MH3100 Real Analysis I

Tutorial 3 Solutions

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Overview

- **Lecture coverage:** Lectures 7–9 — subsequences and the Theorem on Subsequential Limits (Lecture 7), the Divergence Criterion (stated in Lecture 7, completed in Lecture 8), the Bolzano-Weierstrass Theorem (Lecture 8), and the Monotone Convergence Lemma / Monotone Convergence Theorem (Lecture 9).
- **Questions:** 9 questions from Tutorial 3 (Q1–Q5, plus Additional Questions A1–A4).
- **Notation:** $\mathbb{N} = \{1, 2, 3, \ldots\}$ (so $0 \notin \mathbb{N}$). A *subsequence* of $\{a_n\}_{n=1}^{\infty}$ is a sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$, where $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers (Lecture 7, deck p. 6); it is *proper* if $n_k \neq k$ for some $k \in \mathbb{N}$. We use this course definition of "proper" throughout, in particular in Question 1(a).
- **Key facts used repeatedly:** the Divergence Criterion is stated for a *bounded* sequence $\{a_n\}$: it is divergent if and only if it has two subsequences converging to different limits (Lecture 7, deck p. 14). The Monotone Convergence Theorem is proved via Bolzano-Weierstrass together with the Monotone Convergence Lemma (Lecture 9). Every sequence is, by definition, either convergent or divergent, never both – this dichotomy is used repeatedly to turn "not divergent" into "convergent."
- **Question themes:** Q1 subsequences versus convergence of the whole sequence; Q2 sequences with prescribed subsequences; Q3 shuffling two sequences with a common limit; Q4 three overlapping subsequences; Q5 extracting a subsequence converging to 0; A1 $\liminf = \limsup$ characterises convergence; A2 MCT from the Divergence Criterion; A3 Bolzano-Weierstrass from MCT via peak terms; A4 MCT from the Axiom of Completeness.

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Question 1 (True/False – Subsequences and Convergence)

Problem

Determine if each of the following statements is true.

(a) If every proper subsequence of a sequence $\{a_n\}$ converges, then $\{a_n\}$ converges as well.

(b) If every convergent subsequence of a sequence $\{a_n\}$ converges to a, then $\{a_n\}$ converges to a as well.

(c) If a sequence $\{a_n\}$ contains a divergent subsequence, then $\{a_n\}$ diverges.

(Tutorial 3, Question 1; Lectures 7–9. Part (a) is also Sample Midterm Question 1(c).)

Solution

Part (a), Method 1: Tail shift and the definition of limit

Proof. Define $b_k := a_{k+1}$ for $k \in \mathbb{N}$; that is, $b_k = a_{n_k}$ with $n_k := k+1$. For every $k \in \mathbb{N}$, $n_{k+1} = k+2 > k+1 = n_k$. Therefore the indices are strictly increasing, and $\{b_k\}$ is a subsequence of $\{a_n\}$. Since $n_1 = 2 \neq 1$, the subsequence $\{b_k\}$ is proper. Hence, by hypothesis, $\{b_k\}$ converges; let $L := \lim_{k\to\infty} b_k$.

Let $\varepsilon > 0$ be given. Since $L = \lim b_k$, there is $K \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq K$ implies $|b_k - L| < \varepsilon$. Put $N := K+1$, and let $n \in \mathbb{N}$ with $n \geq N$. Then $k := n-1$ satisfies $k \geq N-1 = K \geq 1$, so $k \in \mathbb{N}$ and $k \geq K$. Hence

$$|a_n - L| = |a_{k+1} - L| = |b_k - L| < \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$; that is, $L = \lim a_n$ (def. of limit). Hence $\{a_n\}$ converges. □

True

Part (a), Method 2: Divergence Criterion

Proof. Define $b_k := a_{k+1}$ for $k \in \mathbb{N}$. The indices $k+1$ are strictly increasing and the first one is $2 \neq 1$, so $\{b_k\}$ is a proper subsequence of $\{a_n\}$. Hence, by hypothesis, $\{b_k\}$ converges; let $L := \lim b_k$.

Step 1: $\{a_n\}$ is bounded. Since $\{b_k\}$ converges, it is bounded (Boundedness of Convergent Sequences): there is $M_b > 0$ such that $|b_k| \leq M_b$ for all $k \in \mathbb{N}$. Put $M := \max\{|a_1|, M_b\}$. For $n = 1$, $|a_1| \leq M$. For $n \geq 2$, $n-1 \in \mathbb{N}$, so $|a_n| = |b_{n-1}| \leq M_b \leq M$. Therefore $|a_n| \leq M$ for every $n \in \mathbb{N}$.

Step 2: every convergent subsequence of $\{a_n\}$ has limit L. Let $\{a_{n_k}\}$ be a subsequence with $L' = \lim a_{n_k}$. Put $p_k := n_{k+1} - 1$ for $k \in \mathbb{N}$. By the Subsequence Index Bound, $n_{k+1} \geq k+1$, so $p_k \geq k \geq 1$ and $p_k \in \mathbb{N}$. Also $p_{k+1} = n_{k+2} - 1 > n_{k+1} - 1 = p_k$. Therefore $\{b_{p_k}\}_k$ is a subsequence of $\{b_k\}$, and

$$b_{p_k} = a_{p_k+1} = a_{n_{k+1}} \qquad \text{for every } k \in \mathbb{N}.$$

Hence $L = \lim_{k\to\infty} a_{n_{k+1}}$ (Subsequential Limits, applied to $\{b_k\}$). On the other hand, $k \mapsto k+1$ is strictly increasing, so $\left\{a_{n_{k+1}}\right\}_k$ is a subsequence of $\{a_{n_k}\}_k$; hence $L' = \lim_{k\to\infty} a_{n_{k+1}}$ (Subsequential Limits). Therefore $L' = L$ (Uniqueness of Limits).

