MH3100 Real Analysis I

Midterm Practice 4 (from past Finals) (Solutions)

Past Final questions in Midterm scope (Lectures 1–17)

Quantitative Research Society @NTU

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About This Practice Paper

This is QRS@NTU's fourth MH3100 Midterm practice paper. Instead of a fresh QRS-authored paper, it collects every question part from NTU's three most recent MH3100 Real Analysis I *Final* examinations (AY2022/23, AY2023/24, AY2024/25, Semester 2) that falls inside the **Midterm scope, Lectures 1–17** (construction and completeness of $\mathbb{R}$ through sets of real numbers). Parts that require Lecture 18–20 material (function limits, continuity, uniform continuity), differentiation, integration, uniform convergence of function sequences/series, or open covers/Heine-Borel are outside the Midterm syllabus and are omitted below; the scope table gives the reason for every omission. Since this paper draws only on genuine past-Final material, its total (97 marks across 8 questions) does not match the 24-mark, 60-minute Midterm format – treat it as extra practice, not a timed mock.

Solutions in this paper are held to the same three writing expectations as the Midterm: complete sentences; related statements joined by logical connectives; and inference explicitly marked with "therefore/hence/so".

Overview

Papers used (all MH3100 Real Analysis I, NTU, CLOSED BOOK, 6 questions, 2 hours):

- Semester 2 Examination 2022–2023 (April 2023).
- Semester II Examination 2023–2024 (April 2024).
- Semester II Examination 2024–2025 (April 2025).

In every one of the three papers, Question 1 (15 marks, five True/False parts) and Question 2 (15 marks, three existential parts) mix in-scope and out-of-scope sub-parts, Question 3 (20 marks) is a supremum/infimum proof that is fully in scope, and Questions 4–6 (20+20+10 marks) all hinge on continuity and/or uniform convergence and are fully out of scope. Marks are not printed per sub-part on the original papers for Questions 1–2; QRS distributes each question's total evenly across its parts.

Scope table

AY2022/23 Semester 2 Final

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Question	Status	Reason
Q1(a)–(c)	Partly (3/5)	Monotone Convergence Lemma (L9); Theorem on Subsequential Limits (L7); limit points (L13) – all in scope.
Q1(d)–(e)	Out	Uniform continuity; power series and uniform convergence – beyond L17.
Q2(a)	Partly (1/3)	Absolute convergence + bounded sequence, via Comparison/Absolute Convergence Test (L12).
Q2(b)–(c)	Out	Function limits (L18–20 topic); pointwise convergence of continuous functions – beyond L17.
Q3	In	Definition of supremum (L2), fully in scope.
Q4	Out	Hypothesis "f continuous" is L18–20 material.
Q5	Out	Uniform convergence of a function series – not covered in L1–17.
Q6	Out	Uniform convergence of continuous functions – not covered in L1–17.

AY2023/24 Semester 2 Final

Question	Status	Reason
Q1(a)–(c)	Partly (3/5)	Suprema of sets (L2); Bolzano-Weierstrass Theorem (L8); isolated points/closed sets (L13, L15) – all in scope.
Q1(d)–(e)	Out	Differentiability; uniform convergence of derivatives – beyond L17.
Q2(a)	Partly (1/3)	Series with $\sum a_n$ convergent but $\sum a_n^2$ not (L10–12).
Q2(b)–(c)	Out	Continuity and uniform continuity – beyond L17.
Q3	In	Definition of supremum (L2), fully in scope.
Q4	Out	Continuous functions f, g – L18–20 material.
Q5	Out	Uniform convergence of a function series.
Q6	Out	Function limits (L18–20 topic).

AY2024/25 Semester 2 Final

Question	Status	Reason
Q1(a)–(c)	Partly (3/5)	Sup/inf (L2); Cauchy Criterion for Series (L10) with the Archimedean Property (L3); closure/closed sets (L14–15) – all in scope.
Q1(d)–(e)	Out	Function limits (L18–20 topic); Weierstrass-M-test-style uniform convergence.
Q2	Out (0/3)	(a)–(b) continuous-function hypotheses; (c) uniform convergence – all beyond L17.
Q3	In	Sup/inf via rational upper bounds (L2) + Archimedean Property (L3), fully in scope.
Q4	Out	Continuous functions and compactness combined; the continuity hypothesis is L18–20 material (compactness itself, L16, would be in scope).
Q5	Out	Uniform convergence of a function series.
Q6	Out	Continuity + uniform convergence.

Total marks in this practice paper

Question	1	2	3	4	5	6	7	8	Total
Marks	9	5	20	9	5	20	9	20	97

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Unofficial Solutions by QRS@NTU

Question 1 (9 Marks)

For each of the following statements, determine if it is true or false. You do not need to justify your answer.

(a) A monotone sequence is convergent if and only if it has a convergent subsequence.

(b) A sequence is divergent if it has a divergent subsequence.

(c) If a set has no limit points, then it is finite.

(Source: MH3100 Final AY22-23 Sem 2, Question 1(a)–(c). Question 1(d)–(e), on uniformly continuous functions and uniform convergence of power series, fall outside the Lecture 1–17 Midterm scope and are omitted; see the scope table above. The Final prints only the question total, 15 marks, for all five parts; QRS distributes this evenly, 3 marks per part.)

Solution

Part (a), Method 1: Monotone Convergence Lemma

The statement is true. Let $\{a_n\}$ be a monotone sequence.

Proof. ($\Rightarrow$) Suppose $\{a_n\}$ converges, and let $a := \lim a_n$. Let $n_k := k$ for every $k \in \mathbb{N}$. Then $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers, so $\{a_{n_k}\} = \{a_k\}$ is a subsequence of $\{a_n\}$ (def. of subsequence). This subsequence is the sequence $\{a_n\}$ itself, and therefore it converges to a. Hence $\{a_n\}$ has a convergent subsequence.

($\Leftarrow$) Suppose $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$. Since $\{a_n\}$ is monotone, the tail $\{a_n\}_{n\geq 1}$ is monotone, so $\{a_n\}$ is eventually monotone (with $N_0 = 1$). Thus $\{a_n\}$ is eventually monotone and has a convergent subsequence. Therefore $\{a_n\}$ converges (Monotone Convergence Lemma).

Both directions hold, so a monotone sequence is convergent if and only if it has a convergent subsequence. □

True

Unofficial QRS Solutions

Part (a), Method 2: Bounding by the subsequential limit, then the Monotone Convergence Theorem

The statement is true. Let $\{a_n\}$ be a monotone sequence.

Proof. ($\Rightarrow$) Suppose $\{a_n\}$ converges. With $n_k := k$ for every $k \in \mathbb{N}$, the indices are strictly increasing, so $\{a_{n_k}\} = \{a_n\}$ is a subsequence of $\{a_n\}$, and it converges. Therefore $\{a_n\}$ has a convergent subsequence.

($\Leftarrow$) Suppose $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$ with $L := \lim a_{n_k}$.

Case 1: $\{a_n\}$ is increasing. Fix $n \in \mathbb{N}$. For every $k \in \mathbb{N}$ with $k \geq n$, the Subsequence Index Bound gives $n_k \geq k \geq n$, and since $\{a_n\}$ is increasing,

$$a_n \leq a_{n_k} \qquad \text{for all } k \in \mathbb{N} \text{ with } k \geq n.$$

So the constant sequence $\{c_k\}$ with $c_k := a_n$ for every $k \in \mathbb{N}$ and the sequence $\{a_{n_k}\}_{k\in\mathbb{N}}$ eventually satisfy $c_k \leq a_{n_k}$. Therefore $a_n \leq \lim_{k\to\infty} a_{n_k} = L$ (Order Limit Theorem). Since $n \in \mathbb{N}$ was arbitrary, $a_n \leq L$ for all $n \in \mathbb{N}$. Also $a_1 \leq a_n$ for all $n \in \mathbb{N}$, because $\{a_n\}$ is increasing. Hence

$$a_1 \leq a_n \leq L \qquad \text{for all } n \in \mathbb{N}.$$

Since $-|a_1| \leq a_1$ and $L \leq |L|$, this gives $-|a_1| \leq a_n \leq |L|$, and therefore

$$-M \leq a_n \leq M, \qquad \text{that is,} \qquad |a_n| \leq M := |a_1| + |L| + 1 \qquad \text{for all } n \in \mathbb{N}.$$

Hence $\{a_n\}$ is bounded. Thus $\{a_n\}$ is monotone and bounded, and therefore $\{a_n\}$ converges (MCT).

Case 2: $\{a_n\}$ is decreasing. Let $b_n := -a_n$ for every $n \in \mathbb{N}$. Then $a_n \geq a_{n+1}$ implies $b_n \leq b_{n+1}$, so $\{b_n\}$ is increasing. Also $b_{n_k} = -a_{n_k} \to -L$ (Algebraic Limit Theorem, $c = -1$), so $\{b_n\}$ has a convergent subsequence. By Case 1, $\{b_n\}$ converges, say to b. Therefore $a_n = -b_n \to -b$ (Algebraic Limit Theorem, $c = -1$), so $\{a_n\}$ converges.

In both cases $\{a_n\}$ converges, which proves ($\Leftarrow$). □

$$\boxed{\text{True}}$$

Unofficial Solutions

Part (a), Method 3: Contrapositive via unboundedness of every subsequence

The statement is true. Let $\{a_n\}$ be a monotone sequence.

