MH3100 Real Analysis I

Tutorial 1 Solutions

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Overview

- **Lecture coverage:** Lectures 1–3, namely the construction of $\mathbb{R}$, the Axiom of Completeness, suprema, infima, maxima, minima, the Nested Interval Property, and the Archimedean Property.
- **Questions:** 8 questions in total, namely Q1 through Q5 and Additional Questions A1 through A3.
- **Notation:** $\mathbb{N} = \{1, 2, 3, \ldots\}$, and 0 is *not* a natural number in this course; sup, inf, max, min are written with the built-in operators; ε denotes ε.
- **Question themes:** Q1 sup/inf/max/min of four explicit sets; Q2 supremum of a union; Q3 infimum as the supremum of the lower bounds; Q4 supremum of a sumset; Q5 a $\frac{1}{n}$-characterisation of the supremum; A1 supremum of a countable union; A2 nested and non-nested interval intersections; A3 countability of $\mathbb{N} \times \mathbb{N}$.

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Question 1 (Suprema, infima, maxima, minima of four sets)

Problem

Compute, without proofs, the suprema and infima (if exist) of the following sets, and determine if the suprema and infima (if exist) are, respectively, maxima and minima of the respective sets.

(a) $A = \left\{ \frac{m}{n} : m, n \in \mathbb{N} \text{ with } m \leq n \right\}$.

(b) $B = \left\{ \frac{(-1)^m}{n} : m, n \in \mathbb{N} \right\}$.

(c) $C = \left\{ \frac{n}{3n+1} : n \in \mathbb{N} \right\}$.

(d) $D = \left\{ \frac{m}{m+n} : m, n \in \mathbb{N} \right\}$.

(Tutorial 1, Q1.)

Solution

Part (a)

Method 1: Attained upper bound; ε-Criterion with the Archimedean Property

Proof (supremum). Let $m, n \in \mathbb{N}$ with $m \leq n$. Since $n > 0$, dividing the inequality $m \leq n$ by n preserves it:

$$\frac{m}{n} \leq \frac{n}{n} = 1.$$

Therefore, for all $m, n \in \mathbb{N}$ with $m \leq n$, $\frac{m}{n} \leq 1$; hence 1 is an upper bound for A. Taking $m = n = 1$, which satisfies $m \leq n$, gives

$$\frac{1}{1} = 1 \in A.$$

Therefore $1 \in A$ and 1 is an upper bound for A, so $1 = \max A$ (def. of maximum). Let b be any upper bound for A. Since $1 \in A$, $1 \leq b$. Hence 1 is an upper bound for A that is not greater than any upper bound for A, so $1 = \sup A$ (def. of supremum). □

$$\boxed{\sup A = \max A = 1.}$$

Proof (infimum). Let $m, n \in \mathbb{N}$ with $m \leq n$. Since $m \geq 1$ and $n > 0$,

$$\frac{m}{n} \geq \frac{1}{n} > 0.$$

Therefore every element of A is strictly positive. Hence 0 is a lower bound for A, and $0 \notin A$. Let $\varepsilon > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$ (Archimedean).

Since $1 \leq n$, the choice $m = 1$ gives $\dfrac{1}{n} \in A$, and

$$\frac{1}{n} < \varepsilon = 0 + \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there is an element of A that is less than $0 + \varepsilon$. Since 0 is also a lower bound for A, $0 = \inf A$ (ε-Criterion). Since $0 \notin A$, 0 is not the minimum of A. □

$$\boxed{\inf A = 0, \quad 0 \text{ is not the minimum of } A.}$$

Method 2: ε-Criterion for the supremum; greatest-lower-bound definition with contradiction via the Archimedean Property

Proof (supremum). Let $m, n \in \mathbb{N}$ with $m \leq n$. Since $n > 0$,

$$\frac{m}{n} \leq \frac{n}{n} = 1.$$

Therefore 1 is an upper bound for A. Let $\varepsilon > 0$. The choice $m = n = 1$ gives $a := \dfrac{1}{1} = 1 \in A$, and

$$1 - \varepsilon < 1 = a.$$

Therefore, for every $\varepsilon > 0$ there is $a \in A$ with $a > 1 - \varepsilon$. Since 1 is also an upper bound for A, $1 = \sup A$ (ε-Criterion). Since $1 \in A$ and 1 is an upper bound for A, $1 = \max A$ (def. of maximum). □

$$\boxed{\sup A = \max A = 1.}$$

Proof (infimum). Let $m, n \in \mathbb{N}$ with $m \leq n$. Since $m \geq 1$ and $n > 0$,

$$\frac{m}{n} \geq \frac{1}{n} > 0.$$

Therefore 0 is a lower bound for A, and $0 \notin A$. Let L be a lower bound for A, and suppose, for contradiction, that $L > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\dfrac{1}{n} < L$ (Archimedean). Since $m = 1 \leq n$, $\dfrac{1}{n} \in A$; since L is a lower bound for A,

$$L \leq \frac{1}{n} < L,$$

which is a contradiction. Therefore $L \leq 0$. Hence every lower bound for A is at most 0, and 0 is itself a lower bound for A, so $0 = \inf A$ (def. of infimum). Since $0 \notin A$, 0 is not the minimum of A. □

$$\boxed{\inf A = 0, \quad 0 \text{ is not the minimum of } A.}$$

Part (b)

Method 1: Attained bounds and the definitions of supremum and infimum

Proof (supremum). Let $m, n \in \mathbb{N}$. Then $(-1)^m \in \{-1, 1\}$, so $(-1)^m \leq 1$; dividing by $n > 0$ and using $n \geq 1$,

$$\frac{(-1)^m}{n} \leq \frac{1}{n} \leq 1.$$

Therefore, for all $m, n \in \mathbb{N}$, $\dfrac{(-1)^m}{n} \leq 1$; hence 1 is an upper bound for B. Taking $m = 2$ and $n = 1$ gives

$$\frac{(-1)^2}{1} = 1 \in B.$$

Therefore $1 \in B$ and 1 is an upper bound for B, so $1 = \max B$ (def. of maximum). Let b be any upper bound for B. Since $1 \in B$, $1 \leq b$. Hence $1 = \sup B$ (def. of supremum). □

$$\boxed{\sup B = \max B = 1.}$$

Proof (infimum). Let $m, n \in \mathbb{N}$. Then $(-1)^m \geq -1$; dividing by $n > 0$ gives $\dfrac{(-1)^m}{n} \geq -\dfrac{1}{n}$, and $n \geq 1$ gives $\dfrac{1}{n} \leq 1$, i.e. $-\dfrac{1}{n} \geq -1$. Hence

$$\frac{(-1)^m}{n} \geq -\frac{1}{n} \geq -1.$$

Therefore -1 is a lower bound for B. Taking $m = 1$ and $n = 1$ gives

$$\frac{(-1)^1}{1} = -1 \in B.$$

Therefore $-1 \in B$ and -1 is a lower bound for B, so $-1 = \min B$ (def. of minimum). Let ℓ be any lower bound for B. Since $-1 \in B$, $\ell \leq -1$. Hence $-1 = \inf B$ (def. of infimum). □

$$\boxed{\inf B = \min B = -1.}$$

Method 2: Splitting by the parity of m (Question 2) and the symmetry $B = -B$

Proof (supremum). Let $P^+ := \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\}$ and $P^- := \left\{-\dfrac{1}{n} : n \in \mathbb{N}\right\}$. If m is even then $\dfrac{(-1)^m}{n} = \dfrac{1}{n} \in P^+$, and if m is odd then $\dfrac{(-1)^m}{n} = -\dfrac{1}{n} \in P^-$; conversely $\dfrac{1}{n} = \dfrac{(-1)^2}{n} \in B$ and $-\dfrac{1}{n} = \dfrac{(-1)^1}{n} \in B$. Therefore

$$B = P^+ \cup P^-.$$

For all $n \in \mathbb{N}$, $n \geq 1$ gives $\dfrac{1}{n} \leq 1$, so 1 is an upper bound for P^+; and $1 = \dfrac{1}{1} \in P^+$, so every upper bound b of P^+ satisfies $1 \leq b$. Hence $\sup P^+ = 1$ (def. of supremum). For all $n \in \mathbb{N}$, $-\dfrac{1}{n} < 0$, so 0 is an upper bound for P^-. Let $\varepsilon > 0$; by the Archimedean

Property there is $n \in \mathbb{N}$ with $\dfrac{1}{n} < \varepsilon$, i.e. $-\dfrac{1}{n} > 0 - \varepsilon$. Hence $\sup P^- = 0$ (ε-Criterion). Both P^+ and P^- are nonempty and bounded above, so by Question 2,

$$\sup B = \sup\left(P^+ \cup P^-\right) = \max\{\sup P^+, \sup P^-\} = \max\{1, 0\} = 1.$$