Step 3. Suppose, for contradiction, that $\{a_n\}$ diverges. By Step 1 it is bounded, so the Divergence Criterion gives two subsequences of $\{a_n\}$ converging to limits $L_1 \neq L_2$. By Step 2, $L_1 = L$ and $L_2 = L$, so $L_1 = L_2$: a contradiction. Therefore $\{a_n\}$ is not divergent; hence $\{a_n\}$ converges. □

True

Part (a), Method 3: Cauchy Criterion

Proof. Define $b_k := a_{k+1}$ for $k \in \mathbb{N}$. The indices $k+1$ are strictly increasing and the first one is $2 \neq 1$, so $\{b_k\}$ is a proper subsequence of $\{a_n\}$; hence $\{b_k\}$ converges by hypothesis. Therefore $\{b_k\}$ is Cauchy (Cauchy Criterion).

Let $\varepsilon > 0$ be given. Since $\{b_k\}$ is Cauchy, there is $K \in \mathbb{N}$ such that for all $j, k \in \mathbb{N}$, $j, k \geq K$ implies $|b_j - b_k| < \varepsilon$. Put $N := K + 1$, and let $m, n \in \mathbb{N}$ with $m, n \geq N$. Then $m - 1 \geq K$ and $n - 1 \geq K$, so

$$|a_m - a_n| = |b_{m-1} - b_{n-1}| < \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|a_m - a_n| < \varepsilon$; that is, $\{a_n\}$ is Cauchy. Hence $\{a_n\}$ converges (Cauchy Criterion). □

True

Part (b), Method 1: Counterexample with an unbounded odd part

Counterexample. Define $a_n := n$ if n is odd and $a_n := 0$ if n is even, so that $\{a_n\}$ is $1, 0, 3, 0, 5, 0, \ldots$; take $a := 0$.

Proof. *Step 1: every convergent subsequence converges to* 0. Let $\{a_{n_k}\}$ be a subsequence with $l = \lim a_{n_k}$. Then $\{a_{n_k}\}$ is bounded (Boundedness of Convergent Sequences): there is $M > 0$ such that $|a_{n_k}| \leq M$ for all $k \in \mathbb{N}$. Let $O := \{k \in \mathbb{N} : n_k \text{ is odd}\}$, and suppose O is infinite. Then O is not bounded above: if some $B \in \mathbb{R}$ were an upper bound of O, the Archimedean Property would give $m \in \mathbb{N}$ with $m > B$, and then $O \subseteq \{1, \ldots, m\}$ would be finite. Hence M is not an upper bound of O, so there is $k \in O$ with $k > M$. For this k, n_k is odd, so $a_{n_k} = n_k \geq k > M$ (Subsequence Index Bound). This contradicts $|a_{n_k}| \leq M$. Therefore O is finite. Let $k_0 := \max O$ if $O \neq \emptyset$ and $k_0 := 0$ if $O = \emptyset$. For every $k > k_0$, n_k is even, hence $a_{n_k} = 0$. Let $\varepsilon > 0$ and put $K := k_0 + 1$. For all $k \geq K$, $|a_{n_k} - 0| = 0 < \varepsilon$. Therefore $0 = \lim a_{n_k}$ (def. of limit), and $l = 0$ (Uniqueness of Limits). Moreover, $\{a_{2k}\}$ is a subsequence with $a_{2k} = 0$ for all k, so convergent subsequences do exist.

Step 2: $\{a_n\}$ diverges. Suppose $\{a_n\}$ converges. Then it is bounded: there is $M > 0$ with $|a_n| \leq M$ for all $n \in \mathbb{N}$. By the Archimedean Property there is $m \in \mathbb{N}$ with $m > M$. Then $n := 2m - 1$ is odd and $n = 2m - 1 \geq m > M$, so $a_n = n > M$: a contradiction. Therefore $\{a_n\}$ diverges; in particular $\{a_n\}$ does not converge to $a = 0$.

Hence every convergent subsequence of $\{a_n\}$ converges to 0, but $\{a_n\}$ does not converge to 0. □

False

Part (b), Method 2: Vacuous counterexample $a_n = n$

Proof. Define $a_n := n$ for $n \in \mathbb{N}$, and take $a := 0$. Let $\{a_{n_k}\}$ be any subsequence of $\{a_n\}$, and suppose it converges. Then it is bounded: there is $M > 0$ with $|a_{n_k}| \leq M$ for all $k \in \mathbb{N}$. By the Archimedean Property there is $K \in \mathbb{N}$ with $K > M$. Then

$$a_{n_K} = n_K \geq K > M \qquad \text{(Subsequence Index Bound)},$$

which contradicts $|a_{n_K}| \leq M$. Therefore $\{a_n\}$ has no convergent subsequence. Hence the statement "every convergent subsequence of $\{a_n\}$ converges to 0" is true, since there is no convergent subsequence to check. On the other hand, $\{a_n\}$ is a subsequence of itself ($n_k = k$), so by the previous paragraph $\{a_n\}$ itself does not converge. Therefore the hypothesis holds with $a = 0$, while the conclusion fails. □