Proof. ($\Rightarrow$) If $\{a_n\}$ converges, then $\{a_n\}$ is a convergent subsequence of itself (indices $n_k := k$, strictly increasing).

($\Leftarrow$) We prove the contrapositive: if the monotone sequence $\{a_n\}$ diverges, then no subsequence of $\{a_n\}$ converges. Suppose $\{a_n\}$ is monotone and divergent. If $\{a_n\}$ were bounded, it would converge (MCT); therefore $\{a_n\}$ is unbounded.

Case 1: $\{a_n\}$ is increasing. Then $a_1 \leq a_n$ for all $n \in \mathbb{N}$, so $\{a_n\}$ is bounded below. Since $\{a_n\}$ is unbounded, it is therefore not bounded above. Let $\{a_{n_k}\}$ be any subsequence of $\{a_n\}$, and let $M > 0$. Since M is not an upper bound of $\{a_n : n \in \mathbb{N}\}$, there is $N_1 \in \mathbb{N}$ with $a_{N_1} > M$. For $k := N_1$ the Subsequence Index Bound gives $n_k \geq k = N_1$, and since $\{a_n\}$ is increasing,

$$a_{n_{N_1}} \geq a_{N_1} > M.$$

Hence for every $M > 0$ some term of $\{a_{n_k}\}$ exceeds M, so $\{a_{n_k}\}$ is unbounded. Therefore $\{a_{n_k}\}$ diverges (Boundedness of Convergent Sequences).

Case 2: $\{a_n\}$ is decreasing. Let $b_n := -a_n$; then $\{b_n\}$ is increasing. If $\{b_n\}$ converged, then $a_n = -b_n$ would converge (Algebraic Limit Theorem), so $\{b_n\}$ diverges. Let $\{a_{n_k}\}$ be any subsequence of $\{a_n\}$. By Case 1, the subsequence $\{b_{n_k}\}$ of $\{b_n\}$ diverges. If $\{a_{n_k}\}$ converged, then $b_{n_k} = -a_{n_k}$ would converge (Algebraic Limit Theorem); therefore $\{a_{n_k}\}$ diverges.

In both cases no subsequence of $\{a_n\}$ converges. This is the contrapositive of ($\Leftarrow$), so ($\Leftarrow$) holds. □

True

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Part (b), Method 1: Contradiction via Subsequential Limits

The statement is true. Let $\{a_n\}$ be a sequence with a divergent subsequence $\{a_{n_k}\}$.

Proof. Suppose, for contradiction, that $\{a_n\}$ converges, and let $L := \lim a_n$. Since $\{a_{n_k}\}$ is a subsequence of the convergent sequence $\{a_n\}$, the sequence $\{a_{n_k}\}$ converges to L (Subsequential Limits). This contradicts the hypothesis that $\{a_{n_k}\}$ is divergent. Therefore $\{a_n\}$ is divergent. □

$$\boxed{\text{True}}$$

Part (b), Method 2: Contrapositive via the Cauchy Criterion

The statement is true. We prove the contrapositive: if $\{a_n\}$ converges, then every subsequence of $\{a_n\}$ converges.

Proof. Suppose $\{a_n\}$ converges. Then $\{a_n\}$ is a Cauchy sequence (Cauchy Criterion): for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n \geq N$, $|a_m - a_n| < \varepsilon$.

Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$. Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ be as above. Let $K := N$. Let $j, k \in \mathbb{N}$ with $j, k \geq K$. By the Subsequence Index Bound, $n_j \geq j \geq N$ and $n_k \geq k \geq N$. Therefore

$$\left|a_{n_j} - a_{n_k}\right| < \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $K \in \mathbb{N}$ such that for all $j, k \geq K$, $\left|a_{n_j} - a_{n_k}\right| < \varepsilon$; that is, $\{a_{n_k}\}$ is a Cauchy sequence. Therefore $\{a_{n_k}\}$ converges (Cauchy Criterion).

Thus every subsequence of a convergent sequence converges. By contraposition, if $\{a_n\}$ has a divergent subsequence, then $\{a_n\}$ is divergent. □

$$\boxed{\text{True}}$$

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Part (c), Method 1: The integers, by contradiction with two witnesses

The statement is false. The set $\mathbb{Z}$ is infinite and has no limit points.

Proof. *$\mathbb{Z}$ is infinite.* Every finite nonempty set of real numbers has a maximum, so it is bounded above. If $\mathbb{Z}$ were finite, then $\mathbb{Z}$ would be bounded above, and so would $\mathbb{N} \subseteq \mathbb{Z}$; this contradicts the Archimedean Property. Therefore $\mathbb{Z}$ is infinite.

$\mathbb{Z}$ has no limit points. Let $a \in \mathbb{R}$, and suppose, for contradiction, that a is a limit point of $\mathbb{Z}$. Applying the definition of limit point with $\varepsilon = \frac{1}{2}$, there is $x \in \mathbb{Z} \setminus \{a\}$ with $|x - a| < \frac{1}{2}$. Since $x \neq a$, the number $\varepsilon' := |x - a|$ is positive. Applying the definition of limit point again, with ε', there is $y \in \mathbb{Z} \setminus \{a\}$ with $|y - a| < \varepsilon' = |x - a|$. Then $y \neq x$, because $y = x$ would give $|x - a| < |x - a|$. By the Triangle Inequality,

$$|x - y| \leq |x - a| + |a - y| < \tfrac{1}{2} + \tfrac{1}{2} = 1.$$

But x and y are distinct integers, so $x - y$ is a nonzero integer, and therefore $|x - y| \geq 1$. This is a contradiction. Hence a is not a limit point of $\mathbb{Z}$. Since $a \in \mathbb{R}$ was arbitrary, $\mathbb{Z}$ has no limit points.

Therefore $\mathbb{Z}$ is an infinite set with no limit points, so the statement is false. □

False

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Part (c), Method 2: The natural numbers, via the Sequential Criterion for Limit Points

The statement is false. The set $\mathbb{N}$ is infinite and has no limit points.

Proof. $\mathbb{N}$ *is infinite.* Every finite nonempty set of real numbers has a maximum and is bounded above, whereas $\mathbb{N}$ is not bounded above (Archimedean). Therefore $\mathbb{N}$ is infinite.

$\mathbb{N}$ *has no limit points.* Let $a \in \mathbb{R}$, and suppose, for contradiction, that a is a limit point of $\mathbb{N}$. By the Sequential Criterion for Limit Points, there is a sequence $\{x_n\}$ contained in $\mathbb{N} \setminus \{a\}$ with $a = \lim x_n$. Since $\{x_n\}$ converges, it is a Cauchy sequence (Cauchy Criterion). Applying the Cauchy condition with $\varepsilon = 1$, there is $N_1 \in \mathbb{N}$ such that for all $m, n \geq N_1$, $|x_m - x_n| < 1$. For such m, n, the number $x_m - x_n$ is an integer with absolute value less than 1, so $x_m - x_n = 0$. Hence $x_n = x_{N_1}$ for all $n \geq N_1$, i.e. the tail $\{x_n\}_{n \geq N_1}$ is the constant sequence $c := x_{N_1}$. A constant sequence converges to its constant value, so $\lim x_n = c$ (Tail Shift). By Uniqueness of Limits, $a = c = x_{N_1}$. But $x_{N_1} \in \mathbb{N} \setminus \{a\}$, so $x_{N_1} \neq a$. This is a contradiction. Hence no $a \in \mathbb{R}$ is a limit point of $\mathbb{N}$.

Therefore $\mathbb{N}$ is an infinite set with no limit points, so the statement is false. □

$$\boxed{\text{False}}$$

Marking scheme (9 marks)

QRS suggested marking scheme.

(a) **(3 marks)** 3 marks for $\boxed{\text{True}}$; 0 marks for $\boxed{\text{False}}$. No justification is required on the actual exam.

(b) **(3 marks)** 3 marks for $\boxed{\text{True}}$; 0 marks for $\boxed{\text{False}}$.

(c) **(3 marks)** 3 marks for $\boxed{\text{False}}$; 0 marks for $\boxed{\text{True}}$.

Typical deductions (if justification is examined): in (a), proving only one direction of the "if and only if"; in (b), attempting to prove convergence directly instead of arguing by contradiction via the Theorem on Subsequential Limits; in (c), searching only among finite sets instead of recognising that infinite discrete sets such as $\mathbb{Z}$ or $\mathbb{N}$ already settle the claim.

Question 2 (5 Marks)

For each of the following existential statements, either give an example fulfilling the statement or prove that it is not true.

There is an absolutely convergent series $\sum_{n=1}^{\infty} a_n$ and bounded sequence $\{b_n\}_{n=1}^{\infty}$ for which the series $\sum_{n=1}^{\infty} a_n b_n$ is divergent.

(Source: MH3100 Final AY22-23 Sem 2, Question 2(a). Question 2(b)–(c), on function limits and pointwise convergence of continuous functions, fall outside the Lecture 1–17 Midterm scope and are omitted; see the scope table above. The Final's Question 2 total, 15 marks, is distributed evenly, 5 marks per part.)

