MH3100 Real Analysis I

Tutorial 2 Solutions

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Overview

- **Lecture coverage:** Lectures 4–6 — the definition of the limit of a sequence, the Algebraic Limit Theorem, the Order Limit Theorem, the Squeeze Theorem, and Cauchy sequences.
- **Questions:** 8 questions from Tutorial Question Set No. 2 (main Q1–Q5, Additional Questions A1–A3).
- **Notation:** as in lectures — $\mathbb{N} = \{1, 2, 3, \dots\}$ (does *not* include 0); a sequence is $\{a_n\}$; a limit is written value-first, $a = \lim a_n$; a property holding eventually means it holds on some tail $a_N, a_{N+1}, \dots$.
- **Running restriction:** Q4 and Q5 explicitly forbid the Algebraic Limit Theorem and (for Q5) the Cauchy Criterion; Method 1 and Method 2 in those two questions are therefore built from the raw ε-N definitions only; a method that uses an excluded tool carries a scope note.
- **Question themes:** Q1 examples and counterexamples for convergence; Q2 ε-N verification of three limits; Q3 bounded times null; Q4 definition-only limits of $\frac{2a_n-1}{3}$, $\sqrt{a_n}$, $\frac{1}{a_n}$; Q5 Cauchy sequences under scaling and squaring; A1 a linear recurrence; A2 Cesaro means; A3 double versus iterated limits.

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Question 1 (Eight true/false statements about convergence and divergence)

Problem

Determine if each of the following statements is true.

(a) There is a sequence with an infinite number of 1's and does not converge to 1.

(b) There is a sequence with an infinite number of 1's and converges to a limit not equal to 1.

(c) There is a divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

(d) There are sequences $\{a_n\}$ and $\{b_n\}$, which both diverge, but whose sum $\{a_n + b_n\}$ converges.

(e) There are sequences $\{a_n\}$ and $\{b_n\}$, where $\{a_n\}$ converges, $\{b_n\}$ diverges, and $\{a_n + b_n\}$ converges.

(f) There is a convergent sequence $\{b_n\}$ with $b_n \neq 0$ for each $n \in \mathbb{N}$ such that $\{1/b_n\}$ diverges.

(g) There is an unbounded sequence $\{a_n\}$ and a convergent sequence $\{b_n\}$ with $\{a_n - b_n\}$ bounded.

(h) There are sequences $\{a_n\}$ and $\{b_n\}$, where $\{a_n b_n\}$ and $\{a_n\}$ converge but $\{b_n\}$ does not.

(Tutorial 2, Q1.)

Solution

The eight statements are unrelated, so each part is solved on its own. Throughout, the negation of "$a = \lim a_n$" is used in the form: there is $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there is $n \geq N$ with $|a_n - a| \geq \varepsilon$.

Part (a): True

Method 1: Explicit example; divergence from the definition of limit. Define $a_n := 1$ for odd n and $a_n := 0$ for even n, so $\{a_n\} = 1, 0, 1, 0, \ldots$. Every odd $n \in \mathbb{N}$ gives $a_n = 1$, and there are infinitely many odd natural numbers; hence $\{a_n\}$ has an infinite number of 1's.

Proof. We show that $\{a_n\}$ does not have the limit 1. Take $\varepsilon := \frac{1}{2}$ and let $N \in \mathbb{N}$ be arbitrary. Put $n := 2N$. Then $n \geq N$ and n is even, so $a_n = 0$ and

$$|a_n - 1| = |0 - 1| = 1 \geq \tfrac{1}{2} = \varepsilon.$$

Therefore for every $N \in \mathbb{N}$ there is $n \geq N$ with $|a_n - 1| \geq \varepsilon$. Hence $\{a_n\}$ does not converge to 1 (def. of limit). □

True; $a_n = 1$ (n odd), $a_n = 0$ (n even).

Method 2: Consecutive terms and the Cauchy Criterion. Use the same sequence $a_n = 1$ (n odd), $a_n = 0$ (n even).

Proof. Suppose, for contradiction, that $\{a_n\}$ converges. Then $\{a_n\}$ is Cauchy (Cauchy Criterion, convergent $\implies$ Cauchy). Applying the definition of a Cauchy sequence with $\varepsilon := \frac{1}{2}$, there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \frac{1}{2}$ for all $n, m \geq N$. Take $n := N$ and $m := N + 1$. Exactly one of $N, N + 1$ is odd, so one of a_N, a_{N+1} equals 1 and the other equals 0. Therefore

$$|a_N - a_{N+1}| = 1 \geq \tfrac{1}{2},$$

which contradicts $|a_N - a_{N+1}| < \frac{1}{2}$. Hence $\{a_n\}$ diverges, and in particular $\{a_n\}$ does not converge to 1. □

True; $a_n = 1$ (n odd), $a_n = 0$ (n even).

Part (b): False

Method 1: Contradiction with $\varepsilon = \frac{1}{2}|a - 1|$. Let $\{a_n\}$ be any sequence with $a_n = 1$ for infinitely many $n \in \mathbb{N}$.

Proof. Suppose, for contradiction, that $a = \lim a_n$ with $a \neq 1$. Then $|a - 1| > 0$, so $\varepsilon := \frac{1}{2}|a - 1| > 0$. By the definition of $a = \lim a_n$, there is $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$. The set $\{n \in \mathbb{N} : a_n = 1\}$ is infinite and $\{1, \ldots, N - 1\}$ is finite, so there is $n_0 \geq N$ with $a_{n_0} = 1$. Therefore

$$\begin{aligned} |a - 1| &= |a_{n_0} - a| && (a_{n_0} = 1) \\ &< \varepsilon = \tfrac{1}{2}|a - 1| && (n_0 \geq N) \end{aligned}$$

$\implies$ $\frac{1}{2}|a - 1| < 0$, which contradicts $|a - 1| \geq 0$. Hence $\{a_n\}$ has no limit $a \neq 1$. □

False; a sequence with infinitely many 1's can only converge to 1, never to another limit.

Method 2: Direct proof that $|a - 1| < \varepsilon$ for every $\varepsilon > 0$. Let $\{a_n\}$ have $a_n = 1$ for infinitely many n, and suppose $\{a_n\}$ converges, with $a = \lim a_n$. We show $a = 1$.

Proof. Let $\varepsilon > 0$. By the definition of $a = \lim a_n$, there is $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$. As in Method 1, there is $n_0 \geq N$ with $a_{n_0} = 1$. Hence

$$|1 - a| = |a_{n_0} - a| < \varepsilon.$$

Therefore $0 \leq |a - 1| < \varepsilon$ for every $\varepsilon > 0$. If $|a - 1| > 0$, then the choice $\varepsilon := |a - 1|$ gives $|a - 1| < |a - 1|$, which is false; hence $|a - 1| = 0$, i.e. $a = 1$. Thus every limit of such a sequence equals 1. □

False; a sequence with infinitely many 1's can only converge to 1, never to another limit.

Method 3: Constant subsequence and Subsequential Limits. (Uses Lecture 7: Subsequential Limits; beyond the Lectures 4–6 scope of this tutorial.)

Proof. Let $\{a_n\}$ have $a_n = 1$ for infinitely many n, and suppose $a = \lim a_n$. Let n_1 be the least $n \in \mathbb{N}$ with $a_n = 1$, and for each $k \in \mathbb{N}$ let n_{k+1} be the least $n > n_k$ with $a_n = 1$; such an n exists because infinitely many indices carry the value 1. Then $n_1 < n_2 < \cdots$, so $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, and $a_{n_k} = 1$ for every $k \in \mathbb{N}$. For every $\varepsilon > 0$ and every $k \geq 1$, $|a_{n_k} - 1| = 0 < \varepsilon$; hence $1 = \lim a_{n_k}$. By Subsequential Limits, $a = \lim a_{n_k}$. Therefore $a = 1$ (Uniqueness of Limits). □

False; a sequence with infinitely many 1's can only converge to 1, never to another limit.

Part (c): True

Construction. Put $p_0 := 0$ and $p_k := \dfrac{k(k+3)}{2}$ for $k \in \mathbb{N}$. Then for every $k \in \mathbb{N}$,

$$p_k - p_{k-1} = \frac{k(k+3) - (k-1)(k+2)}{2} = \frac{(k^2+3k) - (k^2+k-2)}{2} = k+1,$$

so $0 = p_0 < p_1 < p_2 < \cdots$, with $p_1 = 2$, $p_2 = 5$, $p_3 = 9$. Define $a_n := 0$ if $n = p_k$ for some $k \in \mathbb{N}$, and $a_n := 1$ otherwise:

$$\{a_n\} : \ 1, 0, 1, 1, 0, 1, 1, 1, 0, \ldots$$

Runs of ones. Let $n \in \mathbb{N}$. The n indices $p_{n-1}+1, \ldots, p_{n-1}+n$ lie strictly between p_{n-1} and $p_n = p_{n-1} + n + 1$. Since the p_k are strictly increasing, none of these indices equals any p_k. Hence $a_j = 1$ for $j = p_{n-1}+1, \ldots, p_{n-1}+n$, i.e. $\{a_n\}$ contains n consecutive 1's.

Two neighbouring indices. For each $N \in \mathbb{N}$, $a_{p_N} = 0$, and $p_N - 1 = p_{N-1} + N$ is the last index of the run just described (with $n = N$), so $a_{p_N - 1} = 1$. Moreover

$$(p_N - 1) - N = \frac{N^2 + 3N - 2 - 2N}{2} = \frac{(N-1)(N+2)}{2} \geq 0,$$

so both $p_N - 1 \geq N$ and $p_N \geq N$.

Method 1: Divergence from the definition with $\varepsilon = \frac{1}{2}$.

Proof. Suppose, for contradiction, that $a = \lim a_n$ for some $a \in \mathbb{R}$. Applying the definition with $\varepsilon := \frac{1}{2}$, there is $N \in \mathbb{N}$ such that $|a_n - a| < \frac{1}{2}$ for all $n \geq N$. By the paragraph above, the indices $p_N - 1$ and p_N are both $\geq N$, with $a_{p_N - 1} = 1$ and $a_{p_N} = 0$. Therefore

$$\begin{aligned} 1 &= |a_{p_N - 1} - a_{p_N}| \\ &\leq |a_{p_N - 1} - a| + |a - a_{p_N}| && \text{(Triangle Inequality)} \\ &< \tfrac{1}{2} + \tfrac{1}{2} = 1, && (p_N - 1,\ p_N \geq N) \end{aligned}$$

i.e. $1 < 1$, a contradiction. Hence $\{a_n\}$ diverges. □

True; growing runs of 1's (length n) separated by single 0's, diverging by the argument above.

Method 2: Cauchy Criterion on the neighbouring indices $p_N - 1,\ p_N$. Use the same sequence and the neighbouring indices from the construction.

Proof. Suppose, for contradiction, that $\{a_n\}$ converges. Then $\{a_n\}$ is Cauchy (Cauchy Criterion), so for $\varepsilon := \frac{1}{2}$ there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \frac{1}{2}$ for all $n, m \geq N$. Take $n := p_N - 1$ and $m := p_N$; both are $\geq N$. Then

$$|a_n - a_m| = |1 - 0| = 1 \geq \tfrac{1}{2},$$

which contradicts $|a_n - a_m| < \frac{1}{2}$. Hence $\{a_n\}$ diverges. □

True; growing runs of 1's (length n) separated by single 0's, diverging by the argument above.

Part (d): True

Method 1: Unboundedness via the Archimedean Property. Let $a_n := n$ and $b_n := -n$ for $n \in \mathbb{N}$.

Proof. Let $M > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $n > M$, so $|a_n| = |b_n| = n > M$. Hence no $M > 0$ satisfies $|a_n| \leq M$ for all n, i.e. $\{a_n\}$ and $\{b_n\}$ are unbounded. Every convergent sequence is bounded (Boundedness of Convergent Sequences); therefore $\{a_n\}$ and $\{b_n\}$ both diverge. Moreover $a_n + b_n = n - n = 0$ for every $n \in \mathbb{N}$, so for every $\varepsilon > 0$ and all $n \geq 1$, $|(a_n + b_n) - 0| = 0 < \varepsilon$. Hence $0 = \lim(a_n + b_n)$ (def. of limit). □

True; $a_n = n,\ b_n = -n,\ a_n + b_n = 0$ for all n.