Since $1 \in P^+ \subseteq B$ and $1 = \sup B$ is an upper bound for B, $1 = \max B$ (def. of maximum). □

$$\boxed{\sup B = \max B = 1.}$$

Proof (infimum). Let $x \in B$, say $x = \dfrac{(-1)^m}{n}$ with $m, n \in \mathbb{N}$. Then $m + 1 \in \mathbb{N}$ and

$$-x = \frac{(-1)^{m+1}}{n} \in B.$$

Therefore $x \in B \implies -x \in B$. Let $x \in B$. Then $-x \in B$, and $\sup B = 1$ is an upper bound for B, so $-x \leq 1$, i.e. $x \geq -1$. Hence -1 is a lower bound for B. Since $1 \in B$, the symmetry gives $-1 \in B$. Therefore $-1 = \min B$ (def. of minimum). Let ℓ be any lower bound for B; since $-1 \in B$, $\ell \leq -1$. Hence $-1 = \inf B$ (def. of infimum). □

$$\boxed{\inf B = \min B = -1.}$$

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Part (c)

Method 1: Monotonicity for the infimum; ε-Criterion with the Archimedean Property for the supremum

Proof (infimum). Let $c_n := \dfrac{n}{3n+1}$ for $n \in \mathbb{N}$, so $C = \{c_n : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$, put both fractions over the common denominator $(3n+4)(3n+1) > 0$:

$$\begin{aligned} c_{n+1} - c_n &= \frac{n+1}{3(n+1)+1} - \frac{n}{3n+1} = \frac{n+1}{3n+4} - \frac{n}{3n+1} \\ &= \frac{(n+1)(3n+1) - n(3n+4)}{(3n+4)(3n+1)} \\ &= \frac{(3n^2+4n+1) - (3n^2+4n)}{(3n+4)(3n+1)} \\ &= \frac{1}{(3n+4)(3n+1)} > 0. \end{aligned}$$

Therefore $c_{n+1} > c_n$ for all $n \in \mathbb{N}$. We show by induction that $c_n \geq c_1$ for all $n \in \mathbb{N}$. For $n = 1$, $c_1 \geq c_1$. If $c_k \geq c_1$ for some $k \in \mathbb{N}$, then $c_{k+1} > c_k \geq c_1$. Hence $c_n \geq c_1$ for all $n \in \mathbb{N}$, where

$$c_1 = \frac{1}{3 \cdot 1 + 1} = \frac{1}{4}.$$

Therefore $\dfrac{1}{4}$ is a lower bound for C and $\dfrac{1}{4} = c_1 \in C$, so $\dfrac{1}{4} = \min C$ (def. of minimum). Let ℓ be any lower bound for C. Since $\dfrac{1}{4} \in C$, $\ell \leq \dfrac{1}{4}$. Hence $\dfrac{1}{4} = \inf C$ (def. of infimum). □

$$\boxed{\inf C = \min C = \tfrac{1}{4}.}$$

Proof (supremum). For $n \in \mathbb{N}$, over the common denominator $3(3n+1) > 0$,

$$\frac{1}{3} - c_n = \frac{1}{3} - \frac{n}{3n+1} = \frac{(3n+1) - 3n}{3(3n+1)} = \frac{1}{3(3n+1)} > 0.$$

Therefore $c_n < \dfrac{1}{3}$ for all $n \in \mathbb{N}$. Hence $\dfrac{1}{3}$ is an upper bound for C, and $\dfrac{1}{3} \notin C$. Let $\varepsilon > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $n > \dfrac{1}{\varepsilon}$, i.e. $\dfrac{1}{n} < \varepsilon$ (Archimedean). Since $3(3n+1) = 9n + 3 > n > 0$,

$$\begin{aligned} \frac{1}{3} - c_n &= \frac{1}{3(3n+1)} \\ &< \frac{1}{n} && (9n+3 > n) \\ &< \varepsilon. \end{aligned}$$

Therefore $c_n \in C$ and $c_n > \dfrac{1}{3} - \varepsilon$. Hence, for every $\varepsilon > 0$ there is an element of C greater than $\dfrac{1}{3} - \varepsilon$; since $\dfrac{1}{3}$ is also an upper bound for C, $\dfrac{1}{3} = \sup C$ (ε-Criterion). Since $\dfrac{1}{3} \notin C$, $\dfrac{1}{3}$ is not the maximum of C. □

$$\boxed{\sup C = \tfrac{1}{3}, \quad \tfrac{1}{3} \text{ is not the maximum of } C.}$$

Method 2: Direct inequality for the infimum; least-upper-bound definition with contradiction via the Archimedean Property

Proof (infimum). Let $c_n := \dfrac{n}{3n+1}$ for $n \in \mathbb{N}$. Over the common denominator $4(3n+1) > 0$,

$$c_n - \frac{1}{4} = \frac{4n-(3n+1)}{4(3n+1)} = \frac{n-1}{4(3n+1)}.$$

Since $n \geq 1$, $n-1 \geq 0$; since also $4(3n+1) > 0$, $c_n - \dfrac{1}{4} \geq 0$. Therefore $c_n \geq \dfrac{1}{4}$ for all $n \in \mathbb{N}$, so $\dfrac{1}{4}$ is a lower bound for C. For $n = 1$ the numerator $n-1$ is 0, so $c_1 = \dfrac{1}{4} \in C$. Therefore $\dfrac{1}{4} = \min C$ (def. of minimum); and since $\dfrac{1}{4} \in C$, every lower bound ℓ for C satisfies $\ell \leq \dfrac{1}{4}$, so $\dfrac{1}{4} = \inf C$ (def. of infimum). □

$$\boxed{\inf C = \min C = \tfrac{1}{4}.}$$

Proof (supremum). For $n \in \mathbb{N}$,

$$\frac{1}{3} - c_n = \frac{(3n+1)-3n}{3(3n+1)} = \frac{1}{3(3n+1)} > 0.$$

Therefore $\dfrac{1}{3}$ is an upper bound for C, and $\dfrac{1}{3} \notin C$. Let b be an upper bound for C, and suppose, for contradiction, that $b < \dfrac{1}{3}$. Put $\delta := \dfrac{1}{3} - b > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\dfrac{1}{n} < \delta$ (Archimedean). Since $9n+3 > n$,

$$\frac{1}{3} - c_n = \frac{1}{9n+3} < \frac{1}{n} < \delta = \frac{1}{3} - b.$$

Therefore $c_n > b$ with $c_n \in C$, which contradicts b being an upper bound for C. Hence $b \geq \dfrac{1}{3}$. So $\dfrac{1}{3}$ is an upper bound for C that is not greater than any upper bound for C, i.e. $\dfrac{1}{3} = \sup C$ (def. of supremum). Since $\dfrac{1}{3} \notin C$, $\dfrac{1}{3}$ is not the maximum of C. □

$$\boxed{\sup C = \tfrac{1}{3}, \quad \tfrac{1}{3} \text{ is not the maximum of } C.}$$

Method 3: Limit of the increasing sequence $\{c_n\}$
Note: this method uses the definition of limit and the Order Limit Theorem (Lectures 4–6), beyond the Lectures 1–3 scope of this tutorial.

Proof (infimum). Let $c_n := \dfrac{n}{3n+1}$ for $n \in \mathbb{N}$. For $n \in \mathbb{N}$,

$$\begin{aligned} c_{n+1} - c_n &= \frac{n+1}{3n+4} - \frac{n}{3n+1} = \frac{(n+1)(3n+1) - n(3n+4)}{(3n+4)(3n+1)} \\ &= \frac{(3n^2+4n+1) - (3n^2+4n)}{(3n+4)(3n+1)} = \frac{1}{(3n+4)(3n+1)} > 0. \end{aligned}$$

Therefore $\{c_n\}$ is strictly increasing. By induction ($c_1 \geq c_1$, and $c_k \geq c_1 \implies c_{k+1} > c_k \geq c_1$), $c_n \geq c_1 = \dfrac{1}{4}$ for all $n \in \mathbb{N}$. Therefore $\dfrac{1}{4}$ is a lower bound for C with $\dfrac{1}{4} \in C$; hence $\dfrac{1}{4} = \min C$ (def. of minimum), and every lower bound ℓ for C satisfies $\ell \leq \dfrac{1}{4}$, so $\dfrac{1}{4} = \inf C$ (def. of infimum). □

$$\boxed{\inf C = \min C = \tfrac{1}{4}.}$$

Proof (supremum). For $n \in \mathbb{N}$,

$$c_n = \frac{1}{3} - \frac{1}{3(3n+1)}.$$

Limit. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $\dfrac{1}{N} < \varepsilon$. For all $n \geq N$, since $9n + 3 > n$,

$$\left| c_n - \frac{1}{3} \right| = \frac{1}{3(3n+1)} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore $\lim c_n = \dfrac{1}{3}$ (def. of limit).