Solution

Method 1: Comparison Test, then the Absolute Convergence Test

The statement is false: for every absolutely convergent series $\sum_{n=1}^{\infty} a_n$ and every bounded sequence $\{b_n\}$, the series $\sum a_n b_n$ converges.

Proof. Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent, so that $\sum |a_n|$ converges, and let $\{b_n\}$ be bounded. By the definition of a bounded sequence, there is $M > 0$ such that $|b_n| \leq M$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we have $|a_n b_n| = |a_n|\,|b_n|$, and multiplying $|b_n| \leq M$ by $|a_n| \geq 0$ gives

$$0 \leq |a_n b_n| = |a_n|\,|b_n| \leq M\,|a_n| \qquad \text{for all } n \in \mathbb{N}.$$

Since $\sum |a_n|$ converges, the series $\sum M\,|a_n|$ converges (Algebraic Limit Theorem for Series, constant multiple M). Since $0 \leq |a_n b_n| \leq M\,|a_n|$ for all $n \in \mathbb{N}$, the series $\sum |a_n b_n|$ converges (Comparison Test). Therefore $\sum a_n b_n$ converges (Absolute Convergence Test).

Hence no absolutely convergent $\sum_{n=1}^{\infty} a_n$ and bounded $\{b_n\}$ make $\sum a_n b_n$ divergent, so the statement is false. □

False: $\sum a_n b_n$ always converges

Method 2: Cauchy Criterion for Series directly

The statement is false. Let $\sum |a_n|$ converge and let $M > 0$ satisfy $|b_n| \leq M$ for all $n \in \mathbb{N}$.

Proof. Let $\varepsilon > 0$. Since $\sum |a_n|$ converges, the Cauchy Criterion for Series applied with $\varepsilon/M > 0$ gives $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$ with $n > m \geq N$,

$$|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \frac{\varepsilon}{M}$$

(the absolute value of this sum of nonnegative terms is the sum itself). Let $n, m \in \mathbb{N}$ with $n > m \geq N$. Then

$$\begin{aligned} |a_{m+1}b_{m+1} + \cdots + a_n b_n| &\leq |a_{m+1}|\,|b_{m+1}| + \cdots + |a_n|\,|b_n| && \text{(Triangle Inequality)} \\ &\leq M\big(|a_{m+1}| + \cdots + |a_n|\big) && (|b_k| \leq M \text{ for every } k) \\ &< M \cdot \frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n > m \geq N$, $|a_{m+1}b_{m+1} + \cdots + a_n b_n| < \varepsilon$. Therefore $\sum a_n b_n$ converges (Cauchy Criterion for Series), and the statement is false. □

False: $\sum a_n b_n$ always converges

Method 3: Monotone Convergence Theorem on the partial sums of the absolute series

The statement is false. Let $\sum |a_n|$ converge and let $M > 0$ satisfy $|b_n| \leq M$ for all $n \in \mathbb{N}$.

Proof. Let $S := \sum_{n=1}^{\infty} |a_n|$, and for $n \in \mathbb{N}$ let

$$S_n := \sum_{k=1}^{n} |a_k| , \qquad T_n := \sum_{k=1}^{n} |a_k b_k| .$$

$S_n \leq S$ for all n. For every $n \in \mathbb{N}$, $S_{n+1} - S_n = |a_{n+1}| \geq 0$, so $\{S_n\}$ is increasing, and $S_n \to S$. Fix $n \in \mathbb{N}$. For all $m \geq n$ we have $S_n \leq S_m$, so $S_n \leq \lim_{m\to\infty} S_m = S$ (Order Limit Theorem).

$\{T_n\}$ is increasing and bounded. For every $n \in \mathbb{N}$, $T_{n+1} - T_n = |a_{n+1}b_{n+1}| \geq 0$, so $\{T_n\}$ is increasing, and $T_n \geq T_1 \geq 0$. Moreover, using $|a_k b_k| = |a_k|\,|b_k| \leq M\,|a_k|$ for each k,

$$T_n = \sum_{k=1}^{n} |a_k|\,|b_k| \leq \sum_{k=1}^{n} M\,|a_k| = MS_n \leq MS \qquad \text{for all } n \in \mathbb{N}.$$

Hence $0 \leq T_n \leq MS$ for all $n \in \mathbb{N}$, so $\{T_n\}$ is bounded.

Since $\{T_n\}$ is monotone and bounded, $\{T_n\}$ converges (MCT); that is, $\sum |a_n b_n|$ converges. Therefore $\sum a_n b_n$ converges (Absolute Convergence Test), and the statement is false. □

False: $\sum a_n b_n$ always converges

Marking scheme (5 marks)

QRS suggested marking scheme.

- **1 mark**: correctly recognising the statement is false, not attempting to fabricate an example.
- **1 mark**: introducing the bound $M > 0$ with $|b_n| \leq M$ and the pointwise estimate $|a_n b_n| \leq M|a_n|$.
- **1 mark**: correct use of the Comparison Test to conclude $\sum |a_n b_n|$ converges.
- **1 mark**: correct use of the Absolute Convergence Test to conclude $\sum a_n b_n$ converges.
- **1 mark**: the three writing expectations are met throughout.

Typical deductions: citing the Comparison Test with the inequality reversed; concluding convergence of $\sum a_n b_n$ without the intermediate absolute-convergence step; using a non-bounded or only conditionally convergent example.

Question 3 (20 Marks)

Use the definition of supremum to prove that if two nonempty sets $A, B \subseteq \mathbb{R}$ that are bounded above satisfy $\sup A < \sup B$, then for all $a \in A$, there exists $b \in B$ such that $a < b$. Give an example to show that the converse is not true.

You do not need to show that the suprema exist.

(Source: MH3100 Final AY22-23 Sem 2, Question 3, verbatim, 20 marks.)

Solution

Throughout, $A, B \subseteq \mathbb{R}$ are nonempty and bounded above, $s := \sup A$ and $t := \sup B$.

Method 1: Definition of supremum, by contradiction

Suppose $s < t$.

Proof. Let $a \in A$. Since s is an upper bound for A (def. of supremum), $a \leq s$. Suppose, for contradiction, that there is no $b \in B$ with $a < b$. Then $b \leq a$ for every $b \in B$, so a is an upper bound for B (def. of upper bound). Since t is not greater than any upper bound for B (def. of supremum), $t \leq a$. Combining the three inequalities,

$$t \leq a \leq s < t,$$

so $t < t$, which is a contradiction. Therefore there is $b \in B$ with $a < b$. Since $a \in A$ was arbitrary, for every $a \in A$ there is $b \in B$ such that $a < b$. □

$$\boxed{\sup A < \sup B \implies \text{for every } a \in A \text{ there is } b \in B \text{ with } a < b}$$

Method 2: Definition of supremum, direct argument via the negation of "upper bound"

Suppose $s < t$.

Proof. Let $a \in A$. Since s is an upper bound for A, $a \leq s$; since also $s < t$, therefore $a < t$. By the definition of supremum, every upper bound u of B satisfies $t \leq u$. Equivalently (contrapositive), every real number u with $u < t$ is not an upper bound for B. Since $a < t$, the number a is not an upper bound for B. The statement "a is an upper bound for B" means "$b \leq a$ for every $b \in B$"; its negation is "there is $b \in B$ with $b > a$". Therefore there is $b \in B$ with $a < b$. Since $a \in A$ was arbitrary, the claim holds for every $a \in A$. □

$$\boxed{\sup A < \sup B \implies \text{for every } a \in A \text{ there is } b \in B \text{ with } a < b}$$

Method 3: ε-Criterion for Suprema

Suppose $s < t$.

Proof. Let $a \in A$. Since s is an upper bound for A, $a \leq s < t$, and therefore $\varepsilon := t - a > 0$. By the ε-Criterion for Suprema applied to $t = \sup B$ with this ε, there is $b \in B$ with $t - \varepsilon < b$. Since $t - \varepsilon = t - (t - a) = a$, this says $a < b$. Since $a \in A$ was arbitrary, for every $a \in A$ there is $b \in B$ such that $a < b$. □

$$\boxed{\sup A < \sup B \implies \text{for every } a \in A \text{ there is } b \in B \text{ with } a < b}$$

The converse, Method 1: Explicit counterexample checked with the definition of supremum

The converse reads: if for every $a \in A$ there is $b \in B$ with $a < b$, then $\sup A < \sup B$. It is false.

Proof (the converse fails). Let $A = B := (0,1)$. Both sets are nonempty ($\frac{1}{2} \in (0,1)$) and bounded above by 1.

The hypothesis of the converse holds. Let $a \in A$, so $0 < a < 1$, and let $b := \dfrac{a+1}{2}$. Then

$$b - a = \frac{1-a}{2} > 0, \qquad 1 - b = \frac{1-a}{2} > 0, \qquad b > a > 0.$$

Therefore $0 < b < 1$, so $b \in B$, and $a < b$.

$\sup A = \sup B = 1$. For every $a \in (0,1)$, $a < 1$; therefore 1 is an upper bound for $(0,1)$. Let u be an upper bound for $(0,1)$, and suppose, for contradiction, that $u < 1$. Since $\frac{1}{2} \in (0,1)$, $u \geq \frac{1}{2} > 0$. Let $c := \dfrac{u+1}{2}$; then $0 < u < c < 1$, so $c \in (0,1)$ and $c > u$, contradicting that u is an upper bound. Therefore every upper bound u satisfies $1 \leq u$, and $\sup(0,1) = 1$ (def. of supremum).