Method 2: Consecutive differences and the Cauchy Criterion. Use the same $a_n = n$, $b_n = -n$.

Proof. Suppose, for contradiction, that $\{a_n\}$ converges. Then $\{a_n\}$ is Cauchy (Cauchy Criterion), so for $\varepsilon := \frac{1}{2}$ there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \frac{1}{2}$ for all $n, m \geq N$. Taking $n := N + 1$ and $m := N$ gives $|a_{N+1} - a_N| = |(N + 1) - N| = 1 \geq \frac{1}{2}$, a contradiction. Hence $\{a_n\}$ diverges. The same argument with $|b_{N+1} - b_N| = |-(N + 1) + N| = 1$ shows that $\{b_n\}$ diverges. Finally $a_n + b_n = 0$ for every n, so $|(a_n + b_n) - 0| = 0 < \varepsilon$ for every $\varepsilon > 0$ and all $n \geq 1$; hence $0 = \lim(a_n + b_n)$ (def. of limit). □

True; $a_n = n,\ b_n = -n,\ a_n + b_n = 0$ for all n.

Part (e): False

Method 1: Algebraic Limit Theorem applied to $b_n = (a_n + b_n) - a_n$.

Proof. Suppose, for contradiction, that $a = \lim a_n$, $s = \lim(a_n + b_n)$, and $\{b_n\}$ diverges. For every $n \in \mathbb{N}$,

$$b_n = (a_n + b_n) + (-1)\, a_n.$$

By the Algebraic Limit Theorem, part (1) with $c = -1$, $-a = \lim(-a_n)$. By part (2), applied to $\{a_n + b_n\}$ and $\{-a_n\}$, $s - a = \lim\big((a_n + b_n) + (-a_n)\big) = \lim b_n$. Therefore $\{b_n\}$

converges, which contradicts the assumption that $\{b_n\}$ diverges. □

False; if $\{a_n\}$ and $\{a_n + b_n\}$ converge then $\{b_n\} = \{(a_n + b_n) - a_n\}$ converges, by the Algebraic Limit Theorem.

Method 2: Direct ε-N proof that $s - a = \lim b_n$.

Proof. Let $a = \lim a_n$ and $s = \lim(a_n + b_n)$, and let $\varepsilon > 0$. By the definition of $s = \lim(a_n + b_n)$, there is $N_1 \in \mathbb{N}$ such that $|(a_n + b_n) - s| < \frac{\varepsilon}{2}$ for all $n \geq N_1$. By the definition of $a = \lim a_n$, there is $N_2 \in \mathbb{N}$ such that $|a_n - a| < \frac{\varepsilon}{2}$ for all $n \geq N_2$. Let $N := \max\{N_1, N_2\}$. For all $n \geq N$,

$$\begin{aligned} |b_n - (s - a)| &= \big|\big((a_n + b_n) - s\big) - (a_n - a)\big| \\ &\leq |(a_n + b_n) - s| + |a_n - a| && \text{(Triangle Inequality)} \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. && (n \geq N_1,\ n \geq N_2) \end{aligned}$$

Therefore $s - a = \lim b_n$ (def. of limit), so $\{b_n\}$ cannot diverge. □

False; if $\{a_n\}$ and $\{a_n + b_n\}$ converge then $\{b_n\} = \{(a_n + b_n) - a_n\}$ converges, by the Algebraic Limit Theorem.

Method 3: Cauchy Criterion.

Proof. Let $\{a_n\}$ and $\{s_n\}$, $s_n := a_n + b_n$, converge. By the Cauchy Criterion both are Cauchy. Let $\varepsilon > 0$. There is $N_1 \in \mathbb{N}$ such that $|s_n - s_m| < \frac{\varepsilon}{2}$ for all $n, m \geq N_1$, and there is $N_2 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\varepsilon}{2}$ for all $n, m \geq N_2$. Let $N := \max\{N_1, N_2\}$. For all $n, m \geq N$,

$$|b_n - b_m| = |(s_n - s_m) - (a_n - a_m)| \leq |s_n - s_m| + |a_n - a_m| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

Therefore $\{b_n\}$ is Cauchy, and hence $\{b_n\}$ converges (Cauchy Criterion). So $\{b_n\}$ cannot diverge. □

False; if $\{a_n\}$ and $\{a_n + b_n\}$ converge then $\{b_n\} = \{(a_n + b_n) - a_n\}$ converges, by the Algebraic Limit Theorem.

Part (f): True

Method 1: $b_n = 1/n$; convergence from the definition, divergence by unboundedness. Let $b_n := \dfrac{1}{n}$ for $n \in \mathbb{N}$; then $b_n \neq 0$ for every n.

Proof. Let $\varepsilon > 0$. By the Archimedean Property choose $N \in \mathbb{N}$ with $N > \dfrac{1}{\varepsilon}$. For all $n \geq N$,

$$|b_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence $0 = \lim b_n$ (def. of limit). Next, $\dfrac{1}{b_n} = n$ for every n. For every $M > 0$ the Archimedean Property gives $n \in \mathbb{N}$ with $n > M$, so $\{1/b_n\}$ is unbounded. Every convergent sequence is bounded (Boundedness of Convergent Sequences); therefore $\{1/b_n\}$ diverges. □

True; $b_n = 1/n \to 0,\ b_n \neq 0,$ but $1/b_n = n$ diverges (unbounded).

Method 2: Contradiction through the product $b_n \cdot (1/b_n) = 1$. Use the same $b_n = \dfrac{1}{n}$, with $0 = \lim b_n$ as in Method 1.

Proof. Suppose, for contradiction, that $L = \lim \dfrac{1}{b_n}$ for some $L \in \mathbb{R}$. For every $n \in \mathbb{N}$, $b_n \cdot \dfrac{1}{b_n} = 1$. By the Algebraic Limit Theorem, part (3), $\lim\left(b_n \cdot \dfrac{1}{b_n}\right) = 0 \cdot L = 0$. The constant sequence $1, 1, 1, \ldots$ also has the limit 1, because $|1 - 1| = 0 < \varepsilon$ for every $\varepsilon > 0$ and all n. By Uniqueness of Limits, $1 = 0$, a contradiction. Therefore $\{1/b_n\}$ diverges. □

True; $b_n = 1/n \to 0,\ b_n \neq 0,$ but $1/b_n = n$ diverges (unbounded).

Part (g): False

Method 1: Contradiction through the Triangle Inequality.

Proof. Suppose, for contradiction, that $\{a_n\}$ is unbounded, $\{b_n\}$ converges, and $\{a_n - b_n\}$ is bounded. Since $\{a_n - b_n\}$ is bounded, there is $M_1 > 0$ with $|a_n - b_n| \leq M_1$ for all $n \in \mathbb{N}$. Since $\{b_n\}$ converges, it is bounded (Boundedness of Convergent Sequences), so there is $M_2 > 0$ with $|b_n| \leq M_2$ for all $n \in \mathbb{N}$. Therefore, for all $n \in \mathbb{N}$,

$$\begin{aligned} |a_n| &= |(a_n - b_n) + b_n| \\ &\leq |a_n - b_n| + |b_n| && \text{(Triangle Inequality)} \\ &\leq M_1 + M_2. \end{aligned}$$

Hence $\{a_n\}$ is bounded by $M_1 + M_2 > 0$, which contradicts the assumption that $\{a_n\}$ is unbounded. □

False; $a_n = (a_n - b_n) + b_n$ is bounded whenever $\{a_n - b_n\}$ and $\{b_n\}$ are, by the triangle inequality.

Part (h): True

Method 1: Divergence of $(-1)^n$ from the definition with $\varepsilon = 1$. Let $a_n := 0$ and $b_n := (-1)^n$ for $n \in \mathbb{N}$.

Proof. Since $a_n = 0$ for every n, $|a_n - 0| = 0 < \varepsilon$ for every $\varepsilon > 0$ and all n, so $0 = \lim a_n$. Likewise $a_n b_n = 0$ for every n, so $0 = \lim a_n b_n$. Suppose, for contradiction, that $b = \lim b_n$. Applying the definition with $\varepsilon := 1$, there is $N \in \mathbb{N}$ such that $|b_n - b| < 1$ for all $n \geq N$.

Take $n_1 := 2N$ (even, so $b_{n_1} = 1$) and $n_2 := 2N + 1$ (odd, so $b_{n_2} = -1$); both are $\geq N$. Then

$$\begin{aligned} 2 &= |b_{n_1} - b_{n_2}| \\ &\leq |b_{n_1} - b| + |b - b_{n_2}| && \text{(Triangle Inequality)} \\ &< 1 + 1 = 2, && (n_1, n_2 \geq N) \end{aligned}$$

i.e. $2 < 2$, a contradiction. Hence $\{b_n\}$ diverges. □

$$\boxed{\text{True; } a_n = 0 \ (\to 0),\ b_n = (-1)^n \text{ (diverges)},\ a_n b_n = 0 \ (\to 0).}$$

Method 2: Consecutive differences and the Cauchy Criterion. Use the same $a_n = 0$, $b_n = (-1)^n$; $0 = \lim a_n$ and $0 = \lim a_n b_n$ as in Method 1.

Proof. Suppose, for contradiction, that $\{b_n\}$ converges. Then $\{b_n\}$ is Cauchy (Cauchy Criterion), so for $\varepsilon := 1$ there is $N \in \mathbb{N}$ such that $|b_n - b_m| < 1$ for all $n, m \geq N$. Taking $n := N$ and $m := N + 1$,

$$|b_N - b_{N+1}| = \left|(-1)^N\right| \, |1 - (-1)| = 2 \geq 1,$$

a contradiction. Hence $\{b_n\}$ diverges. □

$$\boxed{\text{True; } a_n = 0 \ (\to 0),\ b_n = (-1)^n \text{ (diverges)},\ a_n b_n = 0 \ (\to 0).}$$

Method 3: Two subsequences with different limits (Divergence Criterion). (Uses Lecture 7: the Divergence Criterion; beyond the Lectures 4–6 scope of this tutorial.)

Proof. The sequence $\{b_n\}$ is bounded, since $|b_n| = 1 \leq 1$ for all n. The even-indexed subsequence ($n_k = 2k$, strictly increasing) satisfies $b_{2k} = 1$ for all k, so $1 = \lim b_{2k}$. The odd-indexed subsequence ($m_k = 2k - 1$, strictly increasing) satisfies $b_{2k-1} = -1$ for all k, so $-1 = \lim b_{2k-1}$. Since $1 \neq -1$, the bounded sequence $\{b_n\}$ has two subsequences converging to different limits; therefore $\{b_n\}$ diverges (Divergence Criterion). As in Method 1, $0 = \lim a_n$ and $0 = \lim a_n b_n$. □

$$\boxed{\text{True; } a_n = 0 \ (\to 0),\ b_n = (-1)^n \text{ (diverges)},\ a_n b_n = 0 \ (\to 0).}$$

Question 2 (ε-N verification of three limits)

Problem

Verify, using the definition of limits of sequences, that the following sequences converge to the proposed limits.

(a) $\dfrac{2}{5} = \lim \dfrac{2n+1}{5n+4}$.

(b) $0 = \lim \dfrac{2n^2}{n^3+3}$.

(c) $0 = \lim \dfrac{\sin(n^2)}{\sqrt[3]{n}}$.

(Tutorial 2, Q2.)

Solution

Method 1: Definition of limit; bound $|a_n - a|$ by C/n^p and choose N by the Archimedean Property

Recall the definition: $a = \lim a_n$ means $\forall \varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon$. In each part we simplify $|a_n - a|$, bound it above by a multiple of a power of $\frac{1}{n}$, and then choose N so large that this bound is below ε.