Upper bound. Fix $k \in \mathbb{N}$. Since $\{c_n\}$ is increasing, $c_n \geq c_k$ for all $n \geq k$ (induction on n). The constant sequence $\{c_k\}$ has limit c_k, since $|c_k - c_k| = 0 < \varepsilon$ for every $\varepsilon > 0$. Hence $\{c_k\}$ and the sequence $\{c_n\}$ eventually satisfy $c_k \leq c_n$, so

$$c_k = \lim_{n\to\infty} c_k \leq \lim_{n\to\infty} c_n = \frac{1}{3}$$

(Order Limit Theorem). As $k \in \mathbb{N}$ was arbitrary, $\dfrac{1}{3}$ is an upper bound for C.

Least. Let $\varepsilon > 0$. By the definition of limit there is $N \in \mathbb{N}$ such that $\left| c_n - \dfrac{1}{3} \right| < \varepsilon$ for all $n \geq N$; taking $n = N$ gives $\left| c_N - \dfrac{1}{3} \right| < \varepsilon$, hence $c_N > \dfrac{1}{3} - \varepsilon$ with $c_N \in C$. Therefore $\dfrac{1}{3} = \sup C$ (ε-Criterion).

Not attained. Suppose $c_k = \dfrac{1}{3}$ for some $k \in \mathbb{N}$. Then $c_{k+1} > c_k = \dfrac{1}{3}$, which contradicts $\dfrac{1}{3}$ being an upper bound for C. Hence $\dfrac{1}{3} \notin C$, and $\dfrac{1}{3}$ is not the maximum of C. □

$$\boxed{\sup C = \tfrac{1}{3}, \quad \tfrac{1}{3} \text{ is not the maximum of } C.}$$

Part (d)

Method 1: ε-Criterion with the Archimedean Property

Proof (supremum). Let $m, n \in \mathbb{N}$. Since $n \geq 1 > 0$, $m < m+n$; dividing by $m+n > 0$ gives

$$m < m+n \implies \frac{m}{m+n} < 1.$$

Therefore every element of D is strictly less than 1. Hence 1 is an upper bound for D, and $1 \notin D$. Let $\varepsilon > 0$. By the Archimedean Property there is $m \in \mathbb{N}$ with $\frac{1}{m} < \varepsilon$ (Archimedean). The choice $n = 1$ gives $\frac{m}{m+1} \in D$, and

$$\begin{aligned} 1 - \frac{m}{m+1} &= \frac{(m+1)-m}{m+1} = \frac{1}{m+1} \\ &< \frac{1}{m} && (m+1 > m > 0) \\ &< \varepsilon. \end{aligned}$$

Therefore $\frac{m}{m+1} \in D$ and $\frac{m}{m+1} > 1 - \varepsilon$. Hence, for every $\varepsilon > 0$ there is an element of D greater than $1 - \varepsilon$; since 1 is an upper bound for D, $1 = \sup D$ (ε-Criterion). Since $1 \notin D$, 1 is not the maximum of D. □

$$\boxed{\sup D = 1, \quad 1 \text{ is not the maximum of } D.}$$

Proof (infimum). Let $m, n \in \mathbb{N}$. Since $m > 0$ and $m + n > 0$,

$$\frac{m}{m+n} > 0.$$

Therefore 0 is a lower bound for D, and $0 \notin D$. Let $\varepsilon > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$ (Archimedean). The choice $m = 1$ gives $\frac{1}{1+n} \in D$, and

$$\begin{aligned} \frac{1}{1+n} &< \frac{1}{n} && (1+n > n > 0) \\ &< \varepsilon = 0 + \varepsilon. \end{aligned}$$

Hence, for every $\varepsilon > 0$ there is an element of D less than $0 + \varepsilon$; since 0 is a lower bound for D, $0 = \inf D$ (ε-Criterion). Since $0 \notin D$, 0 is not the minimum of D. □

$$\boxed{\inf D = 0, \quad 0 \text{ is not the minimum of } D.}$$

Method 2: Least-upper-bound definition with contradiction via the Archimedean Property; the symmetry $x \mapsto 1 - x$ of D

Proof (supremum). Let $m, n \in \mathbb{N}$. Since $n > 0$, $m < m+n$, and dividing by $m+n > 0$ gives $\dfrac{m}{m+n} < 1$. Therefore 1 is an upper bound for D, and $1 \notin D$. Let b be an upper bound for D, and suppose, for contradiction, that $b < 1$. Put $\delta := 1 - b > 0$. By the Archimedean Property there is $m \in \mathbb{N}$ with $\dfrac{1}{m} < \delta$ (Archimedean). The choice $n = 1$ gives $\dfrac{m}{m+1} \in D$, and

$$1 - \frac{m}{m+1} = \frac{1}{m+1} < \frac{1}{m} < \delta = 1 - b.$$

Therefore $\dfrac{m}{m+1} > b$, which contradicts b being an upper bound for D. Hence $b \geq 1$. So 1 is an upper bound for D that is not greater than any upper bound for D, i.e. $1 = \sup D$ (def. of supremum). Since $1 \notin D$, 1 is not the maximum of D. □

$$\boxed{\sup D = 1, \quad 1 \text{ is not the maximum of } D.}$$

Proof (infimum). Let $x \in D$, say $x = \dfrac{m}{m+n}$ with $m, n \in \mathbb{N}$. Then

$$1 - x = \frac{(m+n) - m}{m+n} = \frac{n}{n+m} \in D,$$

since it has the form $\dfrac{m'}{m'+n'}$ with $m' = n \in \mathbb{N}$ and $n' = m \in \mathbb{N}$. Therefore $x \in D \implies 1 - x \in D$. *Lower bound.* Let $x \in D$. Then $1 - x \in D$, and every element of D is less than 1 (shown above), so $1 - x < 1$, i.e. $x > 0$. Hence 0 is a lower bound for D, and $0 \notin D$. *Greatest.* Let L be a lower bound for D, and suppose, for contradiction, that $L > 0$. Let $x \in D$. Then $1 - x \in D$, so $L \leq 1 - x$, i.e. $x \leq 1 - L$. Therefore $1 - L$ is an upper bound for D, and since $\sup D = 1$ is not greater than any upper bound for D,

$$1 \leq 1 - L \implies L \leq 0,$$

which contradicts $L > 0$. Hence $L \leq 0$. So 0 is a lower bound for D that is not less than any lower bound for D, i.e. $0 = \inf D$ (def. of infimum). Since $0 \notin D$, 0 is not the minimum of D. □

$$\boxed{\inf D = 0, \quad 0 \text{ is not the minimum of } D.}$$

Question 2 (Supremum of a union: $\max\{\sup A, \sup B\} = \sup(A \cup B)$)

Problem

Prove that if the sets $A, B \subseteq \mathbb{R}$ are nonempty and bounded above, then

$$\max\{\sup A, \sup B\} = \sup(A \cup B).$$

(Tutorial 1, Q2; this is also the sample midterm Q3.)

Solution

Method 1: Least-upper-bound definition (two inequalities)

Proof. Since A and B are nonempty and bounded above, $s := \sup A$ and $t := \sup B$ exist (AoC). Let $M := \max\{s, t\}$, so that $s \leq M$ and $t \leq M$.

Step 1: M is an upper bound for $A \cup B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Since s is an upper bound for A and t is an upper bound for B,

$$x \in A \implies x \leq s \leq M,$$
$$x \in B \implies x \leq t \leq M.$$

Therefore $x \leq M$ for all $x \in A \cup B$; hence M is an upper bound for $A \cup B$.

Step 2: $\sup(A \cup B) \leq M$. Since $A \neq \emptyset$ and $A \subseteq A \cup B$, the set $A \cup B$ is nonempty; by Step 1 it is bounded above. Therefore $u := \sup(A \cup B)$ exists (AoC). Since u is not greater than any upper bound for $A \cup B$ and M is such an upper bound, $u \leq M$ (def. of supremum).

Step 3: $M \leq \sup(A \cup B)$. Let $a \in A$. Then $a \in A \cup B$, so $a \leq u$ because u is an upper bound for $A \cup B$. Therefore u is an upper bound for A, and hence $s \leq u$ (def. of supremum, applied to A). Let $b \in B$. Then $b \in A \cup B$, so $b \leq u$. Therefore u is an upper bound for B, and hence $t \leq u$ (def. of supremum, applied to B). Since M equals s or t, $M \leq u$.