Hence $\sup A = 1 = \sup B$, so $\sup A < \sup B$ is false although the hypothesis holds. □

$$\boxed{A = B = (0,1) : \text{ the hypothesis holds, but } \sup A = 1 = \sup B}$$

The converse, Method 2: The hypothesis forces only the non-strict inequality of suprema

Proof (the converse fails). Suppose that for every $a \in A$ there is $b \in B$ with $a < b$. Let $a \in A$ and choose such a $b \in B$. Since t is an upper bound for B, $b \leq t$, and therefore $a < b \leq t$. Hence t is an upper bound for A, and so $s \leq t$ (def. of supremum). In a counterexample the conclusion $s < t$ fails while $s \leq t$ holds; therefore a counterexample to the converse must have $s = t$. Moreover $s \notin A$ in any counterexample: if $s \in A$, the hypothesis would give $b \in B$ with $b > s = t$, contradicting that t is an upper bound for B. So we look for a set A whose supremum is not attained, and take $B := A$; the choice $A = B := (0,1)$ works.

The hypothesis holds. For $a \in (0,1)$, the number $b := \frac{a+1}{2}$ satisfies $0 < a < b < 1$ (because $b - a = \frac{1-a}{2} > 0$ and $1 - b = \frac{1-a}{2} > 0$), so $b \in B$ and $a < b$.

$\sup(0,1) = 1$, *by the ε-Criterion for Suprema.* The number 1 is an upper bound for $(0,1)$. Let $\varepsilon > 0$. If $\varepsilon \geq 1$, then $x := \frac{1}{2} \in (0,1)$ and $x > 0 \geq 1 - \varepsilon$. If $0 < \varepsilon < 1$, then $x := 1 - \frac{\varepsilon}{2} \in (0,1)$ and $x > 1 - \varepsilon$. In both cases some $x \in (0,1)$ satisfies $1 - \varepsilon < x$; therefore $\sup(0,1) = 1$ (ε-Criterion).

Hence $\sup A = 1 = \sup B$, so the conclusion $\sup A < \sup B$ fails. □

$$\boxed{A = B = (0,1) : \text{ the hypothesis holds, but } \sup A = 1 = \sup B}$$

Marking scheme (20 marks)

QRS suggested marking scheme.

- **4 marks**: correct definition of supremum (upper bound; not greater than any upper bound) applied to both $s = \sup A$ and $t = \sup B$.
- **4 marks**: setting up the proof by contradiction, assuming a is an upper bound for B.

- **4 marks**: correctly deriving $t \leq a$ from the minimality of $t = \sup B$ among upper bounds of B.
- **4 marks**: correctly combining $t \leq a \leq s < t$ to reach the contradiction, and concluding the existence of $b \in B$ with $a < b$.
- **3 marks**: a valid example (with justification) showing the converse fails.
- **1 mark**: the three writing expectations are met throughout.

Typical deductions: proving only that a is not an upper bound for B without extracting the required $b \in B$ with $a < b$; choosing a converse example in which $\sup A < \sup B$ still holds.

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Question 4 (9 Marks)

For each of the following statements, determine if it is true or false. You do not need to justify your answer.

(a) If the set A has supremum s and the set B has supremum t, then their intersection $A \cap B$ has supremum $\min\{s, t\}$.

(b) If an unbounded sequence has a bounded subsequence, then it has a convergent subsequence.

(c) If a set contains only isolated points, then it is closed.

(Source: MH3100 Final AY23-24 Sem 2, Question 1(a)–(c). Question 1(d)–(e), on differentiability and uniform convergence of derivatives, fall outside the Lecture 1–17 Midterm scope and are omitted; see the scope table above. Marks are again distributed evenly, 3 marks per part.)

Solution

Part (a), Method 1: Disjoint intervals with empty intersection

The statement is false.

Proof. Let $A := (0, 1)$ and $B := (2, 3)$.

$\sup A = 1$ *and* $\sup B = 3$. Let $c \in \{0, 2\}$; we show $\sup(c, c+1) = c+1$. Every $x \in (c, c+1)$ satisfies $x < c+1$, so $c+1$ is an upper bound for $(c, c+1)$. Let $\varepsilon > 0$ and $x := c+1-\frac{1}{2}\min\{\varepsilon, 1\}$. Since $0 < \frac{1}{2}\min\{\varepsilon, 1\} \leq \frac{1}{2}$, we have $c < c + \frac{1}{2} \leq x < c + 1$, so $x \in (c, c + 1)$; and since $\frac{1}{2}\min\{\varepsilon, 1\} < \varepsilon$, we have $x > c + 1 - \varepsilon$. Therefore $\sup(c, c + 1) = c + 1$ (ε-Criterion for Suprema). With $c = 0$ and $c = 2$: $\sup A = 1$, $\sup B = 3$, and so

$$\min\{\sup A, \sup B\} = \min\{1, 3\} = 1.$$

$A \cap B$ has no supremum. If $x \in A \cap B$, then $x < 1$ and $x > 2$, which is impossible; therefore $A \cap B = \emptyset$. Every $r \in \mathbb{R}$ is an upper bound for $\emptyset$, because there is no element of $\emptyset$ that could exceed r. Given any upper bound r of $\emptyset$, the number $r-1$ is also an upper bound and $r-1 < r$; therefore no upper bound r of $\emptyset$ satisfies $r \leq u$ for every upper bound u of $\emptyset$. Hence $\emptyset = A \cap B$ has no supremum, and in particular $A \cap B$ does not have supremum $\min\{\sup A, \sup B\} = 1$. □

False

Part (a), Method 2: Nonempty intersection whose supremum is smaller

The statement is false even when $A \cap B$ is nonempty.

Proof. Let $A := \{0, 2\}$ and $B := \{0, 1\}$. Since $2 \in A$ and $0 \leq 2$, $2 \leq 2$, the number 2 is the maximum of A, so $\sup A = 2$ (a maximum is the supremum). Likewise $1 \in B$ and $0 \leq 1$, $1 \leq 1$, so $\sup B = \max B = 1$. Therefore $\min\{\sup A, \sup B\} = \min\{2, 1\} = 1$. Next, $0 \in A$ and $0 \in B$, while $2 \notin B$; therefore $A \cap B = \{0\}$, and $\sup(A \cap B) = \max\{0\} = 0$. Since $0 \neq 1$, the set $A \cap B$ does not have supremum $\min\{\sup A, \sup B\}$. □

In general, when $A \cap B \neq \emptyset$, both $\sup A$ and $\sup B$ are upper bounds for $A \cap B$, so $\sup(A \cap B) \leq \min\{\sup A, \sup B\}$; the example shows the inequality can be strict.

False

Part (b), Method 1: Bolzano–Weierstrass applied to the bounded subsequence

The statement is true (the unboundedness of the whole sequence is not needed).

Proof. Let $\{a_n\}$ be a sequence with a bounded subsequence $\{a_{n_k}\}$, where $n_1 < n_2 < n_3 < \cdots$. Since $\{a_{n_k}\}_{k\in\mathbb{N}}$ is a bounded sequence, it has a convergent subsequence (Bolzano–Weierstrass): there are natural numbers $k_1 < k_2 < k_3 < \cdots$ such that $\left\{a_{n_{k_j}}\right\}_{j\in\mathbb{N}}$ converges, say to L. Let $m_j := n_{k_j}$ for $j \in \mathbb{N}$. Since $\{n_k\}$ is strictly increasing, $p < q$ implies $n_p < n_q$ for all $p, q \in \mathbb{N}$; applying this with $p = k_j < k_{j+1} = q$ gives

$$m_j = n_{k_j} < n_{k_{j+1}} = m_{j+1} \qquad \text{for every } j \in \mathbb{N}.$$

Hence $m_1 < m_2 < m_3 < \cdots$, so $\left\{a_{m_j}\right\}$ is a subsequence of $\{a_n\}$, and $\left\{a_{m_j}\right\} = \left\{a_{n_{k_j}}\right\}$ converges to L. Therefore $\{a_n\}$ has a convergent subsequence. □

True

Part (b), Method 2: A monotone subsequence (peak indices) and the Monotone Convergence Theorem

Note: the peak-index construction of a monotone subsequence is beyond the Lecture 1–17 slides; it is proved in full below.

Proof. Let $\{a_{n_k}\}$ be a bounded subsequence of $\{a_n\}$, write $c_k := a_{n_k}$, and let $M > 0$ with $|c_k| \le M$ for all $k \in \mathbb{N}$. Call $p \in \mathbb{N}$ a *peak* if $c_p \ge c_q$ for every $q \in \mathbb{N}$ with $q > p$, and let $P \subseteq \mathbb{N}$ be the set of peaks.