(a) $\dfrac{2}{5} = \lim \dfrac{2n+1}{5n+4}$**.** For every $n \in \mathbb{N}$, put both terms over the common denominator $5(5n+4)$:

$$\frac{2n+1}{5n+4} - \frac{2}{5} = \frac{5(2n+1) - 2(5n+4)}{5(5n+4)} = \frac{10n+5-10n-8}{5(5n+4)} = \frac{-3}{5(5n+4)}.$$

Since $5n+4 > 5n > 0$, we have $5(5n+4) > 25n > 0$, and therefore

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \frac{3}{5(5n+4)} < \frac{3}{25n} \qquad \text{for every } n \in \mathbb{N}.$$

Proof. Let $\varepsilon > 0$. By the Archimedean Property, choose $N \in \mathbb{N}$ with $N > \dfrac{3}{25\varepsilon}$; equivalently, $\dfrac{3}{25N} < \varepsilon$. Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$\begin{aligned}
\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \frac{3}{5(5n+4)} && \text{(computation above)} \\
&< \frac{3}{25n} && (5(5n+4) > 25n) \\
&\leq \frac{3}{25N} && (n \geq N) \\
&< \varepsilon. && \left(N > \tfrac{3}{25\varepsilon}\right)
\end{aligned}$$

Therefore for all $n \geq N$, $\left|\dfrac{2n+1}{5n+4} - \dfrac{2}{5}\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\dfrac{2}{5} = \lim \dfrac{2n+1}{5n+4}$ (def. of limit). □

$$\boxed{\frac{2}{5} = \lim \frac{2n+1}{5n+4} \quad \left(\text{any } N \in \mathbb{N} \text{ with } N > \tfrac{3}{25\varepsilon}\right).}$$

(b) $0 = \lim \dfrac{2n^2}{n^3+3}$**.** For every $n \in \mathbb{N}$, $\dfrac{2n^2}{n^3+3} > 0$, so $\left|\dfrac{2n^2}{n^3+3} - 0\right| = \dfrac{2n^2}{n^3+3}$. Since $n^3 + 3 > n^3 > 0$,

$$\frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} \qquad \text{for every } n \in \mathbb{N}.$$

Proof. Let $\varepsilon > 0$. By the Archimedean Property, choose $N \in \mathbb{N}$ with $N > \dfrac{2}{\varepsilon}$; equivalently, $\dfrac{2}{N} < \varepsilon$. Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$\begin{aligned}
\left|\frac{2n^2}{n^3+3} - 0\right| &= \frac{2n^2}{n^3+3} && \text{(positive)} \\
&< \frac{2}{n} && (n^3 + 3 > n^3) \\
&\leq \frac{2}{N} && (n \geq N) \\
&< \varepsilon. && \left(N > \tfrac{2}{\varepsilon}\right)
\end{aligned}$$

Therefore for all $n \geq N$, $\left|\dfrac{2n^2}{n^3+3} - 0\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $0 = \lim \dfrac{2n^2}{n^3+3}$ (def. of limit). □

$$\boxed{0 = \lim \frac{2n^2}{n^3+3} \quad \left(\text{any } N \in \mathbb{N} \text{ with } N > \tfrac{2}{\varepsilon}\right).}$$

(c) $0 = \lim \dfrac{\sin(n^2)}{\sqrt[3]{n}}$**.** Two facts are used. First, $|\sin(n^2)| \leq 1$ for every $n \in \mathbb{N}$. Second, the cube root is strictly increasing on $(0, \infty)$: if $0 < x < y$ and $\sqrt[3]{x} \geq \sqrt[3]{y} > 0$, then cubing gives $x \geq y$, which is false; hence $0 < x < y \implies \sqrt[3]{x} < \sqrt[3]{y}$.

Proof. Let $\varepsilon > 0$. By the Archimedean Property, choose $N \in \mathbb{N}$ with $N > \dfrac{1}{\varepsilon^3}$. Since the cube root is increasing, $\sqrt[3]{N} > \sqrt[3]{1/\varepsilon^3} = \dfrac{1}{\varepsilon}$, i.e. $\dfrac{1}{\sqrt[3]{N}} < \varepsilon$. Let $n \in \mathbb{N}$ with $n \geq N$; then $\sqrt[3]{n} \geq \sqrt[3]{N} > 0$. Therefore

$$\begin{aligned}
\left|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0\right| &= \frac{|\sin(n^2)|}{\sqrt[3]{n}} \\
&\leq \frac{1}{\sqrt[3]{n}} && (|\sin(n^2)| \leq 1) \\
&\leq \frac{1}{\sqrt[3]{N}} && (\sqrt[3]{n} \geq \sqrt[3]{N}) \\
&< \varepsilon. && \left(\sqrt[3]{N} > \tfrac{1}{\varepsilon}\right)
\end{aligned}$$

Therefore for all $n \geq N$, $\left|\frac{\sin(n^2)}{\sqrt[3]{n}} - 0\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $0 = \lim \frac{\sin(n^2)}{\sqrt[3]{n}}$ (def. of limit). $\square$

$$\boxed{0 = \lim \frac{\sin(n^2)}{\sqrt[3]{n}} \quad \left(\text{any } N \in \mathbb{N} \text{ with } N > \tfrac{1}{\varepsilon^3}\right).}$$

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Method 2: Squeeze Theorem with comparison sequences

The question asks for the definition of limit; this method cites the Algebraic Limit Theorem and the Squeeze Theorem (Lecture 5) and should be used only if the course allows this interpretation.

Two null sequences. $0 = \lim \frac{1}{n}$: given $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$ (Archimedean); then for all $n \geq N$, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. Also $0 = \lim \frac{1}{\sqrt[3]{n}}$: given $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon^3}$; then for all $n \geq N$, $\frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \varepsilon$, exactly as in Method 1(c).

(a). From Method 1, $\frac{2n+1}{5n+4} - \frac{2}{5} = \frac{-3}{5(5n+4)}$ and $0 < \frac{3}{5(5n+4)} < \frac{3}{25n}$ for every $n \in \mathbb{N}$.

Proof. For every $n \in \mathbb{N}$,

$$\frac{2}{5} - \frac{3}{25} \cdot \frac{1}{n} < \frac{2}{5} - \frac{3}{5(5n+4)} = \frac{2n+1}{5n+4} < \frac{2}{5}.$$

By the Algebraic Limit Theorem, parts (1) and (2), $\lim\left(\frac{2}{5} - \frac{3}{25} \cdot \frac{1}{n}\right) = \frac{2}{5} - \frac{3}{25} \cdot 0 = \frac{2}{5}$. The constant sequence $\frac{2}{5}, \frac{2}{5}, \ldots$ has the limit $\frac{2}{5}$. Therefore $\frac{2}{5} = \lim \frac{2n+1}{5n+4}$ (Squeeze Theorem). □

$$\boxed{\frac{2}{5} = \lim \frac{2n+1}{5n+4}.}$$

(b).

Proof. For every $n \in \mathbb{N}$, $n^3 + 3 > n^3 > 0$, hence

$$0 < \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = 2 \cdot \frac{1}{n}.$$

By the Algebraic Limit Theorem, part (1) with $c = 2$, $\lim\left(2 \cdot \frac{1}{n}\right) = 2 \cdot 0 = 0$, and the constant sequence $0, 0, \ldots$ has the limit 0. Therefore $0 = \lim \frac{2n^2}{n^3+3}$ (Squeeze Theorem). □

$$\boxed{0 = \lim \frac{2n^2}{n^3+3}.}$$

(c).

Proof. For every $n \in \mathbb{N}$, $-1 \leq \sin(n^2) \leq 1$ and $\sqrt[3]{n} > 0$; dividing by $\sqrt[3]{n}$ gives

$$-\frac{1}{\sqrt[3]{n}} \leq \frac{\sin(n^2)}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}}.$$

By the two null sequences above, $0 = \lim \dfrac{1}{\sqrt[3]{n}}$, and by the Algebraic Limit Theorem, part (1) with $c = -1$, $0 = \lim\left(-\dfrac{1}{\sqrt[3]{n}}\right)$. Therefore $0 = \lim \dfrac{\sin(n^2)}{\sqrt[3]{n}}$ (Squeeze Theorem). □

$$\boxed{0 = \lim \frac{\sin(n^2)}{\sqrt[3]{n}}.}$$

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Method 3: Algebraic Limit Theorem after dividing by the dominant power; bounded times null

The question asks for the definition of limit; this method cites the Algebraic Limit Theorem (Lecture 5) and Question 3(a) and should be used only if the course allows this interpretation. The null sequences $\left\{\frac{1}{n}\right\}$ and $\left\{\frac{1}{\sqrt[3]{n}}\right\}$ are as in Method 2.

(a).

Proof. Dividing numerator and denominator by $n > 0$, for every $n \in \mathbb{N}$,

$$\frac{2n+1}{5n+4} = \frac{2+\frac{1}{n}}{5+4\cdot\frac{1}{n}} = \left(2+\frac{1}{n}\right)\cdot\frac{1}{5+4\cdot\frac{1}{n}}.$$

Since $0 = \lim\frac{1}{n}$, the Algebraic Limit Theorem gives: $\lim\left(2+\frac{1}{n}\right) = 2+0 = 2$ (part (2)); $\lim\left(5+4\cdot\frac{1}{n}\right) = 5+4\cdot 0 = 5$ (parts (1), (2)). Since $5+4\cdot\frac{1}{n} > 0$ for every n and $5 \neq 0$, part (4) gives $\lim\dfrac{1}{5+4\cdot\frac{1}{n}} = \dfrac{1}{5}$. By part (3), $\lim\dfrac{2n+1}{5n+4} = 2\cdot\dfrac{1}{5} = \dfrac{2}{5}$. □

$$\boxed{\frac{2}{5} = \lim\frac{2n+1}{5n+4}.}$$

(b).

Proof. Dividing numerator and denominator by $n^3 > 0$, for every $n \in \mathbb{N}$,

$$\frac{2n^2}{n^3+3} = \frac{2\cdot\frac{1}{n}}{1+3\cdot\left(\frac{1}{n}\right)^3} = \left(2\cdot\frac{1}{n}\right)\cdot\frac{1}{1+3\left(\frac{1}{n}\right)^3}.$$

By the Algebraic Limit Theorem: $\lim\left(\frac{1}{n}\right)^3 = 0\cdot 0\cdot 0 = 0$ (part (3) twice); $\lim\left(1+3(\frac{1}{n})^3\right) = 1+3\cdot 0 = 1$ (parts (1), (2)); since $1+3(\frac{1}{n})^3 > 0$ for every n and $1 \neq 0$, $\lim\dfrac{1}{1+3(\frac{1}{n})^3} = 1$ (part (4)); $\lim\left(2\cdot\frac{1}{n}\right) = 0$ (part (1)). By part (3), $\lim\dfrac{2n^2}{n^3+3} = 0\cdot 1 = 0$. □

$$\boxed{0 = \lim\frac{2n^2}{n^3+3}.}$$

(c).

Proof. For every $n \in \mathbb{N}$, $\dfrac{\sin(n^2)}{\sqrt[3]{n}} = \sin(n^2)\cdot\dfrac{1}{\sqrt[3]{n}}$. The sequence $\{\sin(n^2)\}$ is bounded, since $|\sin(n^2)| \leq 1$ for all n, and $0 = \lim\dfrac{1}{\sqrt[3]{n}}$. By Question 3(a) (a bounded sequence times a sequence with limit 0 has limit 0), $0 = \lim\dfrac{\sin(n^2)}{\sqrt[3]{n}}$. □

$$\boxed{0 = \lim\frac{\sin(n^2)}{\sqrt[3]{n}}.}$$

Question 3 (Bounded times null, and a warning when the limit is nonzero)

Problem

In this question, $\{a_n\}$ is a bounded sequence and $\{b_n\}$ is a convergent sequence.

(a) Prove that if $\{b_n\}$ has limit 0, then the sequence $\{a_n b_n\}$ has limit 0.

(b) Give an example to show that the sequence $\{a_n b_n\}$ may not have a limit if $\{b_n\}$ has a limit different from 0.

(Tutorial 2, Q3.)

Solution

Method 1: Squeeze Theorem (a); divergence from the definition (b)

(a).