Step 4. From Steps 2 and 3, $u \leq M$ and $M \leq u$. Therefore $u = M$, i.e. $\sup(A \cup B) = \max\{\sup A, \sup B\}$. □

$$\boxed{\max\{\sup A, \sup B\} = \sup(A \cup B).}$$

Method 2: ε-Criterion

Note: the Sample Midterm Q3 phrasing excludes the ε-Criterion; this method answers the Tutorial 1 Q2 phrasing only.

Proof. Since A and B are nonempty and bounded above, $s := \sup A$ and $t := \sup B$ exist (AoC). Let $M := \max\{s, t\}$.

Upper bound. Let $x \in A \cup B$. Since s is an upper bound for A and t is an upper bound for B,

$$\begin{aligned} x \in A &\implies x \leq s \leq M, \\ x \in B &\implies x \leq t \leq M. \end{aligned}$$

Therefore M is an upper bound for $A \cup B$.

Approximation. Let $\varepsilon > 0$. Since $M = \max\{s, t\}$, either $M = s$ or $M = t$.

- If $M = s$: since $s = \sup A$, there is $a \in A$ with $a > s - \varepsilon$ (ε-Criterion for A). Then $a \in A \cup B$ and $a > M - \varepsilon$.
- If $M = t$: since $t = \sup B$, there is $b \in B$ with $b > t - \varepsilon$ (ε-Criterion for B). Then $b \in A \cup B$ and $b > M - \varepsilon$.

Therefore, in both cases, there is an element of $A \cup B$ greater than $M - \varepsilon$. Since $\varepsilon > 0$ was arbitrary and M is an upper bound for $A \cup B$, $M = \sup(A \cup B)$ (ε-Criterion). □

$$\boxed{\max\{\sup A, \sup B\} = \sup(A \cup B).}$$

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Question 3 ($\sup B = \inf A$ for the set of lower bounds)

Problem

Suppose that the set $A \subseteq \mathbb{R}$ is nonempty and bounded below, and define the set

$$B := \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}.$$

Prove that $\sup B = \inf A$. *(Tutorial 1, Q3.)*

Solution

Method 1: Verification against the definition of infimum

Proof. *Step 1: $\sigma := \sup B$ exists.* Since A is bounded below, there is $\ell \in \mathbb{R}$ with $\ell \leq a$ for all $a \in A$; then ℓ is a lower bound for A, so $\ell \in B$ and $B \neq \emptyset$. Since $A \neq \emptyset$, fix $a_0 \in A$. Let $b \in B$. Then b is a lower bound for A and $a_0 \in A$, so $b \leq a_0$. Therefore a_0 is an upper bound for B. Hence B is nonempty and bounded above, so $\sigma := \sup B$ exists (AoC).

Step 2: σ is a lower bound for A. Let $a \in A$. For every $b \in B$, b is a lower bound for A, so $b \leq a$. Therefore a is an upper bound for B, and hence $\sigma \leq a$ (def. of supremum). As $a \in A$ was arbitrary, σ is a lower bound for A.

Step 3: σ is not less than any lower bound for A. Let c be a lower bound for A. Then $c \in B$ by the definition of B. Since σ is an upper bound for B, $c \leq \sigma$.

From Steps 2 and 3, σ is a lower bound for A that is not less than any lower bound for A. Therefore $\sigma = \inf A$ (def. of infimum), i.e. $\sup B = \inf A$. □

$$\boxed{\sup B = \inf A.}$$

Method 2: ε-Criterion for the infimum

Proof. *Step 1: $\sigma := \sup B$ exists.* Since A is bounded below, some $\ell \in \mathbb{R}$ is a lower bound for A, so $\ell \in B$ and $B \neq \emptyset$. Fix $a_0 \in A$. For every $b \in B$, b is a lower bound for A, so $b \leq a_0$. Therefore B is nonempty and bounded above (by a_0), and $\sigma := \sup B$ exists (AoC).

Step 2: σ is a lower bound for A. Let $a \in A$. For every $b \in B$, $b \leq a$ because b is a lower bound for A. Therefore a is an upper bound for B, and hence $\sigma \leq a$ (def. of supremum). Thus σ is a lower bound for A.

Step 3: approximation from above. Let $\varepsilon > 0$. Since σ is an upper bound for B, every $b \in B$ satisfies $b \leq \sigma < \sigma + \varepsilon$. Therefore $\sigma + \varepsilon \notin B$. By the definition of B, $\sigma + \varepsilon$ is not a lower bound for A. Hence there is $a \in A$ with

$$a < \sigma + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary and σ is a lower bound for A (Step 2), $\sigma = \inf A$ (ε-Criterion). Therefore $\sup B = \inf A$. □

$$\boxed{\sup B = \inf A.}$$

Method 3: $\inf A$ is the maximum of B (existence of $\inf A$ by reflection)

Proof. *Step 1:* $\inf A$ *exists.* Let $-A := \{-a : a \in A\}$. Since $A \neq \emptyset$, fix $a_0 \in A$; then $-a_0 \in -A$, so $-A \neq \emptyset$. Since A is bounded below, some $\ell \in \mathbb{R}$ satisfies $\ell \leq a$ for all $a \in A$; multiplying by -1,

$$\ell \leq a \implies -a \leq -\ell \qquad \text{for all } a \in A.$$

Therefore $-\ell$ is an upper bound for $-A$, and $\tau := \sup(-A)$ exists (AoC). Put $i := -\tau$. Let $a \in A$. Then $-a \in -A$, so $-a \leq \tau$, i.e. $a \geq -\tau = i$. Hence i is a lower bound for A. Let c be a lower bound for A. For all $a \in A$, $c \leq a \implies -a \leq -c$; therefore $-c$ is an upper bound for $-A$, so $\tau \leq -c$ (def. of supremum), i.e. $c \leq i$. Hence i is a lower bound for A that is not less than any lower bound for A, so $i = \inf A$ (def. of infimum).

Step 2: $i = \max B$. Since i is a lower bound for A, $i \in B$. Let $b \in B$. Then b is a lower bound for A, so $b \leq i$ by Step 1. Therefore $i \in B$ and i is an upper bound for B, so $i = \max B$ (def. of maximum).

Step 3: $i = \sup B$. By Step 2, i is an upper bound for B. Let u be any upper bound for B. Since $i \in B$, $i \leq u$. Hence $i = \sup B$ (def. of supremum).

Therefore $\sup B = i = \inf A$. □

$$\boxed{\sup B = \inf A.}$$

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Question 4 (Supremum of a sumset: $\sup(A+B) = \sup A + \sup B$)

Problem

Suppose that the sets $A, B \subseteq \mathbb{R}$ are nonempty and bounded above and that $s = \sup A$ and $t = \sup B$. Define the notation

$$A + B := \{a + b : a \in A \text{ and } b \in B\}.$$

(a) Prove that $s + t$ is an upper bound for $A + B$.

(b) Prove that if $u \in \mathbb{R}$ is an upper bound for $A + B$ and $a \in A$, then

$$t \leq u - a.$$

(c) Hence deduce that $\sup(A + B) = s + t$.

(Tutorial 1, Q4.)

Solution

Method 1: Upper-bound chase through (a), (b), (c)

Proof (a). Let $c \in A + B$. By the definition of $A + B$ there are $a \in A$ and $b \in B$ with $c = a + b$. Since $s = \sup A$ is an upper bound for A and $t = \sup B$ is an upper bound for B,

$$a \leq s \text{ and } b \leq t \implies c = a + b \leq s + t.$$

As $c \in A + B$ was arbitrary, $s + t$ is an upper bound for $A + B$. □

$$\boxed{s + t \text{ is an upper bound for } A + B.}$$

Proof (b). Let $u \in \mathbb{R}$ be an upper bound for $A + B$, and let $a \in A$. Let $b \in B$. Then $a + b \in A + B$, and since u is an upper bound for $A + B$,

$$a + b \leq u \implies b \leq u - a.$$

As $b \in B$ was arbitrary, $u - a$ is an upper bound for B. Since $t = \sup B$ is not greater than any upper bound for B, $t \leq u - a$ (def. of supremum). □

$$\boxed{t \leq u - a.}$$

Proof (c). By (a), $s + t$ is an upper bound for $A + B$. Let u be any upper bound for $A + B$. Let $a \in A$. By (b),

$$t \leq u - a \implies a \leq u - t.$$

As $a \in A$ was arbitrary, $u - t$ is an upper bound for A. Since $s = \sup A$ is not greater than any upper bound for A,

$$s \leq u - t \implies s + t \leq u$$

(def. of supremum). Therefore $s + t$ is an upper bound for $A + B$ that is not greater than any upper bound for $A + B$. Hence $\sup(A + B) = s + t$ (def. of supremum). □

$$\boxed{\sup(A + B) = s + t.}$$

Method 2: Contradiction for (a) and (b); ε-Criterion for (c)

