \in A$, so $s - h$ is a rational upper bound of A. Since $s \le u$ for every rational upper bound u of A, we get $s \le s - h$, that is, $h \le 0$. This contradicts $h > 0$.

Both cases lead to a contradiction. Therefore no $s \in \mathbb{Q}$ is a supremum of A in $\mathbb{Q}$. □

False

Part (b), Method 2: Contradiction via the rational map $q \mapsto \frac{2q+2}{q+2}$

Proof. For $q \in \mathbb{Q}$ with $q > 0$ define

$$r(q) := \frac{2q+2}{q+2}.$$

Since $q \in \mathbb{Q}$ and $q + 2 > 0$, we have $r(q) \in \mathbb{Q}$ and $r(q) > 0$. Compute the two differences:

$$\begin{aligned} r(q) - q &= \frac{2q+2-q(q+2)}{q+2} = \frac{2q+2-q^2-2q}{q+2} = \frac{2-q^2}{q+2}, \\ r(q)^2 - 2 &= \frac{(2q+2)^2 - 2(q+2)^2}{(q+2)^2} \\ &= \frac{4q^2+8q+4-2q^2-8q-8}{(q+2)^2} = \frac{2(q^2-2)}{(q+2)^2}. \end{aligned}$$

Since $q + 2 > 0$ and $(q + 2)^2 > 0$, the signs of $r(q) - q$ and $r(q)^2 - 2$ are those of $2 - q^2$ and $q^2 - 2$. Hence, for every $q \in \mathbb{Q}$ with $q > 0$,

$$\begin{aligned} q^2 < 2 &\implies r(q) > q \text{ and } r(q)^2 < 2, && (*) \\ q^2 > 2 &\implies r(q) < q \text{ and } r(q)^2 > 2. && (**) \end{aligned}$$

Suppose, for a contradiction, that $s \in \mathbb{Q}$ is a supremum of A in $\mathbb{Q}$; that is, $q \le s$ for every $q \in A$, and $s \le u$ for every $u \in \mathbb{Q}$ that is an upper bound of A. Since $1 \in A$, we have $s \ge 1 > 0$. Since no rational number squares to 2 (Lecture 1), $s^2 \neq 2$.

Case 1: $s^2 < 2$. By $(*)$, $r(s) \in \mathbb{Q}$, $r(s) > 0$ and $r(s)^2 < 2$, so $r(s) \in A$. Also $r(s) > s$ by $(*)$. This contradicts $q \le s$ for every $q \in A$.

Case 2: $s^2 > 2$. Let $t := r(s)$. By $(**)$, $t \in \mathbb{Q}$, $t > 0$, $t^2 > 2$ and $t < s$. Let $q \in A$. If $q \ge t$, then, since $q > 0$ and $t > 0$,

$$q^2 = q \cdot q \ge q \cdot t \ge t \cdot t = t^2 > 2,$$

which contradicts $q^2 < 2$. Hence $q < t$ for every $q \in A$, so t is a rational upper bound of A. Therefore $s \le t$, which contradicts $t < s$.

Both cases lead to a contradiction. Therefore A has no supremum in $\mathbb{Q}$. □

False

Part (c), Method 1: Decimal truncations of $\sqrt{2}$

Proof. The real number $\sqrt{2} > 0$ satisfies $(\sqrt{2})^2 = 2$ and $\sqrt{2} \notin \mathbb{Q}$ (Lecture 1). For each $n \in \mathbb{N}$ let

$$S_n := \{k \in \mathbb{N}_0 : k \leq \sqrt{2} \cdot 10^n\}.$$

Since $0 \leq \sqrt{2} \cdot 10^n$, we have $0 \in S_n$, so $S_n \neq \emptyset$. Since $\sqrt{2} < 2$, every $k \in S_n$ satisfies $k \leq \sqrt{2} \cdot 10^n < 2 \cdot 10^n$, so $S_n \subseteq \{0, 1, \ldots, 2 \cdot 10^n\}$ is finite. Hence S_n has a largest element, and we define

$$k_n := \max S_n, \qquad a_n := \frac{k_n}{10^n}.$$

Since $k_n \in \mathbb{N}_0$ and $10^n \in \mathbb{N}$, we have $a_n \in \mathbb{Q}$ for every $n \in \mathbb{N}$.

Distance to $\sqrt{2}$. Since $k_n \in S_n$, $k_n \leq \sqrt{2} \cdot 10^n$. Since $k_n + 1 \in \mathbb{N}_0$ and $k_n + 1 > \max S_n$, we have $k_n + 1 \notin S_n$, so $k_n + 1 > \sqrt{2} \cdot 10^n$. Hence

$$k_n \leq \sqrt{2} \cdot 10^n < k_n + 1.$$

Dividing by $10^n > 0$ gives $a_n \leq \sqrt{2} < a_n + 10^{-n}$. Therefore

$$0 \leq \sqrt{2} - a_n < 10^{-n} \qquad \text{for all } n \in \mathbb{N}. \qquad (*)$$

Monotone. Let $n \in \mathbb{N}$. Multiplying $k_n \leq \sqrt{2} \cdot 10^n$ by 10 gives $10k_n \leq \sqrt{2} \cdot 10^{n+1}$, and $10k_n \in \mathbb{N}_0$. Hence $10k_n \in S_{n+1}$, and therefore $10k_n \leq \max S_{n+1} = k_{n+1}$. Dividing by 10^{n+1},

$$a_n = \frac{10k_n}{10^{n+1}} \leq \frac{k_{n+1}}{10^{n+1}} = a_{n+1}.$$

Hence $\{a_n\}$ is increasing.