Proof. Since $\{a_n\}$ is bounded, there is $M > 0$ such that $|a_n| \le M$ for all $n \in \mathbb{N}$. For every $n \in \mathbb{N}$,

$$\begin{aligned} & |a_n b_n| = |a_n|\,|b_n| \le M\,|b_n| \\ \implies \quad & -M\,|b_n| \le a_n b_n \le M\,|b_n|\,. \qquad (|x| \le c \iff -c \le x \le c) \end{aligned}$$

For every $n \in \mathbb{N}$, $\big||b_n| - 0\big| = |b_n - 0|$. Hence, for each $\varepsilon > 0$, any N that witnesses $0 = \lim b_n$ also witnesses $0 = \lim |b_n|$; therefore $0 = \lim |b_n|$ (def. of limit). By the Algebraic Limit Theorem, part (1) with $c = M$ and with $c = -M$,

$$\lim\big(M\,|b_n|\big) = M \cdot 0 = 0, \qquad \lim\big(-M\,|b_n|\big) = -M \cdot 0 = 0.$$

Therefore $0 = \lim a_n b_n$ (Squeeze Theorem). □

$$\boxed{0 = \lim a_n b_n \text{ whenever } \{a_n\} \text{ is bounded and } 0 = \lim b_n.}$$

(b). Let $a_n := (-1)^n$ and $b_n := 1$ for $n \in \mathbb{N}$. Then $|a_n| = 1 \le 1$ for all n, so $\{a_n\}$ is bounded; $|b_n - 1| = 0 < \varepsilon$ for every $\varepsilon > 0$ and all n, so $1 = \lim b_n$ and $1 \ne 0$; and $a_n b_n = (-1)^n$ for every n.

Proof. Suppose, for contradiction, that $L = \lim(-1)^n$ for some $L \in \mathbb{R}$. Applying the definition with $\varepsilon := 1$, there is $N \in \mathbb{N}$ such that $|(-1)^n - L| < 1$ for all $n \ge N$. Take $n_1 := 2N$ and $n_2 := 2N + 1$; both are $\ge N$, $(-1)^{n_1} = 1$ and $(-1)^{n_2} = -1$. Therefore

$$\begin{aligned} 2 = |1 - (-1)| &= \big|\big((-1)^{n_1} - L\big) + \big(L - (-1)^{n_2}\big)\big| \\ &\le |(-1)^{n_1} - L| + |L - (-1)^{n_2}| && \text{(Triangle Inequality)} \\ &< 1 + 1 = 2, && (n_1, n_2 \ge N) \end{aligned}$$

i.e. $2 < 2$, a contradiction. Hence $\{a_n b_n\} = \{(-1)^n\}$ diverges. □

$$\boxed{a_n = (-1)^n,\ b_n = 1\ (\to 1 \ne 0),\ a_n b_n = (-1)^n \text{ diverges.}}$$

Method 2: Direct ε-N (a); Cauchy Criterion (b)

(a).

Proof. Since $\{a_n\}$ is bounded, there is $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then $\dfrac{\varepsilon}{M} > 0$, so by the definition of $0 = \lim b_n$ there is $N \in \mathbb{N}$ such that $|b_n - 0| < \dfrac{\varepsilon}{M}$ for all $n \geq N$. For all $n \geq N$,

$$\begin{aligned} |a_n b_n - 0| &= |a_n|\ |b_n| \\ &\leq M\,|b_n| && (|a_n| \leq M) \\ &< M \cdot \frac{\varepsilon}{M} = \varepsilon. && (n \geq N) \end{aligned}$$

Therefore for all $n \geq N$, $|a_n b_n - 0| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $0 = \lim a_n b_n$ (def. of limit). □

$$\boxed{0 = \lim a_n b_n \text{ whenever } \{a_n\} \text{ is bounded and } 0 = \lim b_n.}$$

(b). Use the same $a_n = (-1)^n$, $b_n = 1$, so $a_n b_n = (-1)^n$.

Proof. Suppose, for contradiction, that $\{(-1)^n\}$ converges. Then it is Cauchy (Cauchy Criterion), so for $\varepsilon := 1$ there is $N \in \mathbb{N}$ such that $|(-1)^n - (-1)^m| < 1$ for all $n, m \geq N$. Taking $n := N$ and $m := N + 1$,

$$\left|(-1)^N - (-1)^{N+1}\right| = \left|(-1)^N\right|\ |1 - (-1)| = 2 \geq 1,$$

a contradiction. Hence $\{a_n b_n\} = \{(-1)^n\}$ diverges. □

$$\boxed{a_n = (-1)^n,\ b_n = 1\ (\to 1 \neq 0),\ a_n b_n = (-1)^n \text{ diverges.}}$$

Unofficial Solutions

Method 3: Contradiction from the negated definition (a); Divergence Criterion (b)

Part (b) of this method uses Lecture 7 (the Divergence Criterion), beyond the Lectures 4–6 scope of this tutorial.

(a).

Proof. Let $M > 0$ with $|a_n| \leq M$ for all $n \in \mathbb{N}$. Suppose, for contradiction, that $0 = \lim a_n b_n$ fails. Then there is $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there is $n \geq N$ with $|a_n b_n| \geq \varepsilon_0$. By the definition of $0 = \lim b_n$ with $\dfrac{\varepsilon_0}{M} > 0$, there is $N_0 \in \mathbb{N}$ such that $|b_n| < \dfrac{\varepsilon_0}{M}$ for all $n \geq N_0$. Choose $n \geq N_0$ with $|a_n b_n| \geq \varepsilon_0$. Then

$$\varepsilon_0 \leq |a_n b_n| = |a_n|\,|b_n| \leq M\,|b_n| < M \cdot \frac{\varepsilon_0}{M} = \varepsilon_0,$$

i.e. $\varepsilon_0 < \varepsilon_0$, a contradiction. Therefore $0 = \lim a_n b_n$. □

$$\boxed{0 = \lim a_n b_n \text{ whenever } \{a_n\} \text{ is bounded and } 0 = \lim b_n.}$$

(b). Use the same $a_n = (-1)^n$, $b_n = 1$, so $a_n b_n = (-1)^n$.

Proof. The sequence $c_n := (-1)^n$ is bounded, since $|c_n| = 1$ for all n. The indices $2k$ ($k \in \mathbb{N}$) are strictly increasing and $c_{2k} = 1$ for every k, so $1 = \lim c_{2k}$. The indices $2k-1$ ($k \in \mathbb{N}$) are strictly increasing and $c_{2k-1} = -1$ for every k, so $-1 = \lim c_{2k-1}$. Since $1 \neq -1$, the bounded sequence $\{c_n\}$ has two subsequences converging to different limits; therefore $\{c_n\} = \{a_n b_n\}$ diverges (Divergence Criterion). □

$$\boxed{a_n = (-1)^n,\ b_n = 1\ (\to 1 \neq 0),\ a_n b_n = (-1)^n \text{ diverges.}}$$

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Question 4 (Three ε-N proofs with the Algebraic Limit Theorem banned)

Problem

In this question, $\{a_n\}$ is a sequence of positive real numbers. Use the definition of limits of sequences to prove the following statements. **Do not use the Algebraic Limit Theorem.**

(a) If $2 = \lim a_n$, then $1 = \lim \dfrac{2a_n - 1}{3}$.

(b) If $2 = \lim a_n$, then $\sqrt{2} = \lim \sqrt{a_n}$.

(c) If $2 = \lim a_n$, then $\dfrac{1}{2} = \lim \dfrac{1}{a_n}$.

(Tutorial 2, Q4.)

Solution

Method 1: Direct ε-N estimates

Throughout, $2 = \lim a_n$ means: for every $\eta > 0$ there is $N \in \mathbb{N}$ such that $|a_n - 2| < \eta$ for all $n \geq N$. In each part we express the target distance in terms of $|a_n - 2|$ and then apply this with a suitable η.

(a) $1 = \lim \dfrac{2a_n - 1}{3}$**.** For every $n \in \mathbb{N}$,

$$\frac{2a_n - 1}{3} - 1 = \frac{2a_n - 1 - 3}{3} = \frac{2a_n - 4}{3} = \frac{2}{3}(a_n - 2),$$

and therefore

$$\left|\frac{2a_n - 1}{3} - 1\right| = \frac{2}{3}|a_n - 2|.$$

Proof. Let $\varepsilon > 0$. Then $\eta := \dfrac{3\varepsilon}{2} > 0$, so by the definition of $2 = \lim a_n$ there is $N \in \mathbb{N}$ such that $|a_n - 2| < \dfrac{3\varepsilon}{2}$ for all $n \geq N$. For all $n \geq N$,

$$\left|\frac{2a_n - 1}{3} - 1\right| = \frac{2}{3}|a_n - 2| < \frac{2}{3} \cdot \frac{3\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $1 = \lim \dfrac{2a_n - 1}{3}$ (def. of limit). □

$$\boxed{1 = \lim \frac{2a_n - 1}{3}.}$$

(b) $\sqrt{2} = \lim \sqrt{a_n}$**.** Since $a_n > 0$, $\sqrt{a_n}$ is defined and $\sqrt{a_n} > 0$, so $\sqrt{a_n} + \sqrt{2} > \sqrt{2} > 0$. Multiplying by the conjugate,

$$\left(\sqrt{a_n} - \sqrt{2}\right)\left(\sqrt{a_n} + \sqrt{2}\right) = a_n - 2 \implies \sqrt{a_n} - \sqrt{2} = \frac{a_n - 2}{\sqrt{a_n} + \sqrt{2}} \qquad (n \in \mathbb{N}).$$

Proof. Let $\varepsilon > 0$. Then $\eta := \sqrt{2}\,\varepsilon > 0$, so by the definition of $2 = \lim a_n$ there is $N \in \mathbb{N}$ such that $|a_n - 2| < \sqrt{2}\,\varepsilon$ for all $n \geq N$. For all $n \geq N$,

$$\begin{aligned}\left|\sqrt{a_n} - \sqrt{2}\right| &= \frac{|a_n - 2|}{\sqrt{a_n} + \sqrt{2}} && \text{(conjugate)}\\ &\leq \frac{|a_n - 2|}{\sqrt{2}} && (\sqrt{a_n} + \sqrt{2} > \sqrt{2})\\ &< \frac{\sqrt{2}\,\varepsilon}{\sqrt{2}} = \varepsilon. && (n \geq N)\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\sqrt{2} = \lim \sqrt{a_n}$ (def. of limit). □

$$\boxed{\sqrt{2} = \lim \sqrt{a_n}.}$$

(c) $\dfrac{1}{2} = \lim \dfrac{1}{a_n}$**.** For every $n \in \mathbb{N}$, $\dfrac{1}{a_n} - \dfrac{1}{2} = \dfrac{2 - a_n}{2a_n}$, so $\left|\dfrac{1}{a_n} - \dfrac{1}{2}\right| = \dfrac{|a_n - 2|}{2a_n}$. The denominator $2a_n$ must first be bounded away from 0.

Proof. *Step 1.* By the definition of $2 = \lim a_n$ with $\eta = 1$, there is $N_1 \in \mathbb{N}$ such that $|a_n - 2| < 1$ for all $n \geq N_1$. Hence $-1 < a_n - 2 < 1$, i.e. $1 < a_n < 3$, and therefore $2a_n > 2$ for all $n \geq N_1$.

Step 2. Let $\varepsilon > 0$. By the definition of $2 = \lim a_n$ with $\eta = 2\varepsilon > 0$, there is $N_2 \in \mathbb{N}$ such that $|a_n - 2| < 2\varepsilon$ for all $n \geq N_2$. Let $N := \max\{N_1, N_2\}$. For all $n \geq N$,

$$\begin{aligned}\left|\frac{1}{a_n} - \frac{1}{2}\right| &= \frac{|a_n - 2|}{2a_n}\\ &\leq \frac{|a_n - 2|}{2} && (2a_n > 2,\ n \geq N_1)\\ &< \frac{2\varepsilon}{2} = \varepsilon. && (n \geq N_2)\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\dfrac{1}{2} = \lim \dfrac{1}{a_n}$ (def. of limit). □

$$\boxed{\frac{1}{2} = \lim \frac{1}{a_n}.}$$

Method 2: Contradiction from the negated definition

For a sequence $\{x_n\}$ and $x \in \mathbb{R}$, "$x = \lim x_n$ fails" means: there is $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there is $n \geq N$ with $|x_n - x| \geq \varepsilon_0$. In each part we assume this and contradict $2 = \lim a_n$, using only the identities displayed in Method 1.

(a).