Proof (a). Suppose, for contradiction, that $s+t$ is not an upper bound for $A+B$. Then there is $c \in A+B$ with $c > s+t$. Write $c = a+b$ with $a \in A$ and $b \in B$. Since s is an upper bound for A and t is an upper bound for B, $a \le s$ and $b \le t$, so

$$c = a + b \le s + t,$$

which contradicts $c > s+t$. Hence $s+t$ is an upper bound for $A+B$. □

$$\boxed{s+t \text{ is an upper bound for } A+B.}$$

Proof (b). Let u be an upper bound for $A+B$ and let $a \in A$. Suppose, for contradiction, that $t > u-a$, and put $\varepsilon := t-(u-a) > 0$. Since $t = \sup B$, there is $b \in B$ with

$$b > t - \varepsilon = u - a$$

(ε-Criterion). Then $a+b \in A+B$ and $a+b > u$, which contradicts u being an upper bound for $A+B$. Hence $t \le u-a$. □

$$\boxed{t \le u - a.}$$

Proof (c). By (a), $s+t$ is an upper bound for $A+B$. Let $\varepsilon > 0$. Since $s = \sup A$ and $\dfrac{\varepsilon}{2} > 0$, there is $a \in A$ with $a > s - \dfrac{\varepsilon}{2}$ (ε-Criterion for A). Since $t = \sup B$, there is $b \in B$ with $b > t - \dfrac{\varepsilon}{2}$ (ε-Criterion for B). Adding the two inequalities,

$$a + b > \left(s - \frac{\varepsilon}{2}\right) + \left(t - \frac{\varepsilon}{2}\right) = (s+t) - \varepsilon,$$

and $a+b \in A+B$. Therefore, for every $\varepsilon > 0$ there is an element of $A+B$ greater than $(s+t)-\varepsilon$. Since $s+t$ is an upper bound for $A+B$, $\sup(A+B) = s+t$ (ε-Criterion). □

$$\boxed{\sup(A+B) = s+t.}$$

Question 5 (An ε-characterisation of the supremum)

Problem

Prove that if $A \subseteq \mathbb{R}$ is nonempty and bounded above, and $s \in \mathbb{R}$ has the property that

$$\forall n \in \mathbb{N} \quad s + \frac{1}{n} \text{ is an upper bound for } A \text{ and } s - \frac{1}{n} \text{ is not an upper bound for } A,$$

then $s = \sup A$. *(Tutorial 1, Q5.)*

Solution

Method 1: Contradiction via the Archimedean Property

Proof. Since A is nonempty and bounded above, $s_0 := \sup A$ exists (AoC). Suppose, for contradiction, that $s \neq s_0$. Then $s < s_0$ or $s > s_0$.

Case $s < s_0$. Then $s_0 - s > 0$, so by the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < s_0 - s$ (Archimedean). Rearranging,

$$s + \frac{1}{n} < s_0.$$

By hypothesis $s + \frac{1}{n}$ is an upper bound for A. Since $s_0 = \sup A$ is not greater than any upper bound for A, $s_0 \leq s + \frac{1}{n}$ (def. of supremum). This contradicts $s + \frac{1}{n} < s_0$.

Case $s > s_0$. Then $s - s_0 > 0$, so there is $n \in \mathbb{N}$ with $\frac{1}{n} < s - s_0$ (Archimedean), i.e. $s_0 < s - \frac{1}{n}$. Since s_0 is an upper bound for A,

$$\forall a \in A : \quad a \leq s_0 < s - \frac{1}{n}.$$

Therefore $s - \frac{1}{n}$ is an upper bound for A. This contradicts the hypothesis that $s - \frac{1}{n}$ is not an upper bound for A.

Both cases lead to a contradiction. Therefore $s = s_0 = \sup A$. □

$$\boxed{s = \sup A.}$$

Method 2: ε-Criterion

Proof. *Step 1: s is an upper bound for A.* Let $a \in A$, and suppose, for contradiction, that $a > s$. Then $a - s > 0$, so by the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < a - s$ (Archimedean), i.e.

$$s + \frac{1}{n} < a.$$

By hypothesis $s + \frac{1}{n}$ is an upper bound for A, so $a \leq s + \frac{1}{n}$, a contradiction. Hence $a \leq s$ for all $a \in A$, i.e. s is an upper bound for A.

Step 2: approximation from below. Let $\varepsilon > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$ (Archimedean). By hypothesis $s - \frac{1}{n}$ is not an upper bound for A, so there is $a \in A$ with $a > s - \frac{1}{n}$. Since $\frac{1}{n} < \varepsilon$,

$$a > s - \frac{1}{n} > s - \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there is $a \in A$ with $a > s - \varepsilon$.

By Steps 1 and 2, $s = \sup A$ (ε-Criterion). □

$$\boxed{s = \sup A.}$$

Method 3: Least-upper-bound definition

Proof. *Step 1: s is an upper bound for A.* Let $a \in A$ and suppose, for contradiction, that $a > s$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < a - s$ (Archimedean), so $s + \frac{1}{n} < a$. This contradicts the hypothesis that $s + \frac{1}{n}$ is an upper bound for A. Hence $a \leq s$ for all $a \in A$.

Step 2: s is not greater than any upper bound for A. Let b be an upper bound for A, and suppose, for contradiction, that $b < s$. Then $s - b > 0$, so there is $n \in \mathbb{N}$ with $\frac{1}{n} < s - b$ (Archimedean), i.e. $b < s - \frac{1}{n}$. By hypothesis $s - \frac{1}{n}$ is not an upper bound for A, so there is $a \in A$ with

$$a > s - \frac{1}{n} > b.$$

This contradicts b being an upper bound for A. Hence $s \leq b$.

By Steps 1 and 2, s is an upper bound for A that is not greater than any upper bound for A. Therefore $s = \sup A$ (def. of supremum). □

$$\boxed{s = \sup A.}$$

Method 4: Squeeze $s - \frac{1}{n} < \sup A \leq s + \frac{1}{n}$ and the Order Limit Theorem

Note: this method uses the definition of limit and the Order Limit Theorem (Lectures 4–6), beyond the Lectures 1–3 scope of this tutorial.

Proof. Since A is nonempty and bounded above, $s_0 := \sup A$ exists (AoC). Fix $n \in \mathbb{N}$. By hypothesis $s + \dfrac{1}{n}$ is an upper bound for A, so $s_0 \leq s + \dfrac{1}{n}$ (def. of supremum). If $s_0 \leq s - \dfrac{1}{n}$, then every $a \in A$ would satisfy $a \leq s_0 \leq s - \dfrac{1}{n}$, making $s - \dfrac{1}{n}$ an upper bound for A, contrary to the hypothesis; hence $s - \dfrac{1}{n} < s_0$. Therefore

$$s - \frac{1}{n} < s_0 \leq s + \frac{1}{n} \qquad \text{for all } n \in \mathbb{N}.$$

Limits. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $\dfrac{1}{N} < \varepsilon$. For all $n \geq N$,

$$\left| \left(s \pm \frac{1}{n} \right) - s \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore $\lim \left(s - \dfrac{1}{n} \right) = s$ and $\lim \left(s + \dfrac{1}{n} \right) = s$ (def. of limit). The constant sequence $\{s_0\}$ has limit s_0. *Passing to the limit.* The sequence $\left\{ s - \frac{1}{n} \right\}$ satisfies $s - \dfrac{1}{n} \leq s_0$ for all n, so

$$s = \lim \left(s - \frac{1}{n} \right) \leq s_0$$

(Order Limit Theorem). The sequences $\{s_0\}$ and $\left\{ s + \frac{1}{n} \right\}$ satisfy $s_0 \leq s + \dfrac{1}{n}$ for all n, so

$$s_0 = \lim s_0 \leq \lim \left(s + \frac{1}{n} \right) = s$$

(Order Limit Theorem). Therefore $s \leq s_0 \leq s$, i.e. $s = s_0 = \sup A$. □

$$\boxed{s = \sup A.}$$

Question A1 (Supremum of a countable union)

Problem

Suppose that $A_1, A_2, A_3, \ldots$ is a list of nonempty sets, each of which is bounded above. Find a formula for $\sup\left(\bigcup_{n=1}^{\infty} A_n\right)$ in terms of $\sup A_1, \sup A_2, \sup A_3, \ldots$. *(Tutorial 1, Additional Question A1.)*

Solution

Method 1: Two inequalities, case split on boundedness

Proof. Let $U := \bigcup_{n=1}^{\infty} A_n$ and $S := \{\sup A_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, A_n is nonempty and bounded above, so $\sup A_n$ exists (AoC); therefore S is a well-defined nonempty set. Since $A_1 \neq \emptyset$ and $A_1 \subseteq U$, also $U \neq \emptyset$. We use the convention $\sup X = \infty$ for a nonempty set X that is not bounded above.