Case 1: P is infinite. Since $P \neq \emptyset$, let $r_1 := \min P$ (Well-Ordering Principle). Suppose $r_j \in P$ has been chosen. The set $P_j := \{p \in P : p > r_j\}$ is nonempty, because otherwise $P \subseteq \{1, 2, \ldots, r_j\}$ and P would be finite; let $r_{j+1} := \min P_j$ (Well-Ordering Principle). This defines, by induction, $r_j \in P$ with $r_{j+1} > r_j$ for every $j \in \mathbb{N}$, so $r_1 < r_2 < r_3 < \cdots$. For each $j \in \mathbb{N}$, r_j is a peak and $r_{j+1} > r_j$; therefore $c_{r_j} \ge c_{r_{j+1}}$. Hence $\left\{c_{r_j}\right\}_{j\in\mathbb{N}}$ is a decreasing subsequence of $\{c_k\}$.

Case 2: P is finite. Let $r_1 := 1$ if $P = \emptyset$, and $r_1 := 1 + \max P$ otherwise; in both cases $r_1 > p$ for every $p \in P$. Suppose $r_j \in \mathbb{N}$ has been chosen with $r_j > p$ for every $p \in P$. Then $r_j \notin P$, so r_j is not a peak; by the negation of the definition of peak, there is $q \in \mathbb{N}$ with $q > r_j$ and $c_q > c_{r_j}$. Let r_{j+1} be such a q. Then $r_{j+1} > r_j > p$ for every $p \in P$, so r_{j+1} again exceeds every peak, and $c_{r_{j+1}} > c_{r_j}$. By induction this defines $r_1 < r_2 < r_3 < \cdots$ with $c_{r_j} < c_{r_{j+1}}$ for every $j \in \mathbb{N}$. Hence $\left\{c_{r_j}\right\}_{j\in\mathbb{N}}$ is an increasing subsequence of $\{c_k\}$.

Conclusion. In either case $\{c_k\}$ has a monotone subsequence $\left\{c_{r_j}\right\}_{j\in\mathbb{N}}$ with $r_1 < r_2 < r_3 < \cdots$. Since $\left|c_{r_j}\right| \le M$ for every $j \in \mathbb{N}$, this subsequence is bounded. Therefore $\left\{c_{r_j}\right\}$ converges (MCT). Finally $c_{r_j} = a_{n_{r_j}}$ for every $j \in \mathbb{N}$. Since $\{n_k\}$ is strictly increasing and $r_j < r_{j+1}$, we have $n_{r_j} < n_{r_{j+1}}$ for every $j \in \mathbb{N}$, so $n_{r_1} < n_{r_2} < n_{r_3} < \cdots$. Hence $\left\{a_{n_{r_j}}\right\}_{j\in\mathbb{N}} = \left\{c_{r_j}\right\}_{j\in\mathbb{N}}$ is a convergent subsequence of $\{a_n\}$. □

True

Part (c), Method 1: Reciprocals of the naturals, with explicit gaps and the definition of closed set

The statement is false. Let $F := \{\frac{1}{n} : n \in \mathbb{N}\}$.

Proof. *Every point of F is isolated.* Let $n \in \mathbb{N}$, $a := \frac{1}{n} \in F$, and $\varepsilon := \dfrac{1}{n(n+1)} > 0$. Let $x \in F \setminus \{a\}$; then $x = \frac{1}{m}$ for some $m \in \mathbb{N}$ with $m \neq n$.

- If $m > n$, then $m \geq n+1$, so $\frac{1}{m} \leq \frac{1}{n+1}$, and therefore

$$a - x = \frac{1}{n} - \frac{1}{m} \geq \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} = \varepsilon.$$

- If $m < n$ (possible only when $n \geq 2$), then $m \leq n-1$, so $\frac{1}{m} \geq \frac{1}{n-1}$, and therefore

$$x - a = \frac{1}{m} - \frac{1}{n} \geq \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} > \frac{1}{n(n+1)} = \varepsilon.$$

In both cases $|x - a| \geq \varepsilon$. Hence there is no $x \in F \setminus \{a\}$ with $|x - a| < \varepsilon$, so a is not a limit point of F. Since $a \in F$, the point a is isolated (def. of isolated point). Since $n \in \mathbb{N}$ was arbitrary, every point of F is isolated.

F is not closed. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$. Then $\frac{1}{N} \in F$, $\frac{1}{N} \neq 0$ (as $\frac{1}{N} > 0$), and $\left|\frac{1}{N} - 0\right| = \frac{1}{N} < \varepsilon$. Hence for every $\varepsilon > 0$ there is $x \in F \setminus \{0\}$ with $|x - 0| < \varepsilon$, so 0 is a limit point of F. But $\frac{1}{n} > 0$ for every $n \in \mathbb{N}$, so $0 \notin F$. Therefore F does not contain all of its limit points, and F is not closed (def. of closed set). □

False

Part (c), Method 2: Sequential Criteria for Limit Points and for Closed Sets

The statement is false. Let $F := \{\frac{1}{n} : n \in \mathbb{N}\}$.

Proof. *Every point of F is isolated.* Let $n \in \mathbb{N}$, and suppose, for contradiction, that $\frac{1}{n}$ is a limit point of F. By the Sequential Criterion for Limit Points there is a sequence $\{x_k\}$ contained in $F \setminus \{\frac{1}{n}\}$ with $x_k \to \frac{1}{n}$. Write $x_k = \frac{1}{m_k}$ with $m_k \in \mathbb{N}$ and $m_k \neq n$. Since $x_k \neq 0$ for every k and $\lim x_k = \frac{1}{n} \neq 0$, the Algebraic Limit Theorem (reciprocals) gives

$$m_k = \frac{1}{x_k} \longrightarrow \frac{1}{1/n} = n.$$

Applying the definition of limit with $\varepsilon = 1$, there is $K \in \mathbb{N}$ such that for all $k \geq K$, $|m_k - n| < 1$. For such k, $m_k - n$ is an integer with absolute value less than 1, so $m_k = n$; this contradicts $m_k \neq n$. Hence $\frac{1}{n}$ is not a limit point of F; since $\frac{1}{n} \in F$, it is isolated. Since $n \in \mathbb{N}$ was arbitrary, every point of F is isolated.

F is not closed. Let $y_k := \frac{1}{k} \in F$ for $k \in \mathbb{N}$. Let $\varepsilon > 0$ and choose $K \in \mathbb{N}$ with $K > \frac{1}{\varepsilon}$ (Archimedean). For all $k \geq K$,

$$|y_k - 0| = \frac{1}{k} \leq \frac{1}{K} < \varepsilon,$$

so $y_k \to 0$. Thus $\{y_k\}$ is a convergent sequence contained in F whose limit 0 is not in F (every element of F is positive). Therefore F is not closed (Sequential Criterion for Closed Sets). □

$$\boxed{\text{False}}$$

Marking scheme (9 marks)

QRS suggested marking scheme.

(a) **(3 marks)** 3 marks for $\boxed{\text{False}}$; 0 marks for $\boxed{\text{True}}$.

(b) **(3 marks)** 3 marks for $\boxed{\text{True}}$; 0 marks for $\boxed{\text{False}}$.

(c) **(3 marks)** 3 marks for $\boxed{\text{False}}$; 0 marks for $\boxed{\text{True}}$.

Typical deductions: in (a), forgetting that an empty intersection has no real supremum; in (b), forgetting that a subsequence of a subsequence is still a subsequence of the original sequence; in (c), confusing "every point is isolated within the set*" with "the set has no limit points at all outside itself".*

Question 5 (5 Marks)

For each of the following existential statements, either give an example fulfilling the statement or prove that it is not true.

There is a convergent series $\sum_{n=1}^{\infty} a_n$ such that the series $\sum_{n=1}^{\infty} a_n^2$ is <u>not</u> convergent.

(Source: MH3100 Final AY23-24 Sem 2, Question 2(a). Question 2(b)–(c), on continuous functions and Cauchy sequences and on uniform continuity, fall outside the Lecture 1–17 Midterm scope and are omitted; see the scope table above. Marks distributed evenly, 5 marks per part.)

Solution

Method 1: Alternating Series Test, and the Cauchy Criterion for the harmonic series

The statement is true. Take $a_n := \dfrac{(-1)^n}{\sqrt{n}}$ for $n \in \mathbb{N}$, and let $b_n := \dfrac{1}{\sqrt{n}} > 0$, so that $a_n = (-1)^n b_n$.

Proof. *Step 1: $\{b_n\}$ is decreasing.* Let $n \in \mathbb{N}$, and suppose $\sqrt{n+1} \le \sqrt{n}$. Squaring these nonnegative numbers gives $n+1 \le n$, which is false. Therefore $\sqrt{n} < \sqrt{n+1}$, and taking reciprocals of positive numbers,

$$b_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = b_n.$$

Step 2: $b_n \to 0$. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon^2}$. Let $n \in \mathbb{N}$ with $n \ge N$. Then $n \ge N > \frac{1}{\varepsilon^2}$, so $\sqrt{n} > \frac{1}{\varepsilon}$ (square roots preserve order), and therefore

$$|b_n - 0| = \frac{1}{\sqrt{n}} < \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \ge N$, $|b_n - 0| < \varepsilon$; so $b_n \to 0$ (def. of limit).

Step 3: $\sum a_n$ converges. By Steps 1–2, $\{b_n\}$ is decreasing and converges to 0, so $\sum (-1)^{n-1} b_n$ converges (Alternating Series Test). Since $a_n = (-1)^n b_n = (-1) \cdot (-1)^{n-1} b_n$ for every n, the series $\sum a_n$ converges (Algebraic Limit Theorem for Series, constant multiple -1).