Proof. Suppose, for contradiction, that $1 = \lim \dfrac{2a_n - 1}{3}$ fails, with $\varepsilon_0 > 0$ as above. By the definition of $2 = \lim a_n$ with $\eta = \dfrac{3\varepsilon_0}{2} > 0$, there is $N \in \mathbb{N}$ such that $|a_n - 2| < \dfrac{3\varepsilon_0}{2}$ for all $n \geq N$. Choose $n \geq N$ with $\left|\dfrac{2a_n - 1}{3} - 1\right| \geq \varepsilon_0$. Then

$$\varepsilon_0 \leq \left|\frac{2a_n - 1}{3} - 1\right| = \frac{2}{3}|a_n - 2| < \frac{2}{3} \cdot \frac{3\varepsilon_0}{2} = \varepsilon_0,$$

i.e. $\varepsilon_0 < \varepsilon_0$, a contradiction. Therefore $1 = \lim \dfrac{2a_n - 1}{3}$. □

$$\boxed{1 = \lim \frac{2a_n - 1}{3}.}$$

(b).

Proof. Suppose, for contradiction, that $\sqrt{2} = \lim \sqrt{a_n}$ fails, with $\varepsilon_0 > 0$ as above. By the definition of $2 = \lim a_n$ with $\eta = \sqrt{2}\,\varepsilon_0 > 0$, there is $N \in \mathbb{N}$ such that $|a_n - 2| < \sqrt{2}\,\varepsilon_0$ for all $n \geq N$. Choose $n \geq N$ with $\left|\sqrt{a_n} - \sqrt{2}\right| \geq \varepsilon_0$. Then

$$\varepsilon_0 \leq \left|\sqrt{a_n} - \sqrt{2}\right| = \frac{|a_n - 2|}{\sqrt{a_n} + \sqrt{2}} \leq \frac{|a_n - 2|}{\sqrt{2}} < \frac{\sqrt{2}\,\varepsilon_0}{\sqrt{2}} = \varepsilon_0,$$

i.e. $\varepsilon_0 < \varepsilon_0$, a contradiction. Therefore $\sqrt{2} = \lim \sqrt{a_n}$. □

$$\boxed{\sqrt{2} = \lim \sqrt{a_n}.}$$

(c).

Proof. Suppose, for contradiction, that $\dfrac{1}{2} = \lim \dfrac{1}{a_n}$ fails, with $\varepsilon_0 > 0$ as above. Let $\eta := \min\{1, 2\varepsilon_0\} > 0$. By the definition of $2 = \lim a_n$, there is $N \in \mathbb{N}$ such that $|a_n - 2| < \eta$ for all $n \geq N$. Choose $n \geq N$ with $\left|\dfrac{1}{a_n} - \dfrac{1}{2}\right| \geq \varepsilon_0$. Since $|a_n - 2| < \eta \leq 1$, we get $a_n > 1$, so $2a_n > 2$. Therefore

$$\varepsilon_0 \leq \left|\frac{1}{a_n} - \frac{1}{2}\right| = \frac{|a_n - 2|}{2a_n} \leq \frac{|a_n - 2|}{2} < \frac{\eta}{2} \leq \frac{2\varepsilon_0}{2} = \varepsilon_0,$$

i.e. $\varepsilon_0 < \varepsilon_0$, a contradiction. Therefore $\dfrac{1}{2} = \lim \dfrac{1}{a_n}$. □

$$\boxed{\frac{1}{2} = \lim \frac{1}{a_n}.}$$

Method 3: The inequality $\left|\sqrt{x}-\sqrt{y}\right| \le \sqrt{|x-y|}$ (part (b))

This method applies to part (b).

Lemma. For all $x, y \ge 0$, $\left|\sqrt{x}-\sqrt{y}\right| \le \sqrt{|x-y|}$. By symmetry assume $x \ge y \ge 0$; then $\sqrt{x} \ge \sqrt{y} \ge 0$ and $\sqrt{x}\sqrt{y} \ge \sqrt{y}\sqrt{y} = y$. Hence

$$\left(\sqrt{x}-\sqrt{y}\right)^2 = x - 2\sqrt{x}\sqrt{y} + y \le x - 2y + y = x - y = |x-y|\,.$$

Both $\sqrt{x}-\sqrt{y}$ and $\sqrt{|x-y|}$ are ≥ 0, and the square root is increasing on $[0,\infty)$, so $\sqrt{x}-\sqrt{y} \le \sqrt{|x-y|}$.

Proof. Let $\varepsilon > 0$. Then $\eta := \varepsilon^2 > 0$, so by the definition of $2 = \lim a_n$ there is $N \in \mathbb{N}$ such that $|a_n - 2| < \varepsilon^2$ for all $n \ge N$. For all $n \ge N$ (with $a_n > 0$ and $2 > 0$),

$$\begin{aligned} \left|\sqrt{a_n}-\sqrt{2}\right| &\le \sqrt{|a_n-2|} && \text{(Lemma)} \\ &< \sqrt{\varepsilon^2} = \varepsilon. && (|a_n-2|<\varepsilon^2,\ \sqrt{\ } \text{ increasing}) \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\sqrt{2} = \lim \sqrt{a_n}$ (def. of limit). □

$$\boxed{\sqrt{2} = \lim \sqrt{a_n}.}$$

Unofficial QRS@NTU Community Note

Question 5 (Cauchy sequences: scalar multiples and squares, without the Cauchy Criterion)

Problem

In this question, $\{a_n\}$ is a Cauchy sequence and c is a real number.

(a) Use the definition of Cauchy sequence to show that $\{ca_n\}$ is Cauchy. **Do not use the Cauchy Criterion or the Algebraic Limit Theorem.**

(b) Use the definition of Cauchy sequence to show that $\{a_n^2\}$ is Cauchy. **Do not use the Cauchy Criterion or the Algebraic Limit Theorem.**

[Note: A proof without using the Cauchy Criterion is applicable to sequences in $\mathbb{Q}$ (or any ordered field) for which the Cauchy Criterion does not hold.] *(Tutorial 2, Q5; part (b) is also the sample midterm Q4.)*

Solution

Recall the definition: a sequence $\{x_n\}$ is Cauchy if

$$\forall \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ such that } \forall n, m \geq N,\ |x_n - x_m| < \varepsilon.$$

Methods 1 and 2 use only this definition and the Triangle Inequality, so they also work for sequences in $\mathbb{Q}$.

Method 1: Case split on c (a); global bound on $\{a_n\}$ (b)

(a) $\{ca_n\}$ is Cauchy.

Proof. *Case* $c = 0$. Then $ca_n = 0$ for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ and take $N := 1$. For all $n, m \geq 1$, $|ca_n - ca_m| = |0 - 0| = 0 < \varepsilon$. Therefore $\{ca_n\}$ is Cauchy.

Case $c \neq 0$. Then $|c| > 0$. Let $\varepsilon > 0$; then $\dfrac{\varepsilon}{|c|} > 0$. By the definition of $\{a_n\}$ being Cauchy, there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \dfrac{\varepsilon}{|c|}$ for all $n, m \geq N$. For all $n, m \geq N$,

$$|ca_n - ca_m| = |c\,(a_n - a_m)| = |c|\ |a_n - a_m| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\{ca_n\}$ is Cauchy (def. of Cauchy). The two cases cover every $c \in \mathbb{R}$. □

$$\boxed{\{ca_n\} \text{ is Cauchy, for every } c \in \mathbb{R}.}$$

(b) $\{a_n^2\}$ is Cauchy.

Proof. *Step 1:* $\{a_n\}$ *is bounded.* By the definition of $\{a_n\}$ being Cauchy with $\varepsilon = 1$, there is $N_0 \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N_0$. Taking $m = N_0$, for all $n \geq N_0$,

$$\begin{aligned} |a_n| &= |(a_n - a_{N_0}) + a_{N_0}| \\ &\leq |a_n - a_{N_0}| + |a_{N_0}| && \text{(Triangle Inequality)} \\ &< 1 + |a_{N_0}|\,. \end{aligned}$$

Let $M := \max\{|a_1|, \ldots, |a_{N_0-1}|, 1+|a_{N_0}|\}$ (the list $|a_1|, \ldots, |a_{N_0-1}|$ is empty if $N_0 = 1$). Then $M \geq 1 > 0$. For $n < N_0$, $|a_n|$ is in the list, so $|a_n| \leq M$; for $n \geq N_0$, $|a_n| < 1 + |a_{N_0}| \leq M$. Therefore $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Step 2: the Cauchy estimate. Let $\varepsilon > 0$. Then $\dfrac{\varepsilon}{2M} > 0$, so by the definition of $\{a_n\}$ being Cauchy there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \dfrac{\varepsilon}{2M}$ for all $n, m \geq N$. For all $n, m \geq N$,

$$\begin{aligned}
\left|a_n^2 - a_m^2\right| &= |a_n - a_m|\,|a_n + a_m| \\
&\leq |a_n - a_m|\left(|a_n| + |a_m|\right) && \text{(Triangle Inequality)} \\
&\leq 2M\,|a_n - a_m| && \text{(Step 1)} \\
&< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon. && (n, m \geq N)
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\{a_n^2\}$ is Cauchy (def. of Cauchy). □

$\{a_n^2\}$ is Cauchy (boundedness of $\{a_n\}$, proved directly from the Cauchy definition, is the key extra ingredient).

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Method 2: Uniform scaling by $|c|+1$ (a); tail bound through the fixed term a_{N_1} (b)

(a).

Proof. Let $c \in \mathbb{R}$ and $\varepsilon > 0$. Then $\dfrac{\varepsilon}{|c|+1} > 0$, so by the definition of $\{a_n\}$ being Cauchy there is $N \in \mathbb{N}$ such that $|a_n - a_m| < \dfrac{\varepsilon}{|c|+1}$ for all $n, m \geq N$. For all $n, m \geq N$,

$$\begin{aligned} |ca_n - ca_m| &= |c|\ |a_n - a_m| \\ &\leq |c| \cdot \frac{\varepsilon}{|c|+1} && \left(|a_n - a_m| < \tfrac{\varepsilon}{|c|+1},\ |c| \geq 0\right) \\ &< \varepsilon. && \left(\tfrac{|c|}{|c|+1} < 1\right) \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\{ca_n\}$ is Cauchy (def. of Cauchy); no case split on c is needed. □

$$\boxed{\{ca_n\} \text{ is Cauchy, for every } c \in \mathbb{R}.}$$

(b). Instead of bounding every term, bound only the terms beyond a fixed index, through the identity $a_n + a_m = (a_n - a_{N_1}) + (a_m - a_{N_1}) + 2a_{N_1}$.

Proof. By the definition of $\{a_n\}$ being Cauchy with $\varepsilon = 1$, there is $N_1 \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N_1$. Let $K := 2 + 2\,|a_{N_1}| > 0$. For all $n, m \geq N_1$,

$$\begin{aligned} |a_n + a_m| &= |(a_n - a_{N_1}) + (a_m - a_{N_1}) + 2a_{N_1}| \\ &\leq |a_n - a_{N_1}| + |a_m - a_{N_1}| + 2\,|a_{N_1}| && \text{(Triangle Inequality)} \\ &< 1 + 1 + 2\,|a_{N_1}| = K. && (n, m \geq N_1) \end{aligned}$$

Let $\varepsilon > 0$. Then $\dfrac{\varepsilon}{K} > 0$, so there is $N_2 \in \mathbb{N}$ such that $|a_n - a_m| < \dfrac{\varepsilon}{K}$ for all $n, m \geq N_2$. Let $N := \max\{N_1, N_2\}$. For all $n, m \geq N$,

$$\begin{aligned} \left|a_n^2 - a_m^2\right| &= |a_n - a_m|\ |a_n + a_m| \\ &\leq K\,|a_n - a_m| && (n, m \geq N_1) \\ &< K \cdot \frac{\varepsilon}{K} = \varepsilon. && (n, m \geq N_2) \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $\{a_n^2\}$ is Cauchy (def. of Cauchy). □

$$\boxed{\left\{a_n^2\right\} \text{ is Cauchy.}}$$

Method 3: Cauchy Criterion with the Algebraic Limit Theorem

This method uses the Cauchy Criterion and the Algebraic Limit Theorem, which the question excludes, and it is valid in $\mathbb{R}$ but not in $\mathbb{Q}$; it should be used only if the course allows this interpretation.