False

Part (c), Method 1: Contrapositive via Subsequential Limits

Proof. We prove the contrapositive: if $\{a_n\}$ converges, then every subsequence of $\{a_n\}$ converges. Suppose $\{a_n\}$ converges, and let $a := \lim a_n$. Let $\{a_{n_k}\}$ be any subsequence of $\{a_n\}$. By Subsequential Limits, $a = \lim a_{n_k}$; therefore $\{a_{n_k}\}$ converges. Hence a convergent sequence has no divergent subsequence. Therefore, if $\{a_n\}$ contains a divergent subsequence, then $\{a_n\}$ is not convergent, i.e. $\{a_n\}$ diverges. □

True

Part (c), Method 2: Direct proof from the negation of the definition of limit

Proof. Let $\{a_{n_k}\}$ be a divergent subsequence of $\{a_n\}$, and let $a \in \mathbb{R}$ be arbitrary. Since $\{a_{n_k}\}$ does not converge to a, the negation of the definition of $a = \lim a_{n_k}$ gives $\varepsilon_0 > 0$ such that for every $K \in \mathbb{N}$ there is $k \in \mathbb{N}$ with $k \geq K$ and $|a_{n_k} - a| \geq \varepsilon_0$. Let $N \in \mathbb{N}$. Applying this with $K := N$ gives $k \geq N$ with $|a_{n_k} - a| \geq \varepsilon_0$. Put $n := n_k$. Then $n = n_k \geq k \geq N$ (Subsequence Index Bound) and $|a_n - a| \geq \varepsilon_0$. Therefore there is $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there is $n \geq N$ with $|a_n - a| \geq \varepsilon_0$; hence a is not the limit of $\{a_n\}$. Since $a \in \mathbb{R}$ was arbitrary, $\{a_n\}$ converges to no real number; that is, $\{a_n\}$ diverges. □

True

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Question 2 (True/False – Existence of Sequences with Prescribed Subsequences)

Problem

Determine if each of the following statements is true.

(a) There is a sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(b) There is a sequence with a subsequence that is bounded but contains no subsequence that converges.

(c) There is a monotone sequence that contains two subsequences converging to different limits.

(Tutorial 3, Question 2; Lectures 7–9. Part (b) is also Sample Midterm Question 1(b).)

Solution

Part (a), Method 1: Interleaving two sequences

Construction. For $n \in \mathbb{N}$ define

$$a_{2n-1} := \frac{1}{n+1}, \qquad a_{2n} := 1 - \frac{1}{n+1},$$

so that $\{a_m\}$ is $\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \ldots$.

Proof. *No term is* 0 *or* 1. Let $n \in \mathbb{N}$. Since $n+1 \geq 2$, we have $0 < \dfrac{1}{n+1} \leq \dfrac{1}{2} < 1$. Therefore $0 < a_{2n-1} < 1$. Subtracting from 1, $0 < 1 - \dfrac{1}{n+1} < 1$. Therefore $0 < a_{2n} < 1$. Every $m \in \mathbb{N}$ equals $2n-1$ or $2n$ for some $n \in \mathbb{N}$; hence $0 < a_m < 1$ for all $m \in \mathbb{N}$, and no term equals 0 or 1.

The two subsequences. The indices $2n-1$ and $2n$ are strictly increasing in n, so $\{a_{2n-1}\}_n$ and $\{a_{2n}\}_n$ are subsequences of $\{a_m\}$. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $N > \dfrac{1}{\varepsilon}$. For all $n \geq N$,

$$|a_{2n-1} - 0| = \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore $0 = \lim_{n\to\infty} a_{2n-1}$ (def. of limit). Since $a_{2n} = 1 - a_{2n-1}$, the Algebraic Limit Theorem gives $\lim_{n\to\infty} a_{2n} = 1 - 0 = 1$. Hence $\{a_m\}$ has no term equal to 0 or 1, and it has subsequences converging to 0 and to 1. □

True

Part (a), Method 2: Enumerating the fractions j/m in $(0,1)$

Construction. List, block by block for $m = 2, 3, 4, \ldots$, the fractions $\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}$:

$$\underbrace{\tfrac{1}{2}}_{m=2},\ \underbrace{\tfrac{1}{3}, \tfrac{2}{3}}_{m=3},\ \underbrace{\tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}}_{m=4},\ \cdots$$

and call the resulting sequence $\{a_n\}$. Block m has $m-1$ terms, so blocks $2, \ldots, m$ together have

$$1 + 2 + \cdots + (m-1) = \frac{m(m-1)}{2}$$

terms. Put $k := m - 1 \in \mathbb{N}$. The last term of block $m = k + 1$, namely $\frac{k}{k+1}$, sits at position $q_k := \frac{k(k+1)}{2}$, and the first term of the same block, namely $\frac{1}{k+1}$, sits at position $p_k := \frac{k(k-1)}{2} + 1$.

Proof. *No term is* 0 *or* 1. Every term is $\frac{j}{m}$ with $1 \le j \le m-1$; since $0 < j < m$, we have $0 < \frac{j}{m} < 1$.