Case 1: S is bounded above. Then $M := \sup S$ exists (AoC). Let $x \in U$. Then $x \in A_n$ for some $n \in \mathbb{N}$. Since $\sup A_n$ is an upper bound for A_n, and M is an upper bound for $S \ni \sup A_n$,

$$x \leq \sup A_n \leq M.$$

Therefore M is an upper bound for U. Since U is nonempty and bounded above, $\sup U$ exists (AoC), and $\sup U \leq M$ (def. of supremum). Fix $n \in \mathbb{N}$. Every $x \in A_n$ lies in U, so $x \leq \sup U$; therefore $\sup U$ is an upper bound for A_n, and hence $\sup A_n \leq \sup U$ (def. of supremum). As $n \in \mathbb{N}$ was arbitrary, $\sup U$ is an upper bound for S, and hence $M \leq \sup U$ (def. of supremum). Therefore $\sup U \leq M$ and $M \leq \sup U$, so $\sup U = M = \sup S$.

Case 2: S is not bounded above. Let $K \in \mathbb{R}$. Since K is not an upper bound for S, there is $n \in \mathbb{N}$ with $\sup A_n > K$. If K were an upper bound for A_n, then $\sup A_n \leq K$ (def. of supremum), which is false. Therefore K is not an upper bound for A_n, so there is $x \in A_n$ with $x > K$. Since $x \in A_n \subseteq U$, K is not an upper bound for U. As $K \in \mathbb{R}$ was arbitrary, U is not bounded above. Hence $\sup U = \infty = \sup S$.

Therefore $\sup U = \sup S$ in both cases. □

$$\boxed{\sup\left(\bigcup_{n=1}^{\infty} A_n\right) = \sup\{\sup A_n : n \in \mathbb{N}\}.}$$

Method 2: ε-Criterion and the contrapositive

Proof. Let $U := \bigcup_{n=1}^{\infty} A_n$ and $S := \{\sup A_n : n \in \mathbb{N}\}$. Each $\sup A_n$ exists (AoC), so $S \neq \emptyset$; and $U \supseteq A_1 \neq \emptyset$. We use the convention $\sup X = \infty$ for a nonempty set X that is not bounded above.

Case 1: S is bounded above. Then $M := \sup S$ exists (AoC). *Upper bound.* Let $x \in U$, so $x \in A_n$ for some $n \in \mathbb{N}$. Then $x \leq \sup A_n \leq M$, since $\sup A_n$ is an upper bound for A_n and M is an upper bound for S. Therefore M is an upper bound for U. *Approximation.* Let $\varepsilon > 0$. Since $M = \sup S$ and $\dfrac{\varepsilon}{2} > 0$, there is $n \in \mathbb{N}$ with

$$\sup A_n > M - \frac{\varepsilon}{2}$$

(ε-Criterion for S). Since $\sup A_n$ is the supremum of A_n, there is $x \in A_n$ with

$$x > \sup A_n - \frac{\varepsilon}{2}$$

(ε-Criterion for A_n). Adding, $x > M - \dfrac{\varepsilon}{2} - \dfrac{\varepsilon}{2} = M - \varepsilon$, and $x \in A_n \subseteq U$. Therefore, for every $\varepsilon > 0$ there is $x \in U$ with $x > M - \varepsilon$. Since M is an upper bound for U, $\sup U = M = \sup S$ (ε-Criterion).

Case 2: S is not bounded above. We prove the contrapositive: if U is bounded above, then S is bounded above. Suppose $K \in \mathbb{R}$ is an upper bound for U. Fix $n \in \mathbb{N}$. Since $A_n \subseteq U$, every $x \in A_n$ satisfies $x \leq K$, so K is an upper bound for A_n, and hence $\sup A_n \leq K$ (def. of supremum). As n was arbitrary, K is an upper bound for S. Therefore, since S is not bounded above, U is not bounded above, and $\sup U = \infty = \sup S$.

Therefore $\sup U = \sup S$ in both cases. □

$$\boxed{\sup\left(\bigcup_{n=1}^{\infty} A_n\right) = \sup\{\sup A_n : n \in \mathbb{N}\}.}$$

Question A2 (True/false: nested and non-nested interval intersections)

Problem

Determine if each of the following statements is true.

(a) There is a sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

(b) There is a sequence of closed and bounded (but not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

(Tutorial 1, Additional Question A2.)

Solution

Part (a): True

Method 1: Explicit example $J_n = (-1/n, 1/n)$ with the Archimedean Property

Proof. For $n \in \mathbb{N}$ let $J_n := \left(-\frac{1}{n}, \frac{1}{n}\right)$; since $-\frac{1}{n} < \frac{1}{n}$, each J_n is a nonempty open interval. *Nested.* Let $n \in \mathbb{N}$. Since $0 < n < n+1$, $\frac{1}{n+1} < \frac{1}{n}$ and hence $-\frac{1}{n} < -\frac{1}{n+1}$. Let $x \in J_{n+1}$. Then

$$-\frac{1}{n} < -\frac{1}{n+1} < x < \frac{1}{n+1} < \frac{1}{n},$$

so $x \in J_n$. Therefore $J_{n+1} \subseteq J_n$ for all $n \in \mathbb{N}$, i.e. $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$. *0 lies in the intersection.* For all $n \in \mathbb{N}$, $-\frac{1}{n} < 0 < \frac{1}{n}$, so $0 \in J_n$. Therefore $0 \in \bigcap_{n=1}^{\infty} J_n$. *No other point lies in the intersection.* Let $x \in \mathbb{R}$ with $x \neq 0$, so $|x| > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < |x|$ (Archimedean). If $x > 0$, then $x = |x| > \frac{1}{n}$; if $x < 0$, then $x = -|x| < -\frac{1}{n}$. In either case $x \notin J_n$, and therefore $x \notin \bigcap_{n=1}^{\infty} J_n$. Hence $\bigcap_{n=1}^{\infty} J_n = \{0\}$, which is nonempty and has exactly one element. So this nested sequence of open intervals witnesses statement (a). □

Statement (a) is TRUE; witnessed by $J_n = (-1/n, 1/n)$, $\bigcap_{n=1}^{\infty} J_n = \{0\}$.

Method 2: $|x|$ as a lower bound for $\{1/n : n \in \mathbb{N}\}$, whose infimum is 0

Proof. For $n \in \mathbb{N}$ let $J_n := \left(-\frac{1}{n}, \frac{1}{n}\right)$, a nonempty open interval. For all $n \in \mathbb{N}$, $\frac{1}{n+1} < \frac{1}{n}$, so every x with $-\frac{1}{n+1} < x < \frac{1}{n+1}$ also satisfies $-\frac{1}{n} < x < \frac{1}{n}$; hence $J_{n+1} \subseteq J_n$, and the sequence is nested. Let $P := \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. $\inf P = 0$. For all $n \in \mathbb{N}$, $\frac{1}{n} > 0$, so 0 is a lower bound for P. Let $\varepsilon > 0$; by the Archimedean Property

there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon = 0 + \varepsilon$ (Archimedean). Hence $\inf P = 0$ (ε-Criterion). *The intersection.* For $x \in \mathbb{R}$,

$$x \in \bigcap_{n=1}^{\infty} J_n \iff \forall n \in \mathbb{N} : \ -\frac{1}{n} < x < \frac{1}{n} \iff \forall n \in \mathbb{N} : \ |x| < \frac{1}{n}.$$

Let $x \in \bigcap_{n=1}^{\infty} J_n$. Then $|x| < \frac{1}{n}$ for all $n \in \mathbb{N}$, so $|x|$ is a lower bound for P. Since $\inf P = 0$ is not less than any lower bound for P, $|x| \leq 0$ (def. of infimum). Therefore $|x| = 0$, i.e. $x = 0$. Conversely $|0| = 0 < \frac{1}{n}$ for all $n \in \mathbb{N}$, so $0 \in \bigcap_{n=1}^{\infty} J_n$. Hence $\bigcap_{n=1}^{\infty} J_n = \{0\}$, which is nonempty with exactly one element. □

Statement (a) is TRUE; witnessed by $J_n = (-1/n, 1/n)$, $\bigcap_{n=1}^{\infty} J_n = \{0\}$.