Proof. Since $\{a_n\}$ is a Cauchy sequence of real numbers, it converges (Cauchy Criterion); let $a := \lim a_n$. By the Algebraic Limit Theorem, part (1), $ca = \lim(ca_n)$ for every $c \in \mathbb{R}$; by part (3) applied to $\{a_n\}$ and $\{a_n\}$, $a^2 = \lim(a_n \cdot a_n) = \lim a_n^2$. Hence $\{ca_n\}$ and $\{a_n^2\}$ converge, and therefore both are Cauchy (Cauchy Criterion, convergent $\implies$ Cauchy). □

$$\boxed{\{ca_n\} \text{ is Cauchy, for every } c \in \mathbb{R}.}$$

$$\boxed{\left\{a_n^2\right\} \text{ is Cauchy.}}$$

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Question A1 (A linear recurrence $b_n = a_{n+1} + ra_n$ forces $\{a_n\}$ to converge)

Problem

Prove that for any sequence $\{a_n\}$ and any real number r with $|r| < 1$, if the sequence $\{b_n := a_{n+1} + ra_n\}$ is convergent, then the sequence $\{a_n\}$ is convergent. *(Tutorial 2, Additional Question A1.)*

Solution

Set-up (used by both methods). Let $\rho := |r|$, so $0 \le \rho < 1$, and let $b := \lim b_n$. Since $1 + r \ge 1 - |r| = 1 - \rho > 0$, the number $a := \dfrac{b}{1+r}$ is defined, and $b = (1+r)a = a + ra$. Put $e_n := a_n - a$ for $n \in \mathbb{N}$. For every $n \in \mathbb{N}$,

$$\begin{aligned} b_n - b &= (a_{n+1} + ra_n) - (a + ra) \\ &= (a_{n+1} - a) + r(a_n - a) = e_{n+1} + re_n \\ \implies \quad e_{n+1} &= (b_n - b) - re_n. \end{aligned}$$

Hence, by the Triangle Inequality, for every $n \in \mathbb{N}$,

$$|e_{n+1}| \le |b_n - b| + |r|\ |e_n| = \rho\,|e_n| + |b_n - b|\,.$$

The goal is $0 = \lim e_n$, i.e. $a = \lim a_n$.

Method 1: Inductive geometric-decay estimate with Bernoulli's Inequality

Proof. *Lemma A (Bernoulli's Inequality).* For $h > 0$ and $k \in \mathbb{N}_0$, $(1+h)^k \ge 1 + kh$. For $k = 0$ both sides equal 1. If $(1+h)^k \ge 1 + kh$ for some $k \in \mathbb{N}_0$, then, since $1 + h > 0$,

$$(1+h)^{k+1} = (1+h)^k(1+h) \ge (1+kh)(1+h) = 1 + (k+1)h + kh^2 \ge 1 + (k+1)h.$$

Hence the inequality holds for every $k \in \mathbb{N}_0$ (induction).

Lemma B ($0 = \lim_k \rho^k$). If $\rho = 0$, then $\rho^k = 0$ for every $k \in \mathbb{N}$, so $\left|\rho^k - 0\right| = 0 < \varepsilon$ for every $\varepsilon > 0$. If $0 < \rho < 1$, let $h := \dfrac{1}{\rho} - 1 > 0$, so $\rho = \dfrac{1}{1+h}$. By Lemma A, $(1+h)^k \ge 1 + kh > kh$ for every $k \in \mathbb{N}$, hence

$$0 < \rho^k = \frac{1}{(1+h)^k} < \frac{1}{kh} \qquad (k \in \mathbb{N}).$$

Let $\varepsilon > 0$. By the Archimedean Property choose $K \in \mathbb{N}$ with $K > \dfrac{1}{h\varepsilon}$. For all $k \ge K$, $\left|\rho^k - 0\right| < \dfrac{1}{kh} \le \dfrac{1}{Kh} < \varepsilon$. Therefore $0 = \lim_k \rho^k$ (def. of limit).

Step 1 (a bound along a tail). Let $\eta > 0$, and let $N_0 \in \mathbb{N}$ be such that $|b_n - b| < \eta$ for all $n \ge N_0$ (def. of $b = \lim b_n$). We claim that for all $k \in \mathbb{N}_0$,

$$|e_{N_0+k}| \le \rho^k\,|e_{N_0}| + \frac{\eta}{1-\rho}.$$

For $k = 0$: $|e_{N_0}| \le |e_{N_0}| + \dfrac{\eta}{1-\rho}$ because $\dfrac{\eta}{1-\rho} > 0$. Suppose the claim holds for some $k \in \mathbb{N}_0$. Since $N_0 + k \ge N_0$, the set-up inequality and the induction hypothesis (multiplied by $\rho \ge 0$) give

$$\begin{aligned}
|e_{N_0+k+1}| &\le \rho\,|e_{N_0+k}| + |b_{N_0+k} - b| \\
&< \rho\,|e_{N_0+k}| + \eta && (N_0 + k \ge N_0) \\
&\le \rho\Big(\rho^k\,|e_{N_0}| + \frac{\eta}{1-\rho}\Big) + \eta && (\text{induction hypothesis}) \\
&= \rho^{k+1}\,|e_{N_0}| + \eta\Big(\frac{\rho}{1-\rho} + 1\Big) \\
&= \rho^{k+1}\,|e_{N_0}| + \frac{\eta}{1-\rho}.
\end{aligned}$$

Hence the claim holds for every $k \in \mathbb{N}_0$ (induction).

Step 2 (choice of N). Let $\varepsilon > 0$. Take $\eta := \frac{1}{2}\varepsilon(1-\rho) > 0$, so $\dfrac{\eta}{1-\rho} = \dfrac{\varepsilon}{2}$, and let N_0 be as in Step 1. By Lemma B applied with $\dfrac{\varepsilon}{2\,|e_{N_0}| + 1} > 0$, choose $K \in \mathbb{N}$ with $\rho^K < \dfrac{\varepsilon}{2\,|e_{N_0}| + 1}$. Multiplying by $|e_{N_0}| \ge 0$ (which may be 0, so the first inequality is non-strict), and using $2\,|e_{N_0}| < 2\,|e_{N_0}| + 1$,

$$\rho^K\,|e_{N_0}| \le \frac{\varepsilon\,|e_{N_0}|}{2\,|e_{N_0}| + 1} < \frac{\varepsilon\,(2\,|e_{N_0}| + 1)}{2\,(2\,|e_{N_0}| + 1)} = \frac{\varepsilon}{2}.$$

Let $N := N_0 + K$, and let $n \ge N$. Put $k := n - N_0 \ge K$. Since $0 \le \rho < 1$, $\rho^k = \rho^K \rho^{k-K} \le \rho^K$. By Step 1,

$$\begin{aligned}
|a_n - a| = |e_{N_0+k}| &\le \rho^k\,|e_{N_0}| + \frac{\eta}{1-\rho} \\
&\le \rho^K\,|e_{N_0}| + \frac{\varepsilon}{2} && (\rho^k \le \rho^K) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Therefore for all $n \ge N$, $|a_n - a| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $a = \lim a_n$ (def. of limit). □

$$\boxed{\{a_n\} \text{ converges, with } \lim a_n = \frac{\lim b_n}{1+r}.}$$

Method 2: Tail suprema and the Monotone Convergence Theorem

(Uses Lecture 9: the Monotone Convergence Theorem; beyond the Lectures 4–6 scope of this tutorial.)

Proof. We use the set-up above: $0 \leq \rho < 1$, $a = \dfrac{b}{1+r}$, $e_n = a_n - a$ and $|e_{n+1}| \leq \rho\,|e_n| + |b_n - b|$ for every $n \in \mathbb{N}$.

Step 1 ($\{e_n\}$ is bounded). The sequence $\{b_n\}$ converges, so it is bounded (Boundedness of Convergent Sequences): there is $B_0 > 0$ with $|b_n| \leq B_0$ for all $n \in \mathbb{N}$. Let $B := B_0 + |b| > 0$. Then, by the Triangle Inequality, $|b_n - b| \leq |b_n| + |b| \leq B_0 + |b| = B$ for all $n \in \mathbb{N}$. Let $C := \max\left\{|e_1|\,, \dfrac{B}{1-\rho}\right\} > 0$, so $B \leq (1-\rho)C$. Then $|e_1| \leq C$, and if $|e_n| \leq C$ for some $n \in \mathbb{N}$, then

$$|e_{n+1}| \leq \rho\,|e_n| + |b_n - b| \leq \rho C + B \leq \rho C + (1-\rho)C = C.$$

Hence $|e_n| \leq C$ for every $n \in \mathbb{N}$ (induction).

Step 2 (tail suprema). For $N \in \mathbb{N}$ let $s_N := \sup\{|e_n| : n \geq N\}$ and $t_N := \sup\{|b_n - b| : n \geq N\}$; both sets are nonempty and bounded above (by C and B), so both suprema exist (AoC), with $0 \leq s_N \leq C$ and $0 \leq t_N \leq B$. Since $\{n : n \geq N+1\} \subseteq \{n : n \geq N\}$, $s_{N+1} \leq s_N$ for every N. So $\{s_N\}$ is decreasing and bounded, hence converges (MCT); let $s := \lim s_N$. Since $s_N \geq 0$ for all N, $s \geq 0$ (Order Limit Theorem).

Step 3 ($0 = \lim t_N$). Let $\varepsilon > 0$. By the definition of $b = \lim b_n$ there is $N' \in \mathbb{N}$ with $|b_n - b| < \frac{\varepsilon}{2}$ for all $n \geq N'$. For $N \geq N'$, $\frac{\varepsilon}{2}$ is an upper bound of $\{|b_n - b| : n \geq N\}$, so $0 \leq t_N \leq \frac{\varepsilon}{2} < \varepsilon$. Therefore $0 = \lim t_N$ (def. of limit).

Step 4 (a recursive bound). Fix $N \in \mathbb{N}$ and let $n \geq N+1$. Then $n - 1 \geq N$, so $|e_{n-1}| \leq s_N$ and $|b_{n-1} - b| \leq t_N$, and therefore

$$|e_n| \leq \rho\,|e_{n-1}| + |b_{n-1} - b| \leq \rho s_N + t_N.$$

Thus $\rho s_N + t_N$ is an upper bound of $\{|e_n| : n \geq N+1\}$, and hence $s_{N+1} \leq \rho s_N + t_N$ for every $N \in \mathbb{N}$.

Step 5 ($s = 0$). The sequence $\{s_{N+1}\}_N$ has limit s: for $\varepsilon > 0$, any N^* that works for $\{s_N\}$ also works for $\{s_{N+1}\}$, because $N+1 \geq N \geq N^*$. By the Algebraic Limit Theorem, $\lim(\rho s_N + t_N) = \rho s + 0 = \rho s$. By the Order Limit Theorem (part (2)) and Step 4, $s \leq \rho s$, so $(1-\rho)s \leq 0$. Since $1 - \rho > 0$, $s \leq 0$; together with $s \geq 0$, $s = 0$.

Step 6 (conclusion). For every $n \in \mathbb{N}$, $|e_n|$ belongs to $\{|e_k| : k \geq n\}$, so $0 \leq |e_n| \leq s_n$. Since $0 = \lim s_n$, the Squeeze Theorem gives $0 = \lim |e_n|$. As $\big||a_n - a| - 0\big| = |a_n - a|$ for every n, it follows that $a = \lim a_n$ (def. of limit). □

$$\boxed{\{a_n\} \text{ converges, with } \lim a_n = \frac{\lim b_n}{1+r}.}$$

Question A2 (Cesaro means of a convergent sequence)

Problem

Show that if the sequence $\{a_n\}$ has the limit a, then the sequence $\{b_n\}$ with

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$

for every $n \in \mathbb{N}$ converges to a. *(Tutorial 2, Additional Question A2.)*

Solution

For every $n \in \mathbb{N}$, $a = \dfrac{na}{n} = \dfrac{1}{n}\sum_{k=1}^{n} a$, and therefore

$$b_n - a = \frac{1}{n}\sum_{k=1}^{n} a_k - \frac{1}{n}\sum_{k=1}^{n} a = \frac{1}{n}\sum_{k=1}^{n}(a_k - a).$$

Method 1: Bounded head and $\varepsilon/2$-tail

Proof. *A uniform bound.* The sequence $\{a_n\}$ converges, so it is bounded (Boundedness of Convergent Sequences): there is $C > 0$ with $|a_n| \le C$ for all $n \in \mathbb{N}$. Let $C' := C + |a| > 0$. Then for every $k \in \mathbb{N}$, $|a_k - a| \le |a_k| + |a| \le C'$ (Triangle Inequality).