Step 4: $\sum a_n^2$ diverges. For every $n \in \mathbb{N}$,

$$a_n^2 = \frac{\left((-1)^n\right)^2}{(\sqrt{n})^2} = \frac{1}{n}.$$

Let $s_m := \sum_{k=1}^{m} \frac{1}{k}$ for $m \in \mathbb{N}$, and let $\varepsilon := \frac{1}{2}$. Let $N \in \mathbb{N}$ be arbitrary, and take $m := N$, $n := 2N$, so that $n > m \ge N$. Each of the N integers k with $N+1 \le k \le 2N$ satisfies $\frac{1}{k} \ge \frac{1}{2N}$; therefore

$$|s_{2N} - s_N| = \frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N} \ge N \cdot \frac{1}{2N} = \frac{1}{2} = \varepsilon.$$

Hence for $\varepsilon = \frac{1}{2}$ no $N \in \mathbb{N}$ satisfies the condition of the Cauchy Criterion for Series, so $\sum a_n^2 = \sum \frac{1}{n}$ diverges (Cauchy Criterion for Series).

Therefore $\sum a_n$ converges while $\sum a_n^2$ does not, and the statement is true. □

$$\boxed{\text{True: } a_n = \frac{(-1)^n}{\sqrt{n}}}$$

Method 2: Even and odd partial sums (MCT), and the doubling estimate for the harmonic series

The statement is true, with the same $a_n := \dfrac{(-1)^n}{\sqrt{n}}$.

Proof. *Step 1: $\sum a_n$ converges.* Let $t_n := \sum_{k=1}^{n} \dfrac{(-1)^k}{\sqrt{k}}$. Since $\sqrt{k} < \sqrt{k+1}$ for every $k \in \mathbb{N}$ (squaring $\sqrt{k+1} \le \sqrt{k}$ would give $k+1 \le k$), we have $\frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{k}}$. For every $m \in \mathbb{N}$,

$$\begin{aligned} t_{2m+2} - t_{2m} &= -\frac{1}{\sqrt{2m+1}} + \frac{1}{\sqrt{2m+2}} < 0, && \text{so } \{t_{2m}\} \text{ is decreasing;} \\ t_{2m+1} - t_{2m-1} &= \frac{1}{\sqrt{2m}} - \frac{1}{\sqrt{2m+1}} > 0, && \text{so } \{t_{2m-1}\} \text{ is increasing;} \\ t_{2m} - t_{2m-1} &= \frac{1}{\sqrt{2m}} > 0. \end{aligned}$$

Combining these, $t_1 \le t_{2m-1} < t_{2m} \le t_2$ for every $m \in \mathbb{N}$, where $t_1 = -1$ and $t_2 = -1 + \frac{1}{\sqrt{2}}$. So $\{t_{2m}\}$ and $\{t_{2m-1}\}$ are monotone and bounded, and therefore they converge (MCT); let $L_e := \lim t_{2m}$ and $L_o := \lim t_{2m-1}$. Since $\frac{1}{\sqrt{n}} \to 0$ (given $\varepsilon > 0$, any $N \in \mathbb{N}$ with $N > 1/\varepsilon^2$ gives $\frac{1}{\sqrt{n}} \le \frac{1}{\sqrt{N}} < \varepsilon$ for $n \ge N$), its subsequence $\frac{1}{\sqrt{2m}}$ also tends to 0 (Subsequential Limits). Hence, by the Algebraic Limit Theorem,

$$L_e - L_o = \lim_{m\to\infty} \left(t_{2m} - t_{2m-1}\right) = \lim_{m\to\infty} \frac{1}{\sqrt{2m}} = 0,$$

so $L_e = L_o =: L$ (Uniqueness of Limits). Let $\varepsilon > 0$. There are $M_1, M_2 \in \mathbb{N}$ such that $|t_{2m} - L| < \varepsilon$ for all $m \ge M_1$ and $|t_{2m-1} - L| < \varepsilon$ for all $m \ge M_2$. Let $N := \max\{2M_1, 2M_2\}$ and $n \ge N$. If $n = 2m$ is even, then $m = \frac{n}{2} \ge M_1$, so $|t_n - L| < \varepsilon$. If $n = 2m-1$ is odd, then $m = \frac{n+1}{2} > M_2$, so $|t_n - L| < \varepsilon$. Therefore $t_n \to L$, that is, $\sum a_n$ converges.

Step 2: $\sum a_n^2$ diverges. As in Method 1, $a_n^2 = \frac{1}{n}$; let $s_m := \sum_{k=1}^{m} \frac{1}{k}$. We prove $s_{2^j} \ge 1 + \frac{j}{2}$ for every $j \in \mathbb{N}_0$ by induction on j. For $j = 0$, $s_1 = 1$. Suppose $s_{2^j} \ge 1 + \frac{j}{2}$. The block from $k = 2^j + 1$ to $k = 2^{j+1}$ has 2^j terms, each at least $\frac{1}{2^{j+1}}$, so

$$s_{2^{j+1}} - s_{2^j} = \sum_{k=2^j+1}^{2^{j+1}} \frac{1}{k} \ge 2^j \cdot \frac{1}{2^{j+1}} = \frac{1}{2},$$

and therefore $s_{2^{j+1}} \ge 1 + \frac{j}{2} + \frac{1}{2} = 1 + \frac{j+1}{2}$. This completes the induction. Let $M > 0$. By the Archimedean Property there is $j \in \mathbb{N}$ with $j > 2M$, and then $s_{2^j} \ge 1 + \frac{j}{2} > M$. Hence $\{s_m\}$ is not bounded, so it does not converge (Boundedness of Convergent Sequences); that is, $\sum a_n^2$ diverges.

Therefore $\sum a_n$ converges while $\sum a_n^2$ does not, and the statement is true. □

$$\boxed{\text{True: } a_n = \frac{(-1)^n}{\sqrt{n}}}$$

Marking scheme (5 marks)

QRS suggested marking scheme.

- **1 mark**: a valid example built on conditional-but-not-absolute convergence.

- **2 marks**: correct proof that $\sum a_n$ converges (Alternating Series Test, with the hypothesis that the terms decrease to 0 verified).
- **1 mark**: correct proof that $\sum a_n^2 = \sum 1/n$ diverges (Cauchy Criterion for Series, tail estimate).
- **1 mark**: the three writing expectations are met throughout.

Typical deductions: an example where both series behave the same way; omitting the decreasing-terms check before invoking the Alternating Series Test.

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Question 6 (20 Marks)

Use the definition of supremum to prove that if a set $A \subseteq \mathbb{R}$ is nonempty and bounded above and B denotes the set $\{b \in \mathbb{R} : \exists a \in A,\ b < a\}$, then B has a supremum $\sup B$ and the supremum satisfies $\sup B = \sup A$.

(Source: MH3100 Final AY23-24 Sem 2, Question 3, verbatim, 20 marks. Unlike Question 3 of the other two Finals in this paper, this question does not waive the requirement to show the supremum exists.)

Solution

Method 1: Definition of supremum, with the Axiom of Completeness for existence

Proof. Since A is nonempty and bounded above, A has a supremum (AoC); let $s := \sup A$.

Step 1: $B \neq \emptyset$. Since A is nonempty, there is $a_0 \in A$. Since $a_0 - 1 < a_0$ and $a_0 \in A$, the number $a_0 - 1$ satisfies the defining condition of B (with $a = a_0$). Therefore $a_0 - 1 \in B$, and $B \neq \emptyset$.

Step 2: s is an upper bound for B. Let $b \in B$. By the definition of B, there is $a \in A$ with $b < a$. Since s is an upper bound for A (def. of supremum), $a \leq s$. Therefore $b < a \leq s$, and so $b \leq s$. Since $b \in B$ was arbitrary, s is an upper bound for B.

Existence of $\sup B$. By Steps 1 and 2, B is nonempty and bounded above. Therefore B has a supremum (AoC); let $t := \sup B$.

Step 3: $t \leq s$. By the definition of supremum, t is not greater than any upper bound for B. By Step 2, s is an upper bound for B. Therefore $t \leq s$.

Step 4: $s \leq t$. Suppose, for contradiction, that $t < s$. By the definition of supremum, every upper bound u of A satisfies $s \leq u$; since $t < s$, the number t is not an upper bound for A. Therefore there is $a \in A$ with $a > t$. Let

$$b^* := \frac{t + a}{2}.$$

Then $b^* - t = \dfrac{a - t}{2} > 0$ and $a - b^* = \dfrac{a - t}{2} > 0$, so $t < b^* < a$. Since $b^* < a$ and $a \in A$, we have $b^* \in B$ (def. of B). Since t is an upper bound for B, $b^* \leq t$, which contradicts $t < b^*$. Therefore $s \leq t$.

By Steps 3 and 4, $t \leq s$ and $s \leq t$, so $\sup B = t = s = \sup A$. □

$$\boxed{\sup B \text{ exists and } \sup B = \sup A}$$

Method 2: ε-Criterion for Suprema

Proof. Since A is nonempty and bounded above, let $s := \sup A$ (AoC).