The two index sequences. For $k \in \mathbb{N}$, $k(k-1)$ and $k(k+1)$ are products of consecutive integers, hence even; so $p_k, q_k \in \mathbb{N}$. Moreover

$$p_{k+1} - p_k = \frac{(k+1)k}{2} - \frac{k(k-1)}{2} = k \ge 1,$$
$$q_{k+1} - q_k = \frac{(k+1)(k+2)}{2} - \frac{k(k+1)}{2} = k + 1 \ge 1.$$

Therefore $p_1 < p_2 < \cdots$ and $q_1 < q_2 < \cdots$, so $\{a_{p_k}\}$ and $\{a_{q_k}\}$ are subsequences of $\{a_n\}$, with

$$a_{p_k} = \frac{1}{k+1}, \qquad a_{q_k} = \frac{k}{k+1} = 1 - \frac{1}{k+1}.$$

Limits. Let $\varepsilon > 0$. By the Archimedean Property there is $K \in \mathbb{N}$ with $K > \dfrac{1}{\varepsilon}$. For all $k \ge K$,

$$|a_{p_k} - 0| = \frac{1}{k+1} < \frac{1}{k} \le \frac{1}{K} < \varepsilon.$$

Therefore $0 = \lim a_{p_k}$ (def. of limit), and $1 = \lim a_{q_k} = \lim\big(1 - a_{p_k}\big)$ (Algebraic Limit Theorem). Hence $\{a_n\}$ has no term equal to 0 or 1 and has subsequences converging to 0 and to 1. □

True

Part (b), Method 1: Bolzano-Weierstrass on the bounded subsequence

Proof. Let $\{a_n\}$ be any sequence and let $\{a_{n_k}\}_k$ be a subsequence of $\{a_n\}$ that is bounded. By the Bolzano-Weierstrass Theorem, $\{a_{n_k}\}_k$ has a convergent subsequence: there are $k_1 < k_2 < k_3 < \cdots$ in $\mathbb{N}$ such that $\left\{a_{n_{k_j}}\right\}_j$ converges. Let $j < j'$ in $\mathbb{N}$. Then $k_j < k_{j'}$, and since $n_1 < n_2 < \cdots$, this gives $n_{k_j} < n_{k_{j'}}$. Therefore $n_{k_1} < n_{k_2} < \cdots$, so $\left\{a_{n_{k_j}}\right\}_j$ is a subsequence of $\{a_n\}$; and since $k_1 < k_2 < \cdots$, it is also a subsequence of $\{a_{n_k}\}_k$. It is convergent. Hence every bounded subsequence contains a convergent subsequence, and no sequence as described exists. □

False

Part (b), Method 2: Monotone subsequence (Question A3) and the Monotone Convergence Theorem

Proof. Let $\{a_{n_k}\}_k$ be a bounded subsequence of a sequence $\{a_n\}$, and let $M > 0$ with $|a_{n_k}| \leq M$ for all $k \in \mathbb{N}$. Write $x_k := a_{n_k}$. By Question A3, parts (a) and (b) (whose proofs use neither Bolzano-Weierstrass nor the Monotone Convergence Theorem), $\{x_k\}$ has a monotone subsequence $\left\{x_{k_j}\right\}_j$, where $k_1 < k_2 < \cdots$. Since $\left|x_{k_j}\right| \leq M$ for all $j \in \mathbb{N}$, the sequence $\left\{x_{k_j}\right\}_j$ is monotone and bounded; therefore it converges (MCT; Question A4 proves the increasing case from the Axiom of Completeness alone). Since $j < j'$ implies $k_j < k_{j'}$, the sequence $\left\{x_{k_j}\right\}_j = \left\{a_{n_{k_j}}\right\}_j$ is a convergent subsequence of $\{a_{n_k}\}_k$; since $k_j < k_{j'}$ implies $n_{k_j} < n_{k_{j'}}$, it is also a subsequence of $\{a_n\}$. Hence every bounded subsequence contains a convergent subsequence. □

False

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Part (c), Method 1: Case split on boundedness

Proof. Let $\{a_n\}$ be monotone. We first treat the case that $\{a_n\}$ is increasing. Since $a_n \leq a_{n+1}$ for every $n \in \mathbb{N}$, induction on q gives $a_p \leq a_q$ for all $p, q \in \mathbb{N}$ with $p \leq q$.

Case 1: $\{a_n\}$ is bounded. Then $\{a_n\}$ is monotone and bounded, so it converges (MCT); let $L := \lim a_n$. By Subsequential Limits, every subsequence of $\{a_n\}$ converges to L. Therefore no two subsequences converge to different limits.

Case 2: $\{a_n\}$ is unbounded. For every n, $a_1 \leq a_n$. If some $B \in \mathbb{R}$ satisfied $a_n \leq B$ for all n, then $|a_n| \leq \max\{|a_1|, |B|\}$ for all n, contradicting unboundedness. Therefore, for every $B \in \mathbb{R}$ there is $N \in \mathbb{N}$ with $a_N > B$. Let $\{a_{n_k}\}$ be any subsequence and let $B \in \mathbb{R}$. Choose $N \in \mathbb{N}$ with $a_N > B$. For all $k \geq N$, $n_k \geq k \geq N$ (Subsequence Index Bound), so $a_{n_k} \geq a_N > B$. Hence $\{a_{n_k}\}$ is not bounded above, so it is unbounded and therefore divergent (Boundedness of Convergent Sequences). Therefore no subsequence converges, and in particular no two subsequences converge to different limits.

Decreasing case. If $\{a_n\}$ is decreasing, then $\{-a_n\}$ is increasing. If $\{a_{n_k}\} \to A$ and $\{a_{m_k}\} \to B$, then $\{-a_{n_k}\} \to -A$ and $\{-a_{m_k}\} \to -B$ (Algebraic Limit Theorem), and the increasing case gives $-A = -B$, so $A = B$.