Bounded. For every $n \in \mathbb{N}$, $k_n \geq 0$ gives $a_n \geq 0$, and $(*)$ gives $a_n \leq \sqrt{2} < 2$. Hence $0 \leq a_n \leq 2$ for all $n \in \mathbb{N}$, so $\{a_n\}$ is bounded.

Limit. First, $10^n \geq n$ for every $n \in \mathbb{N}$: it holds for $n = 1$ since $10 \geq 1$, and if $10^n \geq n$ then $10^{n+1} = 10 \cdot 10^n \geq 10n \geq n + 1$; hence it holds for all n by induction. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$. Then for every $n \geq N$,

$$\left|a_n - \sqrt{2}\right| = \sqrt{2} - a_n < 10^{-n} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\left|a_n - \sqrt{2}\right| < \varepsilon$ for all $n \geq N$; that is, $\sqrt{2} = \lim a_n$ (def. of limit).

By the Uniqueness of Limits, $\{a_n\}$ converges to no number other than $\sqrt{2}$, and $\sqrt{2} \notin \mathbb{Q}$. Therefore $\{a_n\}$ is a bounded monotone sequence of rational numbers that does not converge to a rational number, and the statement is false. □

False

Part (c), Method 2: Babylonian recursion and the Monotone Convergence Theorem

Proof. Define $\{a_n\}$ recursively by

$$a_1 := 2, \qquad a_{n+1} := \frac{a_n^2 + 2}{2a_n} \quad (n \in \mathbb{N}).$$

Rational and positive. We use induction on n. First, $a_1 = 2 \in \mathbb{Q}$ and $a_1 > 0$. If $a_n \in \mathbb{Q}$ and $a_n > 0$, then $a_n^2 + 2 \in \mathbb{Q}$ and $2a_n \in \mathbb{Q}$ with $2a_n \neq 0$, so $a_{n+1} \in \mathbb{Q}$; and $a_{n+1} > 0$ as a quotient of positive numbers. Hence $a_n \in \mathbb{Q}$ and $a_n > 0$ for every $n \in \mathbb{N}$.

Squares at least 2. First, $a_1^2 = 4 \geq 2$. For every $n \in \mathbb{N}$,

$$\begin{aligned} a_{n+1}^2 - 2 &= \frac{(a_n^2 + 2)^2}{4a_n^2} - 2 = \frac{a_n^4 + 4a_n^2 + 4 - 8a_n^2}{4a_n^2} \\ &= \frac{a_n^4 - 4a_n^2 + 4}{4a_n^2} = \frac{(a_n^2 - 2)^2}{4a_n^2} \geq 0. \end{aligned}$$

Hence $a_n^2 \geq 2$ for every $n \in \mathbb{N}$. If $0 < a_n \leq 1$, then $a_n^2 \leq 1 < 2$; therefore $a_n > 1$ for every $n \in \mathbb{N}$.

Decreasing. For every $n \in \mathbb{N}$,

$$a_n - a_{n+1} = a_n - \frac{a_n^2 + 2}{2a_n} = \frac{2a_n^2 - a_n^2 - 2}{2a_n} = \frac{a_n^2 - 2}{2a_n} \geq 0,$$

since $a_n^2 \geq 2$ and $a_n > 0$. Hence $\{a_n\}$ is decreasing.

Bounded. Since $\{a_n\}$ is decreasing, $a_n \leq a_1 = 2$ for every n; and $a_n > 1$. Hence $1 < a_n \leq 2$ for every $n \in \mathbb{N}$, so $\{a_n\}$ is bounded.

Limit. Since $\{a_n\}$ is monotone and bounded, $L := \lim a_n$ exists (MCT). Since $a_n \geq 1$ for every n, $L \geq 1$ (Order Limit Theorem), so $L \neq 0$. The sequence $\{a_{n+1}\}$ is the subsequence of $\{a_n\}$ with indices $n_k = k + 1$, so $\lim a_{n+1} = L$ (Subsequential Limits). Multiplying the recursion by $2a_n$ gives, for every $n \in \mathbb{N}$,

$$2a_n a_{n+1} = a_n^2 + 2.$$

Taking limits on both sides (Algebraic Limit Theorem),

$$2L \cdot L = L^2 + 2 \implies 2L^2 - L^2 = 2 \implies L^2 = 2.$$

Since no rational number squares to 2 (Lecture 1), $L \notin \mathbb{Q}$. By the Uniqueness of Limits, $\{a_n\}$ converges to no rational number. Therefore $\{a_n\}$ is a bounded monotone sequence of rational numbers whose limit is not rational, and the statement is false. □

False

Marking scheme (6 marks)

QRS suggested marking scheme.