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Part (b): False

Method 1: Running intersections K_N and the Nested Interval Property

Proof. Let $I_n = [a_n, b_n]$, with $a_n \leq b_n$, be closed and bounded intervals such that $K_N := \bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$. We show that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$; hence no sequence with the stated property exists, and statement (b) is false. For $N \in \mathbb{N}$ put $\alpha_N := \max\{a_1, \ldots, a_N\}$ and $\beta_N := \min\{b_1, \ldots, b_N\}$. A real number is $\geq a_n$ for every $n \leq N$ if and only if it is $\geq$ the largest of them, and $\leq b_n$ for every $n \leq N$ if and only if it is $\leq$ the smallest of them. Therefore, for $x \in \mathbb{R}$,

$$x \in K_N \iff a_n \leq x \leq b_n \text{ for } n = 1, \ldots, N \iff \alpha_N \leq x \leq \beta_N.$$

Hence $K_N = [\alpha_N, \beta_N]$. Since $K_N \neq \emptyset$, pick $x_N \in K_N$; then $\alpha_N \leq x_N \leq \beta_N$, so $\alpha_N \leq \beta_N$. Therefore each K_N is a closed and bounded interval. For $N \in \mathbb{N}$,

$$K_{N+1} = \bigcap_{n=1}^{N+1} I_n = K_N \cap I_{N+1} \subseteq K_N.$$

Therefore $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of closed and bounded intervals. By the Nested Interval Property there is $c \in \mathbb{R}$ with $c \in K_N$ for all $N \in \mathbb{N}$ (Nested Interval Property). Fix $n \in \mathbb{N}$. Then $c \in K_n = \bigcap_{j=1}^{n} I_j \subseteq I_n$, so $c \in I_n$. As $n \in \mathbb{N}$ was arbitrary, $c \in \bigcap_{n=1}^{\infty} I_n$, and therefore $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. □

Statement (b) is FALSE (via the Nested Interval Property applied to K_N.)

Method 2: Pairwise intersections and the supremum of the left endpoints (AoC)

Proof. Let $I_n = [a_n, b_n]$, with $a_n \leq b_n$, be closed and bounded intervals such that $\bigcap_{n=1}^{N} I_n \neq \emptyset$ for all $N \in \mathbb{N}$. We show that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$, so statement (b) is false. *Step 1: $a_m \leq b_n$ for all $m, n \in \mathbb{N}$.* Let $m, n \in \mathbb{N}$ and put $N := \max\{m, n\}$. By hypothesis there is $x \in \bigcap_{j=1}^{N} I_j$. Since $m \leq N$ and $n \leq N$, $x \in I_m$ and $x \in I_n$. Therefore

$$a_m \leq x \leq b_n.$$

Step 2: the supremum of the left endpoints. Let $L := \{a_m : m \in \mathbb{N}\}$. Then L is nonempty ($a_1 \in L$), and by Step 1 every b_n is an upper bound for L; in particular L is bounded above by b_1. Therefore $c := \sup L$ exists (AoC). *Step 3: $c \in I_n$ for every n.* Fix $n \in \mathbb{N}$. Since $a_n \in L$ and c is an upper bound for L, $a_n \leq c$. Since b_n is an upper bound for L (Step 1) and c is not greater than any upper bound for L, $c \leq b_n$ (def. of supremum). Therefore

$$a_n \leq c \leq b_n,$$

i.e. $c \in I_n$. As $n \in \mathbb{N}$ was arbitrary, $c \in \bigcap_{n=1}^{\infty} I_n$, so $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. □

Statement (b) is FALSE.

Question A3 (The set N x N is countable)

Problem

Prove that $\mathbb{N} \times \mathbb{N}$ is countable; i.e., elements of $\mathbb{N} \times \mathbb{N}$ can be listed in a sequence indexed by the natural numbers. *(Tutorial 1, Additional Question A3.)*

Solution

Method 1: Diagonal enumeration via triangular numbers

Proof. *Triangular numbers.* For $k \in \mathbb{N}_0$ let $T_k := \dfrac{k(k+1)}{2}$; since one of k, $k+1$ is even, $T_k \in \mathbb{N}_0$, and $T_0 = 0$. For $k \in \mathbb{N}$,

$$T_k - T_{k-1} = \frac{k(k+1) - (k-1)k}{2} = \frac{k\big((k+1) - (k-1)\big)}{2} = k \geq 1.$$

Therefore $0 = T_0 < T_1 < T_2 < \cdots$, and $T_k = T_{k-1} + k \geq k$ for all $k \in \mathbb{N}$. *Blocks.* For $k \in \mathbb{N}$ let $E_k := \{p \in \mathbb{N} : T_{k-1} < p \leq T_k\} = \{T_{k-1} + 1, \ldots, T_{k-1} + k\}$. Let $p \in \mathbb{N}$. Since $T_p \geq p$, the set $\{k \in \mathbb{N} : T_k \geq p\}$ is nonempty; let k be its least element. Then $T_k \geq p$, and $T_{k-1} < p$ (by minimality if $k \geq 2$; because $T_0 = 0 < p$ if $k = 1$). Hence $p \in E_k$. If $p \in E_k \cap E_{k'}$ with $k < k'$, then $k \leq k' - 1$ and $p \leq T_k \leq T_{k'-1} < p$, a contradiction. Therefore every $p \in \mathbb{N}$ lies in exactly one block E_k. *The map.* For $(m, n) \in \mathbb{N} \times \mathbb{N}$ put $k := m + n - 1$; since $m + n \geq 2$, $k \in \mathbb{N}$. Define

$$f(m, n) := \frac{(m+n-2)(m+n-1)}{2} + m = T_{k-1} + m.$$

Since $n \geq 1$, $m = k + 1 - n \leq k$; also $m \geq 1$. Therefore $T_{k-1} < f(m, n) \leq T_{k-1} + k = T_k$, i.e. $f(m, n) \in E_k \subseteq \mathbb{N}$, and $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is well defined. *Injective.* Let $f(m, n) = f(m', n') =: p$, and put $k := m + n - 1$, $k' := m' + n' - 1$. Then $p \in E_k$ and $p \in E_{k'}$, so $k = k'$ (each p lies in exactly one block). Hence $T_{k-1} + m = T_{k-1} + m'$, so $m = m'$, and $n = k + 1 - m = k' + 1 - m' = n'$. *Surjective.* Let $p \in \mathbb{N}$, and let k be the index with $p \in E_k$. Put $m := p - T_{k-1} \in \{1, \ldots, k\}$ and $n := k + 1 - m \in \{1, \ldots, k\} \subseteq \mathbb{N}$. Then $m + n - 1 = k$, so

$$f(m, n) = T_{k-1} + m = p.$$

Therefore f is a bijection from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$. Setting $x_p := f^{-1}(p)$ for $p \in \mathbb{N}$ lists every element of $\mathbb{N} \times \mathbb{N}$ exactly once:

$$x_1 = (1, 1),\ x_2 = (1, 2),\ x_3 = (2, 1),\ x_4 = (1, 3),\ x_5 = (2, 2),\ x_6 = (3, 1),\ \ldots$$

Hence $\mathbb{N} \times \mathbb{N}$ is countable. □

$$\boxed{\mathbb{N} \times \mathbb{N} \text{ is countable, via the bijection } f(m, n) = \tfrac{(m+n-2)(m+n-1)}{2} + m.}$$