$B \neq \emptyset$ and s is an upper bound for B. Pick $a_0 \in A$; then $a_0 - 1 < a_0$, so $a_0 - 1 \in B$. If $b \in B$, then $b < a$ for some $a \in A$, and $a \leq s$ (s is an upper bound for A), so $b < s$. Hence s is an upper bound for B.

The ε-condition for B. Let $\varepsilon > 0$. By the ε-Criterion for Suprema applied to $s = \sup A$ with $\frac{\varepsilon}{2} > 0$, there is $a \in A$ with $s - \frac{\varepsilon}{2} < a$. Let $b := a - \frac{\varepsilon}{2}$. Then $b < a$ with $a \in A$, so $b \in B$; and

$$b = a - \frac{\varepsilon}{2} > \left(s - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} = s - \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $b \in B$ with $s - \varepsilon < b$.

Since s is an upper bound for B and satisfies this ε-condition, $s = \sup B$ (ε-Criterion for Suprema, "if" direction). In particular $\sup B$ exists, and $\sup B = s = \sup A$. □

$$\boxed{\sup B \text{ exists and } \sup B = \sup A}$$

Method 3: Identifying B as the set of all reals below the supremum of A

Proof. Since A is nonempty and bounded above, A has a supremum (AoC); let $s := \sup A$. We first show that

$$B = \{x \in \mathbb{R} : x < s\}.$$

($\subseteq$) Let $b \in B$. Then $b < a$ for some $a \in A$, and $a \leq s$ since s is an upper bound for A; therefore $b < s$.

($\supseteq$) Let $x \in \mathbb{R}$ with $x < s$. Every upper bound u of A satisfies $s \leq u$ (def. of supremum), so x is not an upper bound for A. Therefore there is $a \in A$ with $x < a$, and so $x \in B$.

Next we show $\sup\{x \in \mathbb{R} : x < s\} = s$. The set is nonempty, since $s - 1 < s$. Every element x satisfies $x < s$, so s is an upper bound. Let u be any upper bound of the set, and suppose, for contradiction, that $u < s$. Let $c := \dfrac{u + s}{2}$; then $u < c < s$, so c belongs to the set and $c > u$, contradicting that u is an upper bound. Therefore every upper bound u satisfies $s \leq u$, and s is the supremum of the set (def. of supremum).

Hence $\sup B$ exists and $\sup B = \sup\{x \in \mathbb{R} : x < s\} = s = \sup A$. □

$$\boxed{\sup B \text{ exists and } \sup B = \sup A}$$

Marking scheme (20 marks)

QRS suggested marking scheme.

- **3 marks**: Step 1 (B nonempty).
- **4 marks**: Step 2 (B bounded above by $s = \sup A$), with the definition of B correctly unpacked.
- **3 marks**: correctly invoking the Axiom of Completeness to obtain $t = \sup B$.
- **4 marks**: Step 3 ($t \leq s$).
- **5 marks**: Step 4 ($s \leq t$), including the proof-by-contradiction structure and the midpoint construction b^*.
- **1 mark**: the three writing expectations are met throughout.

Typical deductions: asserting $\sup B$ exists without first showing B is nonempty and bounded above; omitting the contrapositive step "$t < s \implies t$ is not an upper bound for A" in Step 4.

Question 7 (9 Marks)

For each of the following statements, determine if it is true or false. You do not need to justify your answer.

(a) Two sets $A, B \subseteq \mathbb{R}$ that satisfy $(\forall a \in A,\, \forall b \in B,\, a < b)$ must satisfy $\sup A < \inf B$.

(b) A series $\sum_{n=1}^{\infty} a_n$ that satisfies $(m > n \implies |a_{n+1} + a_{n+2} + \cdots + a_m| < \frac{1}{n})$ for all natural numbers $m, n \in \mathbb{N}$ is convergent.

(c) A set is closed if and only if it is the closure of some set.

(Source: MH3100 Final AY24-25 Sem 2, Question 1(a)–(c). Question 1(d)–(e), on function limits and uniform convergence of a series of functions, fall outside the Lecture 1–17 Midterm scope and are omitted; see the scope table above. Marks distributed evenly, 3 marks per part.)

Solution

Part (a), Method 1: Adjacent open intervals

The statement is false.

Proof. Let $A := (0, 1)$ and $B := (1, 2)$.

The hypothesis holds. Let $a \in A$ and $b \in B$. Then $a < 1$ and $1 < b$, so $a < b$.

$\sup A = 1$. Every $a \in A$ satisfies $a < 1$, so 1 is an upper bound for A. Let $\varepsilon > 0$ and $x := 1 - \frac{1}{2}\min\{\varepsilon, 1\}$. Since $0 < \frac{1}{2}\min\{\varepsilon, 1\} \leq \frac{1}{2}$, we have $0 < \frac{1}{2} \leq x < 1$, so $x \in A$; and $x > 1 - \varepsilon$ because $\frac{1}{2}\min\{\varepsilon, 1\} < \varepsilon$. Therefore $\sup A = 1$ (ε-Criterion for Suprema).

$\inf B = 1$. Every $b \in B$ satisfies $b > 1$, so 1 is a lower bound for B. Let $\varepsilon > 0$ and $y := 1 + \frac{1}{2}\min\{\varepsilon, 1\}$. Then $1 < y \leq \frac{3}{2} < 2$, so $y \in B$; and $y < 1 + \varepsilon$. Therefore $\inf B = 1$ (ε-Criterion for Infima).

Hence $\sup A = 1 = \inf B$, and the conclusion $\sup A < \inf B$ reads $1 < 1$, which is false. □

False

Part (a), Method 2: The hypothesis forces only the non-strict inequality; equality with a singleton

The statement is false.

Proof. *What the hypothesis does give.* Let A, B be nonempty, with $\sup A$ and $\inf B$ existing, and suppose $a < b$ for all $a \in A$ and $b \in B$. Fix $b \in B$. Every $a \in A$ satisfies $a < b$, so b is an upper bound for A, and therefore $\sup A \leq b$ (def. of supremum). Since $b \in B$ was arbitrary, $\sup A$ is a lower bound for B, and therefore $\sup A \leq \inf B$ (def. of infimum). So only the non-strict inequality is forced; we now exhibit equality.

An example with equality. Let $A := \{0\}$ and $B := \{\frac{1}{n} : n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$, $0 < \frac{1}{n}$; therefore the hypothesis holds. Since $0 \in A$ is the only element of A, $\sup A = \max A = 0$. Every $\frac{1}{n} > 0$, so 0 is a lower bound for B. Let $\varepsilon > 0$; by the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon = 0 + \varepsilon$. Therefore $\inf B = 0$ (ε-Criterion for Infima). Hence $\sup A = 0 = \inf B$, and $\sup A < \inf B$ is false. □

False

Part (b), Method 1: Cauchy Criterion for Series with the Archimedean Property

The statement is true.

Proof. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$, so that $\frac{1}{N} < \varepsilon$. Let $m, n \in \mathbb{N}$ with $m > n \geq N$. By the hypothesis applied to this pair,

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \frac{1}{n}.$$

Since $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N}$. Therefore

$$|a_{n+1} + a_{n+2} + \cdots + a_m| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m > n \geq N$, $|a_{n+1} + \cdots + a_m| < \varepsilon$. Therefore $\sum_{n=1}^{\infty} a_n$ converges (Cauchy Criterion for Series). □

True

Part (b), Method 2: Bounded partial sums, Bolzano–Weierstrass, and the whole sequence of partial sums

The statement is true.

Proof. Let $s_n := a_1 + a_2 + \cdots + a_n$ for $n \in \mathbb{N}$. For $m > n$, the first n terms cancel in $s_m - s_n$, so the hypothesis reads

$$|s_m - s_n| = |a_{n+1} + \cdots + a_m| < \frac{1}{n} \qquad \text{for all } m, n \in \mathbb{N} \text{ with } m > n. \tag{$*$}$$

$\{s_n\}$ *is bounded.* For $m > 1$, $(*)$ with $n = 1$ gives $|s_m - s_1| < 1$, so $|s_m| \leq |s_m - s_1| + |s_1| < 1 + |s_1|$ (Triangle Inequality). Also $|s_1| < 1 + |s_1|$. Hence $|s_m| < 1 + |s_1|$ for all $m \in \mathbb{N}$.

A convergent subsequence. By Bolzano–Weierstrass, $\{s_n\}$ has a subsequence $\{s_{m_k}\}$ converging to some $L \in \mathbb{R}$.

$s_n \to L$. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $\frac{1}{N} < \frac{\varepsilon}{2}$. Since $s_{m_k} \to L$, there is $K_0 \in \mathbb{N}$ such that $|s_{m_k} - L| < \frac{\varepsilon}{2}$ for all $k \geq K_0$. Let $k^* := \max\{K_0, N + 1\}$ and $p := m_{k^*}$. Then $|s_p - L| < \frac{\varepsilon}{2}$, and $p \geq k^* \geq N + 1 > N$ (Subsequence Index Bound). Let $n \in \mathbb{N}$ with $n \geq N$.

- If $n = p$, then $|s_n - L| < \frac{\varepsilon}{2} < \varepsilon$.
- If $n < p$, then $(*)$ with the pair (p, n) gives $|s_p - s_n| < \frac{1}{n} \leq \frac{1}{N} < \frac{\varepsilon}{2}$.
- If $n > p$, then $(*)$ with the pair (n, p) gives $|s_n - s_p| < \frac{1}{p} < \frac{1}{N} < \frac{\varepsilon}{2}$.