- **(2 marks)** Part (a): correct answer True.
- **(2 marks)** Part (b): correct answer False.
- **(2 marks)** Part (c): correct answer False.

No justification is required for full marks on this question; the proofs above are provided for the solutions build only.

Question 2 (6 Marks)

Is each of the following statement true or false? You do NOT need to justify your answer.

(a) (2 marks) Every infinite and bounded subset of $\mathbb{R}$ has a limit point.

(b) (2 marks) The closure of a bounded subset of $\mathbb{R}$ is (sequentially) compact.

(c) (2 marks) Every set with no limit points is (sequentially) compact.

Solution

Part (a), Method 1: Distinct terms and Bolzano–Weierstrass

Proof. Let $S \subseteq \mathbb{R}$ be infinite and bounded. Since S is bounded, there is $M > 0$ such that $|s| \leq M$ for every $s \in S$.

A sequence of distinct points of S. Since S is infinite, $S \neq \emptyset$; choose $a_1 \in S$. Suppose distinct points $a_1, \ldots, a_{n-1} \in S$ have been chosen. If $S \setminus \{a_1, \ldots, a_{n-1}\}$ were empty, then $S \subseteq \{a_1, \ldots, a_{n-1}\}$ would be finite, contradicting that S is infinite. Hence $S \setminus \{a_1, \ldots, a_{n-1}\} \neq \emptyset$; choose a_n in it. Then $a_1, \ldots, a_n$ are distinct points of S. By recursion, this defines a sequence $\{a_n\}$ in S with $a_m \neq a_n$ whenever $m \neq n$.

A convergent subsequence. Since $a_n \in S$, we have $|a_n| \leq M$ for every $n \in \mathbb{N}$, so $\{a_n\}$ is bounded. Hence there are a subsequence $\{a_{n_k}\}$ and a number $a \in \mathbb{R}$ with $a = \lim a_{n_k}$ (Bolzano–Weierstrass).

a is a limit point of S. Let $\varepsilon > 0$. By the definition of limit there is $K \in \mathbb{N}$ such that $|a_{n_k} - a| < \varepsilon$ for all $k \geq K$. Since $n_K < n_{K+1}$ and terms with different indices are distinct, $a_{n_K} \neq a_{n_{K+1}}$, so at most one of them equals a. Hence there is $j \in \{K, K+1\}$ with $a_{n_j} \neq a$. Let $x := a_{n_j}$. Then $x \in S \setminus \{a\}$, and since $j \geq K$,

$$|x - a| = \left|a_{n_j} - a\right| < \varepsilon.$$

Hence for every $\varepsilon > 0$ there exists $x \in S \setminus \{a\}$ with $|x - a| < \varepsilon$; therefore a is a limit point of S (def. of limit point). Since S was an arbitrary infinite bounded set, the statement holds. □

True

Part (a), Method 2: Bisection and the Nested Interval Property

Proof. Let $S \subseteq \mathbb{R}$ be infinite and bounded. There is $M > 0$ such that $|s| \le M$ for every $s \in S$; hence $S \subseteq [-M, M]$.

Nested intervals. Let $I_1 = [c_1, d_1] := [-M, M]$. Then $S \cap I_1 = S$ is infinite and $d_1 - c_1 = 2M$. Suppose $I_n = [c_n, d_n]$ has been chosen with $S \cap I_n$ infinite and $d_n - c_n = \frac{2M}{2^{n-1}}$. Let $m_n := \frac{c_n + d_n}{2}$. Then

$$S \cap I_n = \big(S \cap [c_n, m_n]\big) \cup \big(S \cap [m_n, d_n]\big).$$

A union of two finite sets is finite, so at least one of the two pieces is infinite. Let $I_{n+1} := [c_n, m_n]$ if $S \cap [c_n, m_n]$ is infinite, and $I_{n+1} := [m_n, d_n]$ otherwise. Then $I_{n+1} \subseteq I_n$, $S \cap I_{n+1}$ is infinite, and

$$d_{n+1} - c_{n+1} = \frac{d_n - c_n}{2} = \frac{2M}{2^n}.$$

Hence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ are nested closed bounded intervals, and there is $x \in \mathbb{R}$ with $x \in I_n$ for every $n \in \mathbb{N}$ (Nested Interval Property).