Hence no monotone sequence has two subsequences converging to different limits. □

False

Part (c), Method 2: Monotone Convergence Lemma

Proof. Let $\{a_n\}$ be monotone, and suppose $\{a_{n_k}\}$ and $\{a_{m_k}\}$ are subsequences with $A = \lim a_{n_k}$ and $B = \lim a_{m_k}$. Since $\{a_n\}$ is monotone (in particular eventually monotone) and has the convergent subsequence $\{a_{n_k}\}$, the Monotone Convergence Lemma implies that $\{a_n\}$ converges; let $L := \lim a_n$. By Subsequential Limits, $A = L$ and $B = L$. Therefore $A = B$. Hence no monotone sequence has two subsequences converging to different limits. □

False

Question 3 (Shuffled Sequences and the Divergence Criterion)

Problem

In this question, $\{a_n\}$ and $\{b_n\}$ are two given convergent sequences, and $\{c_n\}$ denotes the "shuffled" sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots$$

where the subsequence indexed by odd numbers coincide with $\{a_n\}$, and the subsequence indexed by even numbers coincide with $\{b_n\}$.

Use the Divergence Criterion to prove that if $\{a_n\}$ and $\{b_n\}$ have the same limit, then $\{c_n\}$ is convergent. **Do not use the definition of limits of sequences.** *(Tutorial 3, Question 3; Lectures 7–9.)*

Solution

Method 1: Contrapositive of the (bounded) Divergence Criterion

Proof. Let $L := \lim a_n = \lim b_n$. By the definition of the shuffle, $c_{2m-1} = a_m$ and $c_{2m} = b_m$ for every $m \in \mathbb{N}$.

Step 1: $\{c_n\}$ is bounded. Since $\{a_n\}$ and $\{b_n\}$ converge, they are bounded (Boundedness of Convergent Sequences): there are $M_a, M_b > 0$ with $|a_m| \le M_a$ and $|b_m| \le M_b$ for all $m \in \mathbb{N}$. Put $M := \max\{M_a, M_b\}$. Let $n \in \mathbb{N}$. If n is odd, then $n = 2m - 1$ with $m \in \mathbb{N}$ and $|c_n| = |a_m| \le M_a \le M$; if n is even, then $n = 2m$ with $m \in \mathbb{N}$ and $|c_n| = |b_m| \le M_b \le M$. Therefore $|c_n| \le M$ for every $n \in \mathbb{N}$.

Step 2: every convergent subsequence of $\{c_n\}$ has limit L. Let $\{c_{n_k}\}$ be a subsequence with $L' = \lim c_{n_k}$. Let $O := \{k \in \mathbb{N} : n_k \text{ odd}\}$ and $E := \{k \in \mathbb{N} : n_k \text{ even}\}$. Then $O \cup E = \mathbb{N}$ is infinite, and the union of two finite sets is finite; therefore O or E is infinite.

Odd case (O infinite). List the elements of O as $k_1 < k_2 < k_3 < \cdots$. For each j, n_{k_j} is odd, so $n_{k_j} = 2m_j - 1$ with $m_j := \dfrac{n_{k_j} + 1}{2} \in \mathbb{N}$. Since $k_j < k_{j+1}$, we get $n_{k_j} < n_{k_{j+1}}$, hence $m_j < m_{j+1}$. Moreover

$$c_{n_{k_j}} = c_{2m_j - 1} = a_{m_j} \qquad \text{for every } j \in \mathbb{N}.$$

Since $\left\{c_{n_{k_j}}\right\}_j$ is a subsequence of $\{c_{n_k}\}_k$, we get $L' = \lim_j c_{n_{k_j}}$ (Subsequential Limits). Since $\left\{a_{m_j}\right\}_j$ is a subsequence of $\{a_m\}$, we get $L = \lim_j a_{m_j}$ (Subsequential Limits). These are the same sequence; therefore $L' = L$ (Uniqueness of Limits).

Even case (E infinite). List the elements of E as $k_1 < k_2 < \cdots$ and write $n_{k_j} = 2m_j$ with $m_j := \dfrac{n_{k_j}}{2} \in \mathbb{N}$; then $m_1 < m_2 < \cdots$ and $c_{n_{k_j}} = c_{2m_j} = b_{m_j}$. As in the odd case, $L' = \lim_j c_{n_{k_j}}$ and $L = \lim_j b_{m_j}$ (Subsequential Limits), so $L' = L$ (Uniqueness of Limits).

Step 3: conclusion. By Step 2, any two convergent subsequences of $\{c_n\}$ both have limit L, so $\{c_n\}$ has no two subsequences converging to different limits. By Step 1, $\{c_n\}$ is bounded, so the Divergence Criterion applies: a bounded sequence is divergent if and only if it has two subsequences converging to different limits. Therefore $\{c_n\}$ is not divergent; hence $\{c_n\}$ is convergent. Finally, $\{c_n\}$ is a convergent subsequence of itself ($n_k = k$), so Step 2 gives $\lim c_n = L$. □

$$\boxed{\{c_n\} \text{ converges, to } L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.}$$

Method 2: Direct ε-N argument from the definition of limit

This method should be used only if the course allows this interpretation: it uses the definition of limit, which the question excludes.

Proof. Let $L := \lim a_n = \lim b_n$, so $c_{2m-1} = a_m$ and $c_{2m} = b_m$ for every $m \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Since $L = \lim a_m$, there is $K_a \in \mathbb{N}$ such that for all $m \geq K_a$, $|a_m - L| < \varepsilon$. Since $L = \lim b_m$, there is $K_b \in \mathbb{N}$ such that for all $m \geq K_b$, $|b_m - L| < \varepsilon$. Put $N := \max\{2K_a - 1, 2K_b\}$, and let $n \in \mathbb{N}$ with $n \geq N$.