In the last two cases, by the Triangle Inequality,

$$|s_n - L| \leq |s_n - s_p| + |s_p - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|s_n - L| < \varepsilon$; so $s_n \to L$, and $\sum_{n=1}^{\infty} a_n$ converges. □

True

Part (c), Method 1: Closure Criterion and Closures of Closures

The statement is true.

Proof. ($\Leftarrow$) Suppose $F = \overline{A}$ for some $A \subseteq \mathbb{R}$. By Closures of Closures, $\overline{F} = \overline{\overline{A}} = \overline{A} = F$. In particular $\overline{F} \subseteq F$. Therefore F is closed (Closure Criterion for Closed Sets).

($\Rightarrow$) Suppose F is closed. By the Closure Criterion for Closed Sets, $\overline{F} = F$. Therefore F is the closure of some set, namely of F itself (take $A := F$).

Both directions hold, so a set is closed if and only if it is the closure of some set. □

True

Part (c), Method 2: Sequential Criterion for Closed Sets and the ε-Criterion for Closures

The statement is true.

Proof. ($\Leftarrow$) Let $A \subseteq \mathbb{R}$ and $F := \overline{A}$. Let $\{x_n\}$ be a sequence contained in F with $x_n \to x$; we show $x \in F$. Let $\varepsilon > 0$. By the definition of limit there is $N \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq N$; in particular $|x_N - x| < \frac{\varepsilon}{2}$. Since $x_N \in \overline{A}$, the ε-Criterion for Closures with $\frac{\varepsilon}{2}$ gives $y \in A$ with $|y - x_N| < \frac{\varepsilon}{2}$. By the Triangle Inequality,

$$|y - x| \leq |y - x_N| + |x_N - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $y \in A$ with $|y - x| < \varepsilon$, so $x \in \overline{A} = F$ (ε-Criterion for Closures). Thus every convergent sequence contained in F has its limit in F, and therefore F is closed (Sequential Criterion for Closed Sets).

($\Rightarrow$) Suppose F is closed, and let $x \in \overline{F}$. By the Sequential Criterion for Closures there is a sequence contained in F converging to x; since F is closed, $x \in F$ (Sequential Criterion for Closed Sets). Hence $\overline{F} \subseteq F$; together with $F \subseteq \overline{F}$ (def. of closure), $F = \overline{F}$. Therefore F is the closure of the set F. □

True

Marking scheme (9 marks)

QRS suggested marking scheme.

(a) **(3 marks)** 3 marks for False; 0 marks for True.

(b) **(3 marks)** 3 marks for True; 0 marks for False.

(c) **(3 marks)** 3 marks for True; 0 marks for False.

Typical deductions: in (a), confusing strict inequality between every pair with strict inequality between the sup and inf; in (b), forgetting to choose N via the Archimedean Property so that $1/N < \varepsilon$; in (c), proving only one direction of the biconditional.

Question 8 (20 Marks)

Use the definitions of suprema and infima to prove that if a set $A \subseteq \mathbb{R}$ and its set $B := \{b \in \mathbb{Q} : \forall a \in A,\ a \leq b\}$ of all rational upper bounds are nonempty, then $\sup A = \inf B$.

You do not need to show that the supremum and infimum exist.

(Source: MH3100 Final AY24-25 Sem 2, Question 3, verbatim, 20 marks.)

Solution

Method 1: Definitions of supremum and infimum, with a rational between two reals

Lemma (a rational between two reals). If $x, y \in \mathbb{R}$ and $x < y$, then there is $q \in \mathbb{Q}$ with $x < q < y$.

Proof of Lemma. Since $y - x > 0$, the Archimedean Property gives $n \in \mathbb{N}$ with $\frac{1}{n} < y - x$. By the Archimedean Property again, there is $M \in \mathbb{N}$ with $M > |nx|$; therefore $nx + M \geq M - |nx| > 0$. Let $S := \{k \in \mathbb{N} : k > nx + M\}$. By the Archimedean Property, $S \neq \emptyset$, so S has a least element k_0 (Well-Ordering Principle).

- If $k_0 \geq 2$, then $k_0 - 1 \in \mathbb{N}$ and $k_0 - 1 \notin S$ (it is smaller than the least element), so $k_0 - 1 \leq nx + M$.
- If $k_0 = 1$, then $k_0 - 1 = 0 < nx + M$.

In both cases $k_0 - 1 \leq nx + M < k_0$, the right-hand inequality because $k_0 \in S$. Let $k := k_0 - 1 - M \in \mathbb{Z}$. Subtracting M throughout gives $k \leq nx < k + 1$. Let $q := \dfrac{k+1}{n} \in \mathbb{Q}$. Then

$$nx < k + 1 \implies x < \frac{k+1}{n} = q,$$
$$k \leq nx \implies q = \frac{k}{n} + \frac{1}{n} \leq x + \frac{1}{n} < x + (y - x) = y.$$

Therefore $x < q < y$. □

Proof. Let $s := \sup A$ and $t := \inf B$.

Step 1: $s \leq t$. Let $b \in B$. By the definition of B, $a \leq b$ for all $a \in A$, so b is an upper bound for A. Since s is not greater than any upper bound for A (def. of supremum), $s \leq b$. Since $b \in B$ was arbitrary, s is a lower bound for B. Since t is not less than any lower bound for B (def. of infimum), $s \leq t$.

Step 2: $t \leq s$. Suppose, for contradiction, that $s < t$. By the Lemma with $x = s$ and $y = t$, there is $q \in \mathbb{Q}$ with $s < q < t$. For every $a \in A$, $a \leq s$ (s is an upper bound for A) and $s < q$, so $a \leq q$. Therefore q is a rational upper bound for A, that is, $q \in B$. Since t is a lower bound for B (def. of infimum), $t \leq q$. Combining,

$$t \leq q < t,$$

so $t < t$, which is a contradiction. Therefore $t \leq s$.

By Steps 1 and 2, $s \leq t$ and $t \leq s$, so $\sup A = s = t = \inf B$. □

$$\boxed{\sup A = \inf B}$$

Method 2: ε-Criterion for Infima

This method uses the Lemma proved in Method 1.

Proof. Let $s := \sup A$.

s is a lower bound for B. Let $b \in B$. Then $a \le b$ for all $a \in A$, so b is an upper bound for A, and therefore $s \le b$ (def. of supremum). Since $b \in B$ was arbitrary, s is a lower bound for B.

The ε-condition. Let $\varepsilon > 0$. By the Lemma with $x = s$ and $y = s + \varepsilon$, there is $q \in \mathbb{Q}$ with $s < q < s + \varepsilon$. For every $a \in A$, $a \le s < q$; therefore q is a rational upper bound for A, so $q \in B$. Also $q < s + \varepsilon$. Hence for every $\varepsilon > 0$ there is $q \in B$ with $q < s + \varepsilon$.

Since s is a lower bound for B and satisfies this ε-condition, $s = \inf B$ (ε-Criterion for Infima). Therefore $\sup A = s = \inf B$. □

$$\boxed{\sup A = \inf B}$$

Method 3: A sequence of rational upper bounds and the Order Limit Theorem

This method uses the Lemma proved in Method 1.

Proof. Let $s := \sup A$ and $t := \inf B$.

$s \le t$. As in Step 1 of Method 1, each $b \in B$ is an upper bound for A, so $s \le b$; hence s is a lower bound for B, and $s \le t$ (def. of infimum).

$t \le s$. For each $n \in \mathbb{N}$, the Lemma with $x = s$ and $y = s + \frac{1}{n}$ gives $q_n \in \mathbb{Q}$ with $s < q_n < s + \frac{1}{n}$. For every $a \in A$, $a \le s < q_n$, so q_n is a rational upper bound for A; that is, $q_n \in B$. Since t is a lower bound for B, $t \le q_n$ for every $n \in \mathbb{N}$.

Next, $q_n \to s$: let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$ (Archimedean). For all $n \ge N$,

$$|q_n - s| = q_n - s < \frac{1}{n} \le \frac{1}{N} < \varepsilon.$$

So the constant sequence $\{t\}$ and the sequence $\{q_n\}$ satisfy $t \le q_n$ for all $n \in \mathbb{N}$, and therefore $t \le \lim q_n = s$ (Order Limit Theorem).

Hence $s \le t$ and $t \le s$, so $\sup A = s = t = \inf B$. □

$$\boxed{\sup A = \inf B}$$

Marking scheme (20 marks)

QRS suggested marking scheme.

- **3 marks**: Step 1 ($s \le t$) complete.
- **3 marks**: Lemma correctly set up (choosing n with $1/n < y - x$, and M with $nx + M > 0$).
- **6 marks**: Lemma's construction of k via the least element of S, and the verification $k \le nx < k + 1$.
- **3 marks**: correctly applying the Lemma in Step 2 (the rational q satisfies $s < q < t$, is an upper bound for A, and hence $q \in B$).
- **4 marks**: deriving the contradiction $t \le q < t$ and concluding $t \le s$.
- **1 mark**: the three writing expectations are met throughout.

Typical deductions: skipping the existence argument for the least element k_0 of S; using a real (not necessarily rational) q in Step 2, which does not lie in B.