x is a limit point of S. First, $2^{n-1} \ge n$ for every $n \in \mathbb{N}$: it holds for $n = 1$, and if $2^{n-1} \ge n$ then $2^n = 2 \cdot 2^{n-1} \ge 2n \ge n + 1$. Let $\varepsilon > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $n > \frac{2M}{\varepsilon}$. Then

$$d_n - c_n = \frac{2M}{2^{n-1}} \le \frac{2M}{n} < \varepsilon.$$

Since $S \cap I_n$ is infinite, removing the single point x leaves a nonempty set; choose $y \in (S \cap I_n) \setminus \{x\}$. Then $y \in S \setminus \{x\}$, and since $x, y \in [c_n, d_n]$,

$$|y - x| \le d_n - c_n < \varepsilon.$$

Hence for every $\varepsilon > 0$ there exists $y \in S \setminus \{x\}$ with $|y - x| < \varepsilon$; therefore x is a limit point of S (def. of limit point). □

True

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Part (a), Method 3: Supremum of the thresholds with infinitely many points above (AoC)

Proof. Let $S \subseteq \mathbb{R}$ be infinite and bounded, with $|s| \leq M$ for every $s \in S$. Define

$$T := \{t \in \mathbb{R} : S \cap [t, \infty) \text{ is infinite}\}.$$

Since $S \cap [-M, \infty) = S$ is infinite, $-M \in T$, so $T \neq \emptyset$. If $t > M$, then every $s \in S$ satisfies $s \leq M < t$, so $S \cap [t, \infty) = \emptyset$ is finite and $t \notin T$. Hence every $t \in T$ satisfies $t \leq M$, so T is bounded above. Therefore $x := \sup T$ exists (AoC).

Let $\varepsilon > 0$. *Infinitely many points from $x - \frac{\varepsilon}{2}$ on.* By the ε-Criterion for Suprema there is $t \in T$ with $t > x - \frac{\varepsilon}{2}$. Then $[t, \infty) \subseteq [x - \frac{\varepsilon}{2}, \infty)$, so $S \cap [x - \frac{\varepsilon}{2}, \infty)$ contains the infinite set $S \cap [t, \infty)$ and is therefore infinite.

Finitely many points from $x + \frac{\varepsilon}{2}$ on. Since $x + \frac{\varepsilon}{2} > x = \sup T$ and x is an upper bound of T, we have $x + \frac{\varepsilon}{2} \notin T$. Hence $S \cap [x + \frac{\varepsilon}{2}, \infty)$ is finite.

A point near x. The infinite set splits as

$$S \cap [x - \tfrac{\varepsilon}{2}, \infty) = \big(S \cap [x - \tfrac{\varepsilon}{2}, x + \tfrac{\varepsilon}{2})\big) \cup \big(S \cap [x + \tfrac{\varepsilon}{2}, \infty)\big).$$

If $S \cap [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})$ were finite, the right-hand side would be a union of two finite sets, hence finite; this is false. Therefore $S \cap [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})$ is infinite, and removing the single point x leaves a nonempty set. Choose $y \in S \cap [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2})$ with $y \neq x$. Then $y \in S \setminus \{x\}$ and

$$|y - x| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Hence for every $\varepsilon > 0$ there exists $y \in S \setminus \{x\}$ with $|y - x| < \varepsilon$; therefore x is a limit point of S (def. of limit point). □

True

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Part (b), Method 1: Closed and bounded, then Sequential Compactness of Closed and Bounded Sets

Proof. Let $A \subseteq \mathbb{R}$ be bounded. Then there is $M > 0$ such that $-M \leq a \leq M$ for every $a \in A$.

$\overline{A}$ is bounded. Let $x \in \overline{A}$. By the Sequential Criterion for Closures, there is a sequence $\{a_n\}$ contained in A with $x = \lim a_n$. Since $a_n \in A$, we have $-M \leq a_n \leq M$ for every $n \in \mathbb{N}$. Hence $-M \leq x \leq M$ (Order Limit Theorem). Therefore $|x| \leq M$ for every $x \in \overline{A}$, so $\overline{A}$ is bounded.

$\overline{A}$ is closed. Let y be a limit point of $\overline{A}$. By the theorem on Limit Points of Closures, y is a limit point of A. By the definition of closure, $\overline{A} = A \cup \{\text{limit points of } A\}$, so $y \in \overline{A}$. Hence $\overline{A}$ contains all of its limit points, so $\overline{A}$ is closed (def. of closed set).

Conclusion. Since $\overline{A}$ is closed and bounded, $\overline{A}$ is (sequentially) compact (Sequential Compactness of Closed and Bounded Sets). Since A was an arbitrary bounded set, the statement holds. □

True

Part (b), Method 2: Approximation by points of A and Bolzano–Weierstrass

Proof. Let $A \subseteq \mathbb{R}$ be bounded, with $|a| \leq M$ for every $a \in A$. Let $\{x_n\}$ be a sequence contained in $\overline{A}$.

Approximating sequence in A. For each $n \in \mathbb{N}$, $x_n \in \overline{A}$, so by the ε-Criterion for Closures with $\varepsilon = \frac{1}{n}$ there is $a_n \in A$ with

$$|a_n - x_n| < \frac{1}{n}.$$

This defines a sequence $\{a_n\}$ contained in A. Since $|a_n| \leq M$ for every n, $\{a_n\}$ is bounded. Hence there are a subsequence $\{a_{n_k}\}$ and $a \in \mathbb{R}$ with $a = \lim a_{n_k}$ (Bolzano–Weierstrass).