If n is odd, write $n = 2m - 1$. Then $2m - 1 \geq 2K_a - 1$, so $m \geq K_a$, and

$$|c_n - L| = |a_m - L| < \varepsilon.$$

If n is even, write $n = 2m$. Then $2m \geq 2K_b$, so $m \geq K_b$, and

$$|c_n - L| = |b_m - L| < \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|c_n - L| < \varepsilon$; that is, $L = \lim c_n$ (def. of limit). □

$$\boxed{\{c_n\} \text{ converges, to } L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.}$$

Unofficial QRS@NTU Question Bank

Question 4 (Three Overlapping Subsequences)

Problem

Prove that if $\{a_n\}$ is a sequence for which the subsequences $\{a_{2k}\}$, $\{a_{2k+1}\}$, and $\{a_{3k}\}$ are convergent, then it is convergent. *(Tutorial 3, Question 4; Lectures 7–9; builds on the shuffled-sequence technique of Question 3.)*

Solution

All three methods start from the same fact: the three given limits coincide. Let $L_1 := \lim_k a_{2k}$, $L_2 := \lim_k a_{2k+1}$ and $L_3 := \lim_k a_{3k}$, where k runs through $\mathbb{N}$.

Proof (the three limits agree). $L_1 = L_3$. For every $k \in \mathbb{N}$,

$$a_{6k} = a_{2(3k)} = a_{3(2k)}.$$

The index $j = 3k$ is strictly increasing in k, so $\{a_{6k}\}_k$ is a subsequence of $\{a_{2j}\}_j$; hence $\lim_k a_{6k} = L_1$ (Subsequential Limits). The index $j = 2k$ is strictly increasing in k, so $\{a_{6k}\}_k$ is a subsequence of $\{a_{3j}\}_j$; hence $\lim_k a_{6k} = L_3$ (Subsequential Limits). Therefore $L_1 = L_3$ (Uniqueness of Limits).

$L_2 = L_3$. For every $k \in \mathbb{N}$,

$$a_{6k+3} = a_{2(3k+1)+1} = a_{3(2k+1)}.$$

The index $j = 3k + 1 \in \mathbb{N}$ is strictly increasing in k, so $\{a_{6k+3}\}_k$ is a subsequence of $\{a_{2j+1}\}_j$; hence $\lim_k a_{6k+3} = L_2$ (Subsequential Limits). The index $j = 2k + 1 \in \mathbb{N}$ is strictly increasing in k, so $\{a_{6k+3}\}_k$ is a subsequence of $\{a_{3j}\}_j$; hence $\lim_k a_{6k+3} = L_3$ (Subsequential Limits). Therefore $L_2 = L_3$ (Uniqueness of Limits).

Hence $L_1 = L_2 = L_3$; call this common value L. □

Method 1: Overlapping the three subsequences, then adapting Question 3

Proof. Let $L = \lim a_{2k} = \lim a_{2k+1} = \lim a_{3k}$, as shown above.

Step 1: $\{a_n\}$ is bounded. Since $\{a_{2k}\}$ and $\{a_{2k+1}\}$ converge, they are bounded (Boundedness of Convergent Sequences): there are $M_1, M_2 > 0$ with $|a_{2k}| \le M_1$ and $|a_{2k+1}| \le M_2$ for all $k \in \mathbb{N}$. Put $M := \max\{M_1, M_2, |a_1|\}$. Let $n \in \mathbb{N}$. If $n = 1$, then $|a_n| \le M$. If $n \ge 2$ is even, then $n = 2k$ with $k = \frac{n}{2} \in \mathbb{N}$, so $|a_n| \le M_1 \le M$. If $n \ge 3$ is odd, then $n = 2k + 1$ with $k = \frac{n-1}{2} \in \mathbb{N}$, so $|a_n| \le M_2 \le M$. Therefore $|a_n| \le M$ for every $n \in \mathbb{N}$.

Step 2: every convergent subsequence of $\{a_n\}$ has limit L. Let $\{a_{n_k}\}$ be a subsequence with $L' = \lim a_{n_k}$. Let $E := \{k \in \mathbb{N} : n_k \text{ even}\}$ and $O := \{k \in \mathbb{N} : n_k \text{ odd and } n_k \ge 3\}$. Since $n_k \ge k$ (Subsequence Index Bound), $n_k = 1$ is possible only for $k = 1$; so $E \cup O \supseteq \{k \in \mathbb{N} : k \ge 2\}$ is infinite. The union of two finite sets is finite; therefore E or O is infinite.

Even case (E infinite). List the elements of E as $k_1 < k_2 < \cdots$ and write $n_{k_j} = 2m_j$ with $m_j := \frac{n_{k_j}}{2} \in \mathbb{N}$. Since $k_j < k_{j+1}$, we get $n_{k_j} < n_{k_{j+1}}$, hence $m_j < m_{j+1}$. So $\left\{a_{n_{k_j}}\right\}_j = \left\{a_{2m_j}\right\}_j$ is a subsequence of $\{a_{n_k}\}_k$ and of $\{a_{2m}\}_m$. Therefore $L' = \lim_j a_{2m_j} = L_1 = L$ (Subsequential Limits, Uniqueness of Limits).

Odd case (O infinite). List the elements of O as $k_1 < k_2 < \cdots$ and write $n_{k_j} = 2m_j+1$ with $m_j := \frac{n_{k_j}-1}{2} \in \mathbb{N}$ (as $n_{k_j} \geq 3$). Again $m_1 < m_2 < \cdots$, so $\left\{a_{n_{k_j}}\right\}_j = \left\{a_{2m_j+1}\right\}_j$ is a subsequence of $\{a_{n_k}\}_k$ and of $\{a_{2m+1}\}_m$. Therefore $L' = L_2 = L$ (Subsequential Limits, Uniqueness of Limits).