Choice of N. Let $\varepsilon > 0$. By the definition of $a = \lim a_n$ with $\frac{\varepsilon}{2}$, there is $M \in \mathbb{N}$ such that $|a_k - a| < \dfrac{\varepsilon}{2}$ for all $k \ge M$. By the Archimedean Property choose $N \in \mathbb{N}$ with $N > M$ and $N > \dfrac{2MC'}{\varepsilon}$; then $\dfrac{MC'}{N} < \dfrac{\varepsilon}{2}$.

Estimate. Let $n \ge N$; then $n > M$, so we can split the sum at M: the head $k = 1, \ldots, M$ has M terms, each at most C'; the tail $k = M+1, \ldots, n$ has $n - M \ge 1$ terms, each less than $\frac{\varepsilon}{2}$ (because $k \ge M + 1 \ge M$). Hence

$$\begin{aligned}
|b_n - a| &= \left|\frac{1}{n}\sum_{k=1}^{n}(a_k - a)\right| \\
&\le \frac{1}{n}\sum_{k=1}^{M}|a_k - a| + \frac{1}{n}\sum_{k=M+1}^{n}|a_k - a| && \text{(Triangle Inequality)} \\
&< \frac{MC'}{n} + \frac{n-M}{n}\cdot\frac{\varepsilon}{2} && \text{(head, tail bounds)} \\
&\le \frac{MC'}{N} + \frac{\varepsilon}{2} && \left(n \ge N,\ \tfrac{n-M}{n} < 1\right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. && \left(N > \tfrac{2MC'}{\varepsilon}\right)
\end{aligned}$$

Therefore for all $n \ge N$, $|b_n - a| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $a = \lim b_n$ (def. of limit). □

$$\boxed{a = \lim b_n, \quad \text{where } b_n = \frac{a_1 + \cdots + a_n}{n}.}$$

Method 2: Sandwich between two sequences whose limits follow from the Algebraic Limit Theorem

Proof. Let $\varepsilon > 0$ and put $\delta := \dfrac{\varepsilon}{2} > 0$. By the definition of $a = \lim a_n$, there is $N_0 \in \mathbb{N}$ such that $|a_k - a| < \delta$, i.e. $a - \delta < a_k < a + \delta$, for all $k \geq N_0$. Let $M := N_0 - 1 \in \mathbb{N}_0$ and $S := \sum_{k=1}^{M} a_k$ (with $S := 0$ if $M = 0$); S is a fixed real number.

Comparison sequences. For $n > M$, the tail $a_{M+1}, \ldots, a_n$ has $n - M$ terms, each in $(a - \delta, a + \delta)$, so

$$(n - M)(a - \delta) < \sum_{k=M+1}^{n} a_k < (n - M)(a + \delta).$$

Adding S and dividing by $n > 0$ gives, for every $n > M$,

$$L_n := \frac{S}{n} + \Big(1 - \frac{M}{n}\Big)(a - \delta) < b_n < \frac{S}{n} + \Big(1 - \frac{M}{n}\Big)(a + \delta) =: U_n.$$

Limits of the comparison sequences. Since $0 = \lim \frac{1}{n}$ (Archimedean), the Algebraic Limit Theorem gives $\lim \frac{S}{n} = S \cdot 0 = 0$ and $\lim\big(1 - \frac{M}{n}\big) = 1 - M \cdot 0 = 1$, hence

$$\lim L_n = 0 + 1 \cdot (a - \delta) = a - \delta, \qquad \lim U_n = 0 + 1 \cdot (a + \delta) = a + \delta.$$

Conclusion. By the definition of $a - \delta = \lim L_n$ with δ, there is $N_1 \in \mathbb{N}$ with $L_n > (a - \delta) - \delta = a - \varepsilon$ for all $n \geq N_1$. By the definition of $a + \delta = \lim U_n$ with δ, there is $N_2 \in \mathbb{N}$ with $U_n < (a + \delta) + \delta = a + \varepsilon$ for all $n \geq N_2$. Let $N := \max\{N_1, N_2, M + 1\}$. For all $n \geq N$,

$$a - \varepsilon < L_n < b_n < U_n < a + \varepsilon,$$

so $|b_n - a| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $a = \lim b_n$ (def. of limit). □

$$\boxed{a = \lim b_n, \quad \text{where } b_n = \frac{a_1 + \cdots + a_n}{n}.}$$

Question A3 (Doubly indexed arrays: iterated limits versus the double limit)

Problem

Given a doubly indexed array a_{mn} where $m, n \in \mathbb{N}$, define $a = \lim_{m,n\to\infty} a_{mn}$ to mean

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall m, n \geq N \quad |a_{mn} - a| < \varepsilon.$$

[Note: a doubly indexed array is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ where we use a_{mn} to denote $f(m, n)$.]

(a) If the array satisfies $a_{mn} = m/(m + n)$ for each $m, n \in \mathbb{N}$, what are the iterated limits $\lim_{n\to\infty}\big(\lim_{m\to\infty} a_{mn}\big)$ and $\lim_{m\to\infty}\big(\lim_{n\to\infty} a_{mn}\big)$?

(b) If the array satisfies $a_{mn} = mn/(m + n)$ for each $m, n \in \mathbb{N}$, does there exist $a \in \mathbb{R}$ for which $a = \lim_{m,n\to\infty} a_{mn}$? Do the two iterated limits exist? How do these three values compare?

(c) Prove that if $a = \lim_{m,n\to\infty} a_{mn}$ and the iterated limit $\lim_{n\to\infty}\big(\lim_{m\to\infty} a_{mn}\big)$ exists, then the iterated limit is a.

(Tutorial 2, Additional Question A3.)

Solution

For a fixed second index n, "$\lim_{m\to\infty} a_{mn}$" is the limit of the ordinary sequence $a_{1n}, a_{2n}, a_{3n}, \ldots$; for a fixed first index m, "$\lim_{n\to\infty} a_{mn}$" is the limit of $a_{m1}, a_{m2}, \ldots$. Part (b) is solved for the array as printed, $a_{mn} = mn/(m + n)$; part (b′) also treats the array $a_{mn} = mn/(m^2 + n^2)$, for which the comparison asked for in (b) is not vacuous.

Method 1: Single-variable ε-N for each inner limit; direct contradiction for the double limit

(a) $a_{mn} = m/(m + n)$**.** For all $m, n \in \mathbb{N}$,

$$\left|\frac{m}{m+n} - 1\right| = \frac{|m - (m+n)|}{m+n} = \frac{n}{m+n} < \frac{n}{m}, \qquad \left|\frac{m}{m+n} - 0\right| = \frac{m}{m+n} < \frac{m}{n}.$$

Proof. *Inner limit in m.* Fix $n \in \mathbb{N}$ and let $\varepsilon > 0$. By the Archimedean Property choose $M \in \mathbb{N}$ with $M > \dfrac{n}{\varepsilon}$. For all $m \geq M$, $|a_{mn} - 1| < \dfrac{n}{m} \leq \dfrac{n}{M} < \varepsilon$. Therefore $\lim_{m\to\infty} a_{mn} = 1$ for every $n \in \mathbb{N}$.

Inner limit in n. Fix $m \in \mathbb{N}$ and let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $N > \dfrac{m}{\varepsilon}$. For all $n \geq N$, $|a_{mn} - 0| < \dfrac{m}{n} \leq \dfrac{m}{N} < \varepsilon$. Therefore $\lim_{n\to\infty} a_{mn} = 0$ for every $m \in \mathbb{N}$.

Outer limits. The constant sequence $1, 1, \ldots$ has the limit 1 and the constant sequence $0, 0, \ldots$ has the limit 0 (each term is at distance $0 < \varepsilon$ from the constant). Hence $\lim_{n\to\infty}\big(\lim_{m\to\infty} a_{mn}\big) = \lim_{n\to\infty} 1 = 1$ and $\lim_{m\to\infty}\big(\lim_{n\to\infty} a_{mn}\big) = \lim_{m\to\infty} 0 = 0$. □

$$\boxed{\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = 1.}$$

$$\boxed{\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = 0.}$$

(b) $a_{mn} = mn/(m+n)$.

Proof: no double limit. Suppose, for contradiction, that $a = \lim_{m,n\to\infty} a_{mn}$ for some $a \in \mathbb{R}$. Applying the definition with $\varepsilon := 1$, there is $N \in \mathbb{N}$ such that $|a_{mn} - a| < 1$, and hence $a_{mn} < a + 1$, for all $m, n \geq N$. By the Archimedean Property choose $n_0 \in \mathbb{N}$ with $n_0 > \max\{N, 2|a| + 2\}$. Then $n_0 \geq N$ and

$$a_{n_0 n_0} = \frac{n_0 \cdot n_0}{n_0 + n_0} = \frac{n_0}{2} > \frac{2|a| + 2}{2} = |a| + 1 \geq a + 1,$$

which contradicts $a_{n_0 n_0} < a + 1$. Hence the double limit does not exist. □

Proof: no iterated limit. Fix $n \in \mathbb{N}$. For all $m \in \mathbb{N}$,

$$\left|\frac{mn}{m+n} - n\right| = \frac{|mn - n(m+n)|}{m+n} = \frac{n^2}{m+n} < \frac{n^2}{m}.$$

Let $\varepsilon > 0$ and choose $M \in \mathbb{N}$ with $M > \dfrac{n^2}{\varepsilon}$ (Archimedean). For all $m \geq M$, $|a_{mn} - n| < \dfrac{n^2}{m} \leq \dfrac{n^2}{M} < \varepsilon$. Therefore $\lim_{m\to\infty} a_{mn} = n$ for every n. The sequence $\{n\}$ is unbounded: for every $K > 0$ there is $n \in \mathbb{N}$ with $n > K$ (Archimedean). Every convergent sequence is bounded, so $\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = \lim_{n\to\infty} n$ does not exist. Since $a_{mn} = a_{nm}$ for all m, n, the same computation with the roles of m and n exchanged gives $\lim_{n\to\infty} a_{mn} = m$ for every m, and $\lim_{m\to\infty} m$ does not exist either. □

$$\boxed{\begin{array}{c}\text{As printed } (a_{mn} = mn/(m+n)) \text{: the double limit and both iterated limits} \\ \text{all fail to exist.}\end{array}}$$

(b′) $a_{mn} = mn/(m^2 + n^2)$. For every $n \in \mathbb{N}$,

$$a_{nn} = \frac{n^2}{n^2 + n^2} = \frac{1}{2}, \qquad a_{2n,n} = \frac{2n \cdot n}{4n^2 + n^2} = \frac{2n^2}{5n^2} = \frac{2}{5}.$$

Proof: no double limit. Suppose, for contradiction, that $a = \lim_{m,n\to\infty} a_{mn}$. Applying the definition with $\varepsilon := \frac{1}{20}$, there is $N \in \mathbb{N}$ such that $|a_{mn} - a| < \frac{1}{20}$ for all $m, n \geq N$. The index pairs (N, N) and $(2N, N)$ have both entries $\geq N$, so $\left|\frac{1}{2} - a\right| < \frac{1}{20}$ and $\left|\frac{2}{5} - a\right| < \frac{1}{20}$. Therefore

$$\begin{aligned} \tfrac{1}{10} &= \left|\tfrac{1}{2} - \tfrac{2}{5}\right| \\ &\leq \left|\tfrac{1}{2} - a\right| + \left|a - \tfrac{2}{5}\right| && \text{(Triangle Inequality)} \\ &< \tfrac{1}{20} + \tfrac{1}{20} = \tfrac{1}{10}, \end{aligned}$$

i.e. $\frac{1}{10} < \frac{1}{10}$, a contradiction. Hence the double limit does not exist. □

Proof: both iterated limits are 0. Fix $n \in \mathbb{N}$. For all $m \in \mathbb{N}$, $m^2 + n^2 \geq m^2 > 0$, so

$$0 < a_{mn} = \frac{mn}{m^2 + n^2} \leq \frac{mn}{m^2} = \frac{n}{m}.$$

Let $\varepsilon > 0$ and choose $M \in \mathbb{N}$ with $M > \dfrac{n}{\varepsilon}$ (Archimedean). For all $m \geq M$, $|a_{mn} - 0| \leq \dfrac{n}{m} \leq \dfrac{n}{M} < \varepsilon$. Therefore $\lim_{m\to\infty} a_{mn} = 0$ for every n, and the outer limit of the constant

sequence $0, 0, \ldots$ is 0: $\lim_{n\to\infty}\big(\lim_{m\to\infty} a_{mn}\big) = 0$. Since $a_{mn} = a_{nm}$, the same argument gives $\lim_{n\to\infty} a_{mn} = 0$ for every m, and $\lim_{m\to\infty}\big(\lim_{n\to\infty} a_{mn}\big) = 0$. □

$a_{mn} = mn/(m^2+n^2)$: double limit does not exist; both iterated limits equal 0.