The limit lies in $\overline{A}$. The sequence $\{a_{n_k}\}$ is contained in A and converges to a, so $a \in \overline{A}$ (Sequential Criterion for Closures).

The subsequence $\{x_{n_k}\}$ converges to a. Let $\varepsilon > 0$. By the definition of limit there is $K_1 \in \mathbb{N}$ such that $|a_{n_k} - a| < \frac{\varepsilon}{2}$ for all $k \geq K_1$. By the Archimedean Property there is $K_2 \in \mathbb{N}$ with $K_2 > \frac{2}{\varepsilon}$. Let $K := \max\{K_1, K_2\}$ and $k \geq K$. Then $n_k \geq k$ (Subsequence Index Bound), so $\frac{1}{n_k} \leq \frac{1}{k} \leq \frac{1}{K_2} < \frac{\varepsilon}{2}$. Therefore

$$\begin{aligned} |x_{n_k} - a| &\leq |x_{n_k} - a_{n_k}| + |a_{n_k} - a| && \text{(Triangle Inequality)} \\ &< \frac{1}{n_k} + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence $a = \lim x_{n_k}$ (def. of limit).

Conclusion. Every sequence contained in $\overline{A}$ has a convergent subsequence whose limit lies in $\overline{A}$. Therefore $\overline{A}$ is (sequentially) compact (def. of compact set). □

True

Part (c), Method 1: Counterexample $\mathbb{N}$, checked from the definitions

Proof. Let $A = \mathbb{N}$.

A has no limit points. Suppose, for a contradiction, that $a \in \mathbb{R}$ is a limit point of A. Applying the definition with $\varepsilon = \frac{1}{2}$, there is $x_1 \in A \setminus \{a\}$ with $|x_1 - a| < \frac{1}{2}$. Since $x_1 \neq a$, the number $|x_1 - a|$ is positive; applying the definition with $\varepsilon = |x_1 - a|$, there is $x_2 \in A \setminus \{a\}$ with $|x_2 - a| < |x_1 - a|$. Hence $x_2 \neq x_1$. Distinct natural numbers differ by at least 1, so

$$\begin{aligned} 1 \leq |x_1 - x_2| &\leq |x_1 - a| + |a - x_2| && \text{(Triangle Inequality)} \\ &< 2\,|x_1 - a| < 2 \cdot \tfrac{1}{2} = 1. \end{aligned}$$

This gives $1 < 1$, a contradiction. Therefore A has no limit points.

A is not compact. Let $a_n := n$ for $n \in \mathbb{N}$; then $\{a_n\}$ is a sequence contained in A. Let $\{a_{n_k}\}$ be any subsequence. By the Subsequence Index Bound, $n_k \geq k$ for every $k \in \mathbb{N}$, so $a_{n_k} = n_k \geq k$. Let $M > 0$. By the Archimedean Property there is $k \in \mathbb{N}$ with $k > M$, and then $|a_{n_k}| \geq k > M$. Hence no $M > 0$ bounds $\{a_{n_k}\}$, so $\{a_{n_k}\}$ is unbounded. Every convergent sequence is bounded (Boundedness of Convergent Sequences), so $\{a_{n_k}\}$ diverges. Hence $\{a_n\}$ is a sequence in A with no convergent subsequence, so A is not (sequentially) compact.

Therefore $A = \mathbb{N}$ has no limit points and is not (sequentially) compact, so the statement is false. □

False

Part (c), Method 2: Counterexample $\mathbb{N}$ via the Cauchy Criterion and the closed-and-bounded characterisation

Proof. Let $A = \mathbb{N}$.

A has no limit points. Suppose, for a contradiction, that $a \in \mathbb{R}$ is a limit point of A. By the Sequential Criterion for Limit Points there is a sequence $\{x_n\}$ contained in $A \setminus \{a\}$ with $a = \lim x_n$. Since $\{x_n\}$ converges, it is a Cauchy sequence (Cauchy Criterion). Applying the definition of a Cauchy sequence with $\varepsilon = 1$, there is $P \in \mathbb{N}$ such that $|x_m - x_n| < 1$ for all $m, n \geq P$. Let $n \geq P$. Taking $m = P$ gives $|x_n - x_P| < 1$. Since $x_n, x_P \in \mathbb{N}$ and distinct natural numbers differ by at least 1, therefore $x_n = x_P$. Hence $x_n = x_P$ for every $n \geq P$. Hence for every $\varepsilon > 0$ and every $n \geq P$, $|x_n - x_P| = 0 < \varepsilon$, so $\lim x_n = x_P$ (def. of limit). By the Uniqueness of Limits, $a = x_P$. But $x_P \in A \setminus \{a\}$, so $x_P \neq a$. This is a contradiction. Therefore A has no limit points.