Step 3: conclusion. By Step 2, $\{a_n\}$ has no two subsequences converging to different limits. By Step 1, $\{a_n\}$ is bounded. Therefore, by the Divergence Criterion, $\{a_n\}$ is not divergent; hence $\{a_n\}$ is convergent. Since $\{a_n\}$ is a convergent subsequence of itself ($n_k = k$), Step 2 gives $\lim a_n = L$. □

$$\boxed{\{a_n\} \text{ converges, to } L = \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} a_{3k}.}$$

Method 2: Reduction to Question 3 and a tail shift

Proof. Let $L = \lim a_{2k} = \lim a_{2k+1} = \lim a_{3k}$, as shown above. For $k \in \mathbb{N}$ put $u_k := a_{2k}$ and $v_k := a_{2k+1}$; both $\{u_k\}$ and $\{v_k\}$ converge to L. Let $\{c_n\}$ be the shuffle $u_1, v_1, u_2, v_2, \ldots$, i.e. $c_{2k-1} = u_k$ and $c_{2k} = v_k$. Then

$$c_{2k-1} = a_{2k} = a_{(2k-1)+1}, \qquad c_{2k} = a_{2k+1} = a_{(2k)+1},$$

so $c_n = a_{n+1}$ for every $n \in \mathbb{N}$, i.e. $\{c_n\}$ is $a_2, a_3, a_4, \ldots$. By Question 3, applied to $\{u_k\}$ and $\{v_k\}$, the sequence $\{c_n\}$ converges to L; that is, $L = \lim_{n\to\infty} a_{n+1}$.

Let $\varepsilon > 0$ be given. There is $K \in \mathbb{N}$ such that for all $k \geq K$, $|a_{k+1} - L| < \varepsilon$. Put $N := K + 1$, and let $n \geq N$. Then $k := n - 1 \geq K \geq 1$, so

$$|a_n - L| = |a_{k+1} - L| < \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$; that is, $L = \lim a_n$ (def. of limit). □

$$\boxed{\{a_n\} \text{ converges, to } L = \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} a_{3k}.}$$

Method 3: Direct ε-N argument on even and odd indices

Proof. Let $L = \lim a_{2k} = \lim a_{2k+1} = \lim a_{3k}$, as shown above. Let $\varepsilon > 0$ be given. Since $L = \lim a_{2k}$, there is $K_1 \in \mathbb{N}$ such that for all $k \geq K_1$, $|a_{2k} - L| < \varepsilon$. Since $L = \lim a_{2k+1}$, there is $K_2 \in \mathbb{N}$ such that for all $k \geq K_2$, $|a_{2k+1} - L| < \varepsilon$. Put $N := \max\{2K_1, 2K_2 + 1\}$, and let $n \in \mathbb{N}$ with $n \geq N$. If n is even, then $n = 2k$ with $2k \geq 2K_1$, so $k \geq K_1$ and $|a_n - L| = |a_{2k} - L| < \varepsilon$. If n is odd, then $n \geq 2K_2 + 1 \geq 3$, so $n = 2k + 1$ with $k \in \mathbb{N}$ and $2k + 1 \geq 2K_2 + 1$; hence $k \geq K_2$ and $|a_n - L| = |a_{2k+1} - L| < \varepsilon$. Therefore, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$; that is, $L = \lim a_n$ (def. of limit). □

$$\boxed{\{a_n\} \text{ converges, to } L = \lim_{k\to\infty} a_{2k} = \lim_{k\to\infty} a_{2k+1} = \lim_{k\to\infty} a_{3k}.}$$

Question 5 (Constructing a Subsequence Converging to Zero)

Problem

Prove that if $\{a_n\}$ is a sequence such that for every positive number $\varepsilon > 0$, there are infinitely many terms satisfying $|a_n| < \varepsilon$, then it has a subsequence $\{a_{n_k}\}$ converging to 0. *(Tutorial 3, Question 5; Lectures 4–6 and 7–9.)*

Solution

Method 1: Recursive construction by Well-Ordering with $\varepsilon_k = 1/k$

Construction. For $k \in \mathbb{N}$ let

$$S_k := \left\{ n \in \mathbb{N} : |a_n| < \frac{1}{k} \right\}.$$

Applying the hypothesis with $\varepsilon = \frac{1}{k} > 0$ shows that S_k is infinite for every $k \in \mathbb{N}$. Since $S_1 \neq \emptyset$, it has a least element (Well-Ordering); put $n_1 := \min S_1$. Suppose n_k has been chosen. If $S_{k+1} \cap \{n \in \mathbb{N} : n > n_k\}$ were empty, then $S_{k+1} \subseteq \{1, \ldots, n_k\}$ would be finite, which is false. Therefore this set is nonempty, and we put

$$n_{k+1} := \min\big(S_{k+1} \cap \{n \in \mathbb{N} : n > n_k\}\big) \qquad \text{(Well-Ordering)}.$$

Then $n_{k+1} > n_k$ and $n_{k+1} \in S_{k+1}$. By induction, $n_1 < n_2 < n_3 < \cdots$ and $n_k \in S_k$, i.e. $|a_{n_k}| < \dfrac{1}{k}$, for every $k \in \mathbb{N}$. So $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$.