Method 2: Injection $(m,n) \mapsto 2^m 3^n$ by parity

Proof. Let $g : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, $g(m,n) := 2^m 3^n$. *Injective.* Let $(m,n),(m',n') \in \mathbb{N}\times\mathbb{N}$ with $g(m,n) = g(m',n')$. If $m > m'$, dividing $2^m 3^n = 2^{m'} 3^{n'}$ by $2^{m'}$ gives

$$2^{m-m'} 3^n = 3^{n'};$$

the left side is even (as $m - m' \geq 1$) and the right side is odd (a product of odd numbers), a contradiction. If $m < m'$, dividing by 2^m gives $3^n = 2^{m'-m} 3^{n'}$, odd = even, again a contradiction. Therefore $m = m'$, and dividing by 2^m gives $3^n = 3^{n'}$. If $n > n'$ then $3^n = 3^{n'} \cdot 3^{n-n'} \geq 3 \cdot 3^{n'} > 3^{n'}$, a contradiction; if $n < n'$, the same argument with the roles exchanged gives a contradiction. Therefore $n = n'$, and g is injective. *A list with repeats.* Let $G := g(\mathbb{N}\times\mathbb{N}) \subseteq \mathbb{N}$. For $k \in \mathbb{N}$ put $x_k := g^{-1}(k)$ if $k \in G$ (well defined since g is injective), and $x_k := (1,1)$ if $k \notin G$. Then

$$(m,n) = x_{g(m,n)} \qquad \text{for all } (m,n) \in \mathbb{N}\times\mathbb{N},$$

so every element of $\mathbb{N}\times\mathbb{N}$ occurs in the sequence $x_1, x_2, x_3, \ldots$. *Removing repeats.* Let $F := \{k \in \mathbb{N} : x_k \notin \{x_1, \ldots, x_{k-1}\}\}$ be the set of first occurrences. For $(m,n) \in \mathbb{N}\times\mathbb{N}$ the set $\{k \in \mathbb{N} : x_k = (m,n)\}$ contains $g(m,n)$, so it has a least element $\varphi(m,n)$; by minimality $x_{\varphi(m,n)} = (m,n)$ differs from $x_1, \ldots, x_{\varphi(m,n)-1}$, so $\varphi(m,n) \in F$. If $\varphi(m,n) = \varphi(m',n')$ then $(m,n) = x_{\varphi(m,n)} = x_{\varphi(m',n')} = (m',n')$, so φ is injective. The pairs $(1,n)$, $n \in \mathbb{N}$, are distinct, so their images $\varphi(1,n)$ are distinct elements of F; therefore F is infinite. Let $k_1 := \min F$ and $k_{j+1} := \min\{k \in F : k > k_j\}$ (nonempty because F is infinite). Then $k_1 < k_2 < k_3 < \cdots$, and $k_j \geq j$ for all $j \in \mathbb{N}$ by induction ($k_1 \geq 1$, and $k_{j+1} > k_j \geq j$ gives $k_{j+1} \geq j+1$). Let $k \in F$. Then $k_1 \leq k$, and $k_j \leq k$ forces $j \leq k_j \leq k$, so there is a largest j with $k_j \leq k$. If $k_j < k$, then $k \in F$ and $k > k_j$ give $k_{j+1} \leq k$, contradicting the choice of j. Hence $k = k_j$; therefore $F = \{k_j : j \in \mathbb{N}\}$. Put $y_j := x_{k_j}$ for $j \in \mathbb{N}$. Every $(m,n) \in \mathbb{N}\times\mathbb{N}$ occurs: $\varphi(m,n) \in F$, so $\varphi(m,n) = k_j$ for some j, and $y_j = x_{\varphi(m,n)} = (m,n)$. No element occurs twice: if $j < j'$ then $k_j < k_{j'}$ and $k_{j'} \in F$, so $y_{j'} = x_{k_{j'}} \notin \{x_1, \ldots, x_{k_{j'}-1}\} \ni x_{k_j} = y_j$, hence $y_{j'} \neq y_j$. Therefore $y_1, y_2, y_3, \ldots$ lists every element of $\mathbb{N}\times\mathbb{N}$ exactly once. Hence $\mathbb{N}\times\mathbb{N}$ is countable. □

$\boxed{\mathbb{N}\times\mathbb{N} \text{ is countable.}}$

Method 3: Bijection $(m,n) \mapsto 2^{m-1}(2n-1)$ (power of two times an odd number)

Proof. Let $h : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, $h(m,n) := 2^{m-1}(2n-1)$; since $m - 1 \geq 0$ and $2n - 1 \geq 1$, $h(m,n) \in \mathbb{N}$. *Surjective.* Let $p \in \mathbb{N}$. Let $Z := \{a \in \mathbb{N}_0 : 2^a \text{ divides } p\}$. Then $0 \in Z$. For all $a \in \mathbb{N}_0$, $2^a > a$ (induction: $2^0 = 1 > 0$, and $2^a \geq a+1$ gives $2^{a+1} = 2^a + 2^a \geq (a+1)+1 > a+1$). If $a \in Z$ then $2^a \leq p$, so $a < 2^a \leq p$; hence $Z \subseteq \{0, 1, \ldots, p-1\}$ is a nonempty finite set, and it has a largest element a. Write $p = 2^a q$ with $q \in \mathbb{N}$. If q were even, then 2^{a+1} would divide p, contradicting the maximality of a; therefore q is odd, $q = 2n-1$ with $n := \dfrac{q+1}{2} \in \mathbb{N}$. Put $m := a+1 \in \mathbb{N}$. Then

$$h(m,n) = 2^a(2n-1) = 2^a q = p.$$

Injective. Let $h(m,n) = h(m',n')$. If $m > m'$, dividing by $2^{m'-1}$ gives

$$2^{m-m'}(2n-1) = 2n' - 1,$$

whose left side is even and right side is odd, a contradiction. If $m < m'$, dividing by 2^{m-1} gives $2n - 1 = 2^{m'-m}(2n'-1)$, odd = even, a contradiction. Therefore $m = m'$; dividing by 2^{m-1} gives $2n - 1 = 2n' - 1$, so $n = n'$. Therefore h is a bijection from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$, and $x_p := h^{-1}(p)$, $p \in \mathbb{N}$, lists every element of $\mathbb{N} \times \mathbb{N}$ exactly once:

$$x_1 = (1,1),\ x_2 = (2,1),\ x_3 = (1,2),\ x_4 = (3,1),\ x_5 = (1,3),\ x_6 = (2,2),\ \ldots$$

Hence $\mathbb{N} \times \mathbb{N}$ is countable. □

$\boxed{\mathbb{N} \times \mathbb{N} \text{ is countable.}}$

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Method Map

Q.	Core topic	Methods
1	Sup/inf/max/min of explicit sets	(a) M1 attained upper bound, ε-Criterion with Archimedean; M2 ε-Criterion, greatest-lower-bound definition with contradiction. (b) M1 attained bounds; M2 parity split (Question 2) and symmetry $B = -B$. (c) M1 monotonicity, ε-Criterion; M2 direct inequality, least-upper-bound definition with contradiction; M3 limit of the increasing sequence (Lectures 4–6). (d) M1 ε-Criterion with Archimedean; M2 least-upper-bound definition with contradiction, symmetry $x \mapsto 1 - x$.
2	$\max\{\sup A, \sup B\} = \sup(A \cup B)$	M1 least-upper-bound definition (two inequalities); M2 ε-Criterion.
3	$\sup B = \inf A$ for the lower-bound set	M1 verification against the definition of infimum; M2 ε-Criterion for the infimum; M3 $\inf A = \max B$ via reflection $-A$.
4	$\sup(A + B) = \sup A + \sup B$	M1 upper-bound chase (a)–(c); M2 contradiction for (a), (b) and ε-Criterion with $\varepsilon/2$ for (c).
5	ε-characterisation of $\sup A$	M1 contradiction via Archimedean; M2 ε-Criterion; M3 least-upper-bound definition; M4 squeeze and Order Limit Theorem (Lectures 4–6).
A1	sup of a countable union	M1 two inequalities, case split on boundedness; M2 ε-Criterion (bounded case), contrapositive (unbounded case).
A2	Interval intersections	(a) M1 explicit example with Archimedean; M2 $\inf\{1/n\} = 0$. (b) M1 running intersections and Nested Interval Property; M2 pairwise intersections and sup of left endpoints (AoC).
A3	Countability of $\mathbb{N} \times \mathbb{N}$	M1 diagonal bijection via triangular numbers; M2 injection $2^m 3^n$ by parity; M3 bijection $2^{m-1}(2n-1)$.

Final Answers

Q1(a) $\sup A = \max A = 1$; $\inf A = 0$, not the minimum.

Q1(b) $\sup B = \max B = 1$; $\inf B = \min B = -1$.

Q1(c) $\sup C = \frac{1}{3}$, not the maximum; $\inf C = \min C = \frac{1}{4}$.

Q1(d) $\sup D = 1$, not the maximum; $\inf D = 0$, not the minimum.

Q2 $\max\{\sup A, \sup B\} = \sup(A \cup B)$.

Q3 $\sup B = \inf A$.

Q4 (a) $s + t$ is an upper bound for $A + B$. (b) $t \leq u - a$. (c) $\sup(A + B) = s + t$.

Q5 $s = \sup A$.

A1 $\sup\left(\bigcup_{n=1}^{\infty} A_n\right) = \sup\{\sup A_n : n \in \mathbb{N}\}$.

A2 (a) True: $J_n = (-1/n, 1/n)$, $\bigcap_{n=1}^{\infty} J_n = \{0\}$. (b) False.

A3 $\mathbb{N} \times \mathbb{N}$ is countable: $f(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m$ is a bijection onto $\mathbb{N}$.

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