(c).

Proof. Let $b_n := \lim_{m\to\infty} a_{mn}$; by hypothesis these inner limits exist and $\lim_{n\to\infty} b_n$ exists. Let $\varepsilon > 0$. By the definition of $a = \lim_{m,n\to\infty} a_{mn}$ with $\frac{\varepsilon}{2}$, there is $N_1 \in \mathbb{N}$ such that $|a_{mn} - a| < \frac{\varepsilon}{2}$ for all $m, n \geq N_1$.

Fix $n \geq N_1$. By the definition of $b_n = \lim_{m\to\infty} a_{mn}$ with $\frac{\varepsilon}{2}$, there is $M_n \in \mathbb{N}$ such that $|a_{mn} - b_n| < \frac{\varepsilon}{2}$ for all $m \geq M_n$. Let $m_0 := \max\{N_1, M_n\}$. Then $m_0 \geq N_1$, $n \geq N_1$ and $m_0 \geq M_n$, so

$$\begin{aligned} |b_n - a| &\leq |b_n - a_{m_0 n}| + |a_{m_0 n} - a| && \text{(Triangle Inequality)} \\ &< \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon. && (m_0 \geq M_n;\ m_0, n \geq N_1) \end{aligned}$$

Therefore $|b_n - a| < \varepsilon$ for all $n \geq N_1$. Since $\varepsilon > 0$ was arbitrary, $a = \lim_{n\to\infty} b_n$ (def. of limit), i.e. the iterated limit is a. □

If $a = \lim_{m,n\to\infty} a_{mn}$ and $\lim_{n\to\infty}\big(\lim_{m\to\infty} a_{mn}\big)$ exists, it equals a.

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Method 2: Algebraic Limit Theorem for the inner limits; diagonal sequences; Order Limit Theorem

(a).

Proof. Fix $n \in \mathbb{N}$. Dividing by $m > 0$, $a_{mn} = \dfrac{m}{m+n} = \dfrac{1}{1+n\cdot\frac{1}{m}}$ for every $m \in \mathbb{N}$. Since $0 = \lim_{m\to\infty}\frac{1}{m}$, the Algebraic Limit Theorem gives $\lim_{m\to\infty}\left(1+n\cdot\frac{1}{m}\right) = 1+n\cdot 0 = 1$; as $1+\frac{n}{m} > 0$ for all m and $1 \neq 0$, part (4) gives $\lim_{m\to\infty} a_{mn} = \frac{1}{1} = 1$.

Fix $m \in \mathbb{N}$. Dividing by $n > 0$, $a_{mn} = \dfrac{m\cdot\frac{1}{n}}{1+m\cdot\frac{1}{n}}$ for every $n \in \mathbb{N}$. By the Algebraic Limit Theorem, the numerator has limit $m\cdot 0 = 0$ and the denominator has limit $1+m\cdot 0 = 1 \neq 0$ (and is positive), so $\lim_{n\to\infty} a_{mn} = 0\cdot\frac{1}{1} = 0$.

The outer limits are limits of the constant sequences $1, 1, \ldots$ and $0, 0, \ldots$, namely 1 and 0. □

$$\boxed{\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right) = 1.}$$

$$\boxed{\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{mn}\right) = 0.}$$

(b).

Proof. *Diagonal sequence.* If $a = \lim_{m,n\to\infty} a_{mn}$, then the diagonal sequence $d_n := a_{nn}$ has limit a: for $\varepsilon > 0$, the N from the double-limit definition satisfies $|a_{nn} - a| < \varepsilon$ for all $n \geq N$, because both entries of (n, n) are $\geq N$. Here $d_n = \dfrac{n^2}{2n} = \dfrac{n}{2}$, and $\{n/2\}$ is unbounded (for every $K > 0$ the Archimedean Property gives $n > 2K$, so $\frac{n}{2} > K$). A convergent sequence is bounded, so $\{d_n\}$ diverges; hence the double limit does not exist.

Inner and outer limits. Fix $n \in \mathbb{N}$. Dividing by $m > 0$, $a_{mn} = \dfrac{n}{1+n\cdot\frac{1}{m}}$; by the Algebraic Limit Theorem the denominator has limit $1 \neq 0$, so $\lim_{m\to\infty} a_{mn} = \frac{n}{1} = n$. The sequence $\{n\}$ is unbounded, hence divergent, so $\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{mn}\right)$ does not exist. By the symmetry $a_{mn} = a_{nm}$, $\lim_{n\to\infty} a_{mn} = m$ and $\lim_{m\to\infty} m$ does not exist. □

As printed ($a_{mn} = mn/(m+n)$): the double limit and both iterated limits all fail to exist.

(b′).

Proof. *Two diagonal sequences and Uniqueness of Limits.* Suppose, for contradiction, that $a = \lim_{m,n\to\infty} a_{mn}$, and put $c_n := a_{nn}$, $d_n := a_{2n,n}$. Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ be given by the double-limit definition, so $|a_{mn} - a| < \varepsilon$ for all $m, n \geq N$. For every $n \geq N$, both (n, n) and $(2n, n)$ have entries $\geq N$, so $|c_n - a| < \varepsilon$ and $|d_n - a| < \varepsilon$. Hence $a = \lim c_n$ and $a = \lim d_n$. But $c_n = \frac{1}{2}$ and $d_n = \frac{2}{5}$ for all n, so $\frac{1}{2} = \lim c_n$ and $\frac{2}{5} = \lim d_n$. By Uniqueness of Limits, $\frac{1}{2} = a = \frac{2}{5}$, which is false. Hence the double limit does not exist.

Inner limits. Fix $n \in \mathbb{N}$. Dividing by $m^2 > 0$, $a_{mn} = \dfrac{n\cdot\frac{1}{m}}{1+n^2\cdot\left(\frac{1}{m}\right)^2}$. By the Algebraic Limit Theorem the numerator has limit $n\cdot 0 = 0$ and the denominator has limit $1+n^2\cdot 0 = 1 \neq 0$, so $\lim_{m\to\infty} a_{mn} = 0$; the outer limit of the constant sequence $0, 0, \ldots$ is 0. By the

symmetry $a_{mn} = a_{nm}$, also $\lim_{m\to\infty}\big(\lim_{n\to\infty} a_{mn}\big) = 0$. □

$a_{mn} = mn/(m^2+n^2)$: double limit does not exist; both iterated limits equal 0.

(c).

Proof. Let $b_n := \lim_{m\to\infty} a_{mn}$ and let $\varepsilon > 0$. By the definition of $a = \lim_{m,n\to\infty} a_{mn}$ with $\frac{\varepsilon}{2}$, there is $N_1 \in \mathbb{N}$ such that $a - \frac{\varepsilon}{2} < a_{mn} < a + \frac{\varepsilon}{2}$ for all $m, n \geq N_1$. Fix $n \geq N_1$. The sequence $\{a_{mn}\}_{m\in\mathbb{N}}$ converges to b_n and satisfies $a_{mn} \leq a + \frac{\varepsilon}{2}$ for all $m \geq N_1$; the constant sequence $a + \frac{\varepsilon}{2}$ has the limit $a + \frac{\varepsilon}{2}$, so the Order Limit Theorem (part (2)) gives $b_n \leq a + \frac{\varepsilon}{2}$. In the same way, $a - \frac{\varepsilon}{2} \leq a_{mn}$ for all $m \geq N_1$ gives $a - \frac{\varepsilon}{2} \leq b_n$. Therefore

$$|b_n - a| \leq \tfrac{\varepsilon}{2} < \varepsilon \qquad \text{for all } n \geq N_1.$$

Since $\varepsilon > 0$ was arbitrary, $a = \lim_{n\to\infty} b_n$ (def. of limit). □

If $a = \lim_{m,n\to\infty} a_{mn}$ and $\lim_{n\to\infty}\big(\lim_{m\to\infty} a_{mn}\big)$ exists, it equals a.

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Method Map

Q.	Core topic	Methods
1	True/false on convergence	Explicit examples with divergence from the definition; Cauchy Criterion; (b) contradiction, direct $\lvert a-1\rvert<\varepsilon$, Subsequential Limits; (e) ALT, direct ε-N, Cauchy Criterion; (f) ALT product contradiction; (g) Triangle Inequality; (h) Divergence Criterion.
2	ε-N verification of three limits	Definition with C/n^p bounds and the Archimedean Property; Squeeze Theorem with comparison sequences; ALT after dividing by the dominant power, bounded times null.
3	Bounded $\times$ null; nonzero-limit warning	Squeeze Theorem, direct ε-N, negated definition for (a); definition, Cauchy Criterion, Divergence Criterion for $(-1)^n$ in (b).
4	Three definition-only proofs	Direct ε-N (conjugate for $\sqrt{a_n}$, two-stage bound for $1/a_n$); contradiction from the negated definition; $\lvert\sqrt{x}-\sqrt{y}\rvert\le\sqrt{\lvert x-y\rvert}$ for (b).
5	Cauchy: scalar multiple, square	Case split on c and global bound M; uniform scaling by $\lvert c\rvert+1$ and tail bound through a_{N_1}; Cauchy Criterion with ALT (scope note).
A1	Recurrence $b_n=a_{n+1}+ra_n$	Induction on $\lvert e_{N_0+k}\rvert$ with Bernoulli's Inequality; tail suprema and the MCT (scope note).
A2	Cesaro means	Bounded head plus $\varepsilon/2$-tail; sandwich between bounds whose limits follow from ALT.
A3	Double vs. iterated limits	Single-variable ε-N, direct contradiction, Triangle Inequality in (c); ALT for inner limits, diagonal sequences with Uniqueness of Limits, Order Limit Theorem in (c).

Final Answers

1. (a) True, (b) False, (c) True, (d) True, (e) False, (f) True, (g) False, (h) True.
2. $\frac{2}{5}=\lim\frac{2n+1}{5n+4}$, $0=\lim\frac{2n^2}{n^3+3}$, $0=\lim\frac{\sin(n^2)}{\sqrt[3]{n}}$; any $N\in\mathbb{N}$ with $N>3/(25\varepsilon)$, $N>2/\varepsilon$, $N>1/\varepsilon^3$ respectively.
3. (a) $0=\lim a_nb_n$. (b) $a_n=(-1)^n$, $b_n=1$ gives $a_nb_n=(-1)^n$, divergent.
4. (a) $1=\lim\frac{2a_n-1}{3}$. (b) $\sqrt{2}=\lim\sqrt{a_n}$. (c) $\frac{1}{2}=\lim\frac{1}{a_n}$.
5. (a) $\{ca_n\}$ is Cauchy for every $c\in\mathbb{R}$. (b) $\{a_n^2\}$ is Cauchy.
6. $\{a_n\}$ converges, with $\lim a_n=(\lim b_n)/(1+r)$.
7. $a=\lim b_n$, where $b_n=(a_1+\cdots+a_n)/n$.

8. (a) $\lim_n(\lim_m a_{mn}) = 1$, $\lim_m(\lim_n a_{mn}) = 0$. (b) $mn/(m+n)$: double limit and both iterated limits fail to exist; $mn/(m^2+n^2)$: double limit fails to exist, both iterated limits equal 0. (c) The iterated limit, when it exists, equals a.

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