A is closed but not bounded. The set of limit points of A is empty, and the empty set is contained in A; hence A is closed (def. of closed set). For every $M > 0$ there is $n \in \mathbb{N}$ with $n > M$ (Archimedean), so no $M > 0$ satisfies $|n| \leq M$ for all $n \in A$. Hence A is not bounded.

A is not compact. By the Sequential Compactness of Closed and Bounded Sets, every (sequentially) compact set is closed and bounded. Since A is not bounded, A is not (sequentially) compact.

Therefore $A = \mathbb{N}$ has no limit points and is not (sequentially) compact, so the statement is false. □

False

Marking scheme (6 marks)

QRS suggested marking scheme.

- **(2 marks)** Part (a): correct answer True.
- **(2 marks)** Part (b): correct answer True.
- **(2 marks)** Part (c): correct answer False.

No justification is required for full marks on this question; the proofs above are provided for the solutions build only.

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Question 3 (6 Marks)

In this question, A and B are sets of real numbers. **Write down the definition** of a limit point of a set, and **use the definition to show** that if x is a limit point of $A \cup B$, then x is a limit point of A or x is a limit point of B. *Do not use the Sequential Criterion for Limit Points.*

Solution

Definition (Limit Point). Let $S \subseteq \mathbb{R}$. A number $a \in \mathbb{R}$ is a *limit point* of S if for every $\varepsilon > 0$ there exists $x \in S \setminus \{a\}$ such that $|x - a| < \varepsilon$.

Method 1: Contrapositive

Proof. We prove the contrapositive: if x is a limit point of neither A nor B, then x is not a limit point of $A \cup B$. Suppose x is not a limit point of A and not a limit point of B.

Since x is not a limit point of A, negating the definition gives $\varepsilon_1 > 0$ such that

$$|y - x| \geq \varepsilon_1 \qquad \text{for every } y \in A \setminus \{x\}.$$

Since x is not a limit point of B, negating the definition gives $\varepsilon_2 > 0$ such that

$$|z - x| \geq \varepsilon_2 \qquad \text{for every } z \in B \setminus \{x\}.$$

Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. Since $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, we have $\varepsilon > 0$, and $\varepsilon \leq \varepsilon_1$, $\varepsilon \leq \varepsilon_2$.

Let $w \in (A \cup B) \setminus \{x\}$. Then $w \neq x$, and $w \in A$ or $w \in B$; hence $w \in A \setminus \{x\}$ or $w \in B \setminus \{x\}$. If $w \in A \setminus \{x\}$, then $|w - x| \geq \varepsilon_1 \geq \varepsilon$. If $w \in B \setminus \{x\}$, then $|w - x| \geq \varepsilon_2 \geq \varepsilon$. In both cases

$$|w - x| \geq \varepsilon.$$

Hence there is $\varepsilon > 0$ such that every $w \in (A \cup B) \setminus \{x\}$ satisfies $|w - x| \geq \varepsilon$. This is the negation of the definition of limit point, so x is not a limit point of $A \cup B$.

Therefore, by contraposition, if x is a limit point of $A \cup B$, then x is a limit point of A or x is a limit point of B. □

Method 2: Direct proof of "not a limit point of A implies a limit point of B"

Proof. Suppose x is a limit point of $A \cup B$. If x is a limit point of A, the conclusion holds. Otherwise x is not a limit point of A, and we show that x is a limit point of B.

Since x is not a limit point of A, negating the definition gives $\varepsilon_1 > 0$ such that

$$|y - x| \geq \varepsilon_1 \qquad \text{for every } y \in A \setminus \{x\}.$$

Let $\varepsilon > 0$, and let $\delta := \min\{\varepsilon, \varepsilon_1\}$; then $\delta > 0$, $\delta \leq \varepsilon$ and $\delta \leq \varepsilon_1$. Since x is a limit point of $A \cup B$, applying the definition with δ gives $w \in (A \cup B) \setminus \{x\}$ with

$$|w - x| < \delta.$$

Suppose $w \in A$. Then $w \in A \setminus \{x\}$, so $|w - x| \geq \varepsilon_1 \geq \delta$, which contradicts $|w - x| < \delta$. Hence $w \notin A$. Since $w \in A \cup B$ and $w \notin A$, we get $w \in B$; since $w \neq x$, $w \in B \setminus \{x\}$. Moreover

$$|w - x| < \delta \leq \varepsilon.$$

Hence for every $\varepsilon > 0$ there exists $w \in B \setminus \{x\}$ with $|w - x| < \varepsilon$, so x is a limit point of B (def. of limit point).

Therefore, in either case, x is a limit point of A or x is a limit point of B. □

Method 3: Pigeonhole on the radii $\frac{1}{n}$

This method applies the definition of limit point at the radii $\varepsilon = \frac{1}{n}$ only; it does not use the Sequential Criterion for Limit Points.

Proof. Suppose x is a limit point of $A \cup B$. For each $n \in \mathbb{N}$, applying the definition with $\varepsilon = \frac{1}{n}$ gives a point $w_n \in (A \cup B) \setminus \{x\}$ with $|w_n - x| < \frac{1}{n}$. Let

$$I_A := \{n \in \mathbb{N} : w_n \in A\}, \qquad I_B := \{n \in \mathbb{N} : w_n \in B\}.$$

Since $w_n \in A \cup B$ for every n, we have $I_A \cup I_B = \mathbb{N}$. If I_A and I_B were both finite, then $\mathbb{N}$ would be a union of two finite sets, hence finite, which is false. Therefore I_A or I_B is infinite. Choose $C \in \{A, B\}$ with $I_C := \{n \in \mathbb{N} : w_n \in C\}$ infinite.

Let $\varepsilon > 0$. By the Archimedean Property there is $m \in \mathbb{N}$ with $m > \frac{1}{\varepsilon}$. Since I_C is infinite, it is not contained in the finite set $\{1, \ldots, m-1\}$, so there is $n \in I_C$ with $n \geq m$. Then $w_n \in C$ and $w_n \neq x$, so $w_n \in C \setminus \{x\}$, and

$$|w_n - x| < \frac{1}{n} \leq \frac{1}{m} < \varepsilon.$$

Hence for every $\varepsilon > 0$ there exists a point of $C \setminus \{x\}$ within ε of x, so x is a limit point of C (def. of limit point). Since $C = A$ or $C = B$, therefore x is a limit point of A or x is a limit point of B. □

Marking scheme (6 marks)

QRS suggested marking scheme.

- **(2 marks)** Correct definition of a limit point, with the quantifiers in the correct order ("for every $\varepsilon > 0$" before "there exists $x \in S \setminus \{a\}$") and the deleted-point set $S \setminus \{a\}$ present. 1 mark if the quantifier order is reversed or $S \setminus \{a\}$ is written as S.

- **(3 marks)** Key steps of the proof: 1 mark for correctly negating the definition of limit point for A (obtaining ε_1); 1 mark for the same for B (obtaining ε_2); 1 mark for taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and correctly concluding that no point of $(A \cup B) \setminus \{x\}$ lies within ε of x.
- **(1 mark)** The three writing expectations: complete sentences, statements joined by connectives, and inference shown with "therefore"/"hence"/"thus"/"so" or $\therefore$. Deduct this mark for a proof written as a bare symbolic chain.

Typical deductions: using the Sequential Criterion for Limit Points despite the prohibition (0 marks for the proof); forgetting to exclude x itself when forming $A \setminus \{x\}$ or $B \setminus \{x\}$; taking $\varepsilon = \varepsilon_1 + \varepsilon_2$ or $\max\{\varepsilon_1, \varepsilon_2\}$ instead of the minimum; asserting the contrapositive conclusion without proving it.

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Question 4 (6 Marks)

Write down the definition of a (sequentially) compact set, and **use the definition to show** that if K_1 and K_2 are compact sets, then $K_1 \cup K_2$ is compact. *Do not use the Sequential Compactness of Closed and Bounded Sets or the Bolzano-Weierstrass Theorem.*

Solution

Definition ((Sequentially) Compact Set). A set $K \subseteq \mathbb{R}$ is *(sequentially) compact* if every sequence contained in K has a convergent subsequence whose limit lies in K.

Method 1: Splitting the index set

Proof. Let K_1 and K_2 be compact, and let $\{a_n\}$ be a sequence contained in $K_1 \cup K_2$. For $i = 1, 2$ let

$$S_i := \{n \in \mathbb{N} : a_n \in K_i\}.$$

One index set is infinite. For every $n \in \mathbb{N}$, $a_n \in K_1 \cup K_2$, so $n \in S_1$ or $n \in S_2$. Hence $S_1 \cup S_2 = \mathbb{N}$. If S_1 and S_2 were both finite, then $\mathbb{N} = S_1 \cup S_2$ would be a union of two finite sets, hence finite, which is false. Therefore S_1 or S_2 is infinite. Without loss of generality S_1 is infinite (otherwise exchange the roles of K_1 and K_2).

A subsequence contained in K_1. Let $n_1 := \min S_1$, which exists because S_1 is a nonempty set of natural numbers. Having chosen $n_k \in S_1$, the set $\{n \in S_1 : n > n_k\}$ is nonempty, because otherwise $S_1 \subseteq \{1, \dots, n_k\}$ would be finite; let n_{k+1} be its minimum. Then $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers with $n_k \in S_1$, that is, $a_{n_k} \in K_1$ for every $k \in \mathbb{N}$. Hence $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ contained in K_1.

Compactness of K_1. Since K_1 is compact and $\{a_{n_k}\}$ is a sequence contained in K_1, there are a strictly increasing sequence $k_1 < k_2 < k_3 < \cdots$ of natural numbers and a number $a \in K_1$ with

$$a = \lim_{j \to \infty} a_{n_{k_j}}.$$

The extracted sequence is a subsequence of $\{a_n\}$. Let $j < j'$ in $\mathbb{N}$. Since $\{k_j\}$ is strictly increasing, $k_j < k_{j'}$; since $\{n_k\}$ is strictly increasing, it follows that $n_{k_j} < n_{k_{j'}}$.

$$j < j' \implies k_j < k_{j'} \implies n_{k_j} < n_{k_{j'}}.$$

Hence $j \mapsto n_{k_j}$ is strictly increasing, so $\left\{a_{n_{k_j}}\right\}$ is a subsequence of $\{a_n\}$ (def. of subsequence).

Conclusion. The sequence $\{a_n\}$ has the subsequence $\left\{a_{n_{k_j}}\right\}$, which converges to $a \in K_1 \subseteq K_1 \cup K_2$. Since $\{a_n\}$ was an arbitrary sequence contained in $K_1 \cup K_2$, the set $K_1 \cup K_2$ is (sequentially) compact. □

Method 2: Case analysis on returns to K_1, with a tail sequence

Proof. Let K_1 and K_2 be compact, and let $\{a_n\}$ be a sequence contained in $K_1 \cup K_2$. Exactly one of the following two cases holds.

- *Case 1:* for every $m \in \mathbb{N}$ there is $n \geq m$ with $a_n \in K_1$.
- *Case 2:* there is $m_0 \in \mathbb{N}$ such that $a_n \notin K_1$ for every $n \geq m_0$.

Case 1. Applying the hypothesis with $m = 1$, choose $n_1 \in \mathbb{N}$ with $a_{n_1} \in K_1$. Having chosen n_k, apply the hypothesis with $m = n_k + 1$ to choose $n_{k+1} \geq n_k + 1 > n_k$ with $a_{n_{k+1}} \in K_1$. Hence $n_1 < n_2 < \cdots$, and $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ contained in K_1. Since K_1 is compact, there are a strictly increasing sequence $k_1 < k_2 < \cdots$ of natural numbers and $a \in K_1$ with $a = \lim_{j\to\infty} a_{n_{k_j}}$. For $j < j'$ we have $k_j < k_{j'}$ and therefore $n_{k_j} < n_{k_{j'}}$, so $\left\{a_{n_{k_j}}\right\}$ is a subsequence of $\{a_n\}$. Its limit a lies in $K_1 \subseteq K_1 \cup K_2$.

Case 2. For every $n \geq m_0$, $a_n \in K_1 \cup K_2$ and $a_n \notin K_1$; hence $a_n \in K_2$. Define the tail sequence

$$b_k := a_{m_0+k-1} \qquad (k \in \mathbb{N}).$$

Since $m_0 + k - 1 \geq m_0$, every b_k lies in K_2, so $\{b_k\}$ is a sequence contained in K_2. Since K_2 is compact, there are a strictly increasing sequence $k_1 < k_2 < \cdots$ of natural numbers and $b \in K_2$ with $b = \lim_{j\to\infty} b_{k_j}$. Now $b_{k_j} = a_{p_j}$ with $p_j := m_0 + k_j - 1 \in \mathbb{N}$, and

$$j < j' \implies k_j < k_{j'} \implies p_j = m_0 + k_j - 1 < m_0 + k_{j'} - 1 = p_{j'}.$$

Hence $\left\{a_{p_j}\right\}$ is a subsequence of $\{a_n\}$, and it converges to $b \in K_2 \subseteq K_1 \cup K_2$.

Conclusion. In both cases $\{a_n\}$ has a convergent subsequence whose limit lies in $K_1 \cup K_2$. Since $\{a_n\}$ was an arbitrary sequence contained in $K_1 \cup K_2$, the set $K_1 \cup K_2$ is (sequentially) compact. □

Marking scheme (6 marks)

QRS suggested marking scheme.

- **(2 marks)** Correct definition of a (sequentially) compact set: every sequence in K has a convergent subsequence with limit in K. 1 mark if "in K" is dropped from either the subsequence's convergence or its limit.
- **(3 marks)** Key steps: 1 mark for correctly splitting the index set $\mathbb{N}$ into two pieces and arguing at least one is infinite; 1 mark for constructing the subsequence $\{a_{n_k}\}$ lying entirely in one K_i with strictly increasing indices; 1 mark for correctly applying the compactness of that K_i to extract the final convergent subsequence and noting it is still a subsequence of $\{a_n\}$.
- **(1 mark)** The three writing expectations: complete sentences, connectives, and explicit inference ("therefore"/"hence"/"thus"/"so" or $\therefore$).

Typical deductions: citing the Sequential Compactness of Closed and Bounded Sets or the Bolzano-Weierstrass Theorem (0 marks for the proof, since both are prohibited); asserting without justification that "infinitely many terms lie in K_1 or K_2"; forgetting to check that the extracted sub-subsequence is a genuine subsequence of the original $\{a_n\}$; omitting "without loss of generality" when only one case is argued.