Proof. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $K \in \mathbb{N}$ with $K > \dfrac{1}{\varepsilon}$, i.e. $\dfrac{1}{K} < \varepsilon$. For all $k \in \mathbb{N}$ with $k \geq K$,

$$|a_{n_k} - 0| = |a_{n_k}| < \frac{1}{k} \leq \frac{1}{K} < \varepsilon.$$

Therefore, for every $\varepsilon > 0$ there is $K \in \mathbb{N}$ such that for all $k \geq K$, $|a_{n_k} - 0| < \varepsilon$; that is, $0 = \lim a_{n_k}$ (def. of limit). □

Such a subsequence $\{a_{n_k}\}$ exists, with $a_{n_k} \to 0$, e.g. via $\varepsilon_k = 1/k$.

Method 2: Unbounded index sets and the Squeeze Theorem

Proof. For $k \in \mathbb{N}$ let $S_k := \{n \in \mathbb{N} : |a_n| < 1/k\}$; by the hypothesis with $\varepsilon = 1/k$, each S_k is infinite. An infinite subset $S \subseteq \mathbb{N}$ is not bounded above: if $B \in \mathbb{R}$ were an upper bound of S, the Archimedean Property would give $m \in \mathbb{N}$ with $m > B$, and then $S \subseteq \{1, \ldots, m\}$ would be finite.

Choose any $n_1 \in S_1$. Having chosen n_k, the number n_k is not an upper bound of S_{k+1}, so there is $n_{k+1} \in S_{k+1}$ with $n_{k+1} > n_k$. This gives $n_1 < n_2 < \cdots$ with $n_k \in S_k$ for every k; hence

$$0 \le |a_{n_k}| < \frac{1}{k} \qquad \text{for every } k \in \mathbb{N}.$$

The sequence $\{1/k\}$ converges to 0: for $\varepsilon > 0$ take $K \in \mathbb{N}$ with $K > 1/\varepsilon$ (Archimedean), so $|1/k - 0| \le 1/K < \varepsilon$ for $k \ge K$. Therefore, by the Squeeze Theorem applied to $0 \le |a_{n_k}| \le 1/k$, $\lim_k |a_{n_k}| = 0$. Since $-|a_{n_k}| \le a_{n_k} \le |a_{n_k}|$ for every k, and $\lim_k \big(-|a_{n_k}|\big) = -0 = 0$ (Algebraic Limit Theorem), the Squeeze Theorem gives $0 = \lim a_{n_k}$. □

Such a subsequence $\{a_{n_k}\}$ exists, with $a_{n_k} \to 0$, e.g. via $\varepsilon_k = 1/k$.

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Question A1 (Limit Superior and Limit Inferior)

Problem

In this question, $\{a_n\}$ denotes a bounded sequence. Recall that the *limit superior* of the sequence $\{a_n\}$ is

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} \sup\{a_k : k \in \mathbb{N} \text{ and } k \geq n\}$$

and the *limit inferior* of the sequence $\{a_n\}$ is

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} \inf\{a_k : k \in \mathbb{N} \text{ and } k \geq n\}.$$

Recall also that $\liminf a_n \leq \limsup a_n$.

Prove that $\{a_n\}$ is convergent if and only if $\liminf a_n = \limsup a_n$, and that in this case $\liminf a_n = \limsup a_n = \lim a_n$. *(Tutorial 3, Additional Question 1; Lectures 2, 5, 7–9.)*

Solution

All three methods use the same notation. Let $M > 0$ satisfy $|a_n| \leq M$ for all $n \in \mathbb{N}$, and for $n \in \mathbb{N}$ let

$$T_n := \{a_k : k \in \mathbb{N} \text{ and } k \geq n\}, \qquad u_n := \sup T_n, \qquad v_n := \inf T_n.$$

Proof (the tail suprema and infima converge). Let $n \in \mathbb{N}$. The set T_n is nonempty ($a_n \in T_n$), bounded above by M and bounded below by $-M$. Therefore $u_n = \sup T_n$ and $v_n = \inf T_n$ exist (AoC, and its version for infima). Since $a_n \in T_n$, $v_n \leq a_n \leq u_n$. Since M is an upper bound and $-M$ a lower bound of T_n, $u_n \leq M$ and $v_n \geq -M$. Hence

$$-M \leq v_n \leq a_n \leq u_n \leq M \qquad \text{for every } n \in \mathbb{N}.$$

Since $T_{n+1} \subseteq T_n$, the number u_n is an upper bound of T_{n+1}, so $u_{n+1} \leq u_n$; likewise v_n is a lower bound of T_{n+1}, so $v_{n+1} \geq v_n$. Therefore $\{u_n\}$ is decreasing and bounded, and $\{v_n\}$ is increasing and bounded; hence both converge (MCT). Put $U := \lim u_n = \limsup a_n$ and $V := \lim v_n = \liminf a_n$. □

Method 1: Squeeze Theorem and bounds on the tail suprema and infima

Proof ("if"). Suppose $V = U$, and call this common value L; thus $L = \lim v_n = \lim u_n$. Since $v_n \leq a_n \leq u_n$ for every $n \in \mathbb{N}$, the Squeeze Theorem gives $L = \lim a_n$. Therefore $\{a_n\}$ converges, and $\lim a_n = L = \liminf a_n = \limsup a_n$. □

Proof ("only if"). Suppose $\{a_n\}$ converges, and let $a := \lim a_n$. Let $\varepsilon > 0$ be given. Since $a = \lim a_n$, there is $N \in \mathbb{N}$ such that for all $k \geq N$, $|a_k - a| < \frac{\varepsilon}{2}$, i.e. $a - \frac{\varepsilon}{2} < a_k < a + \frac{\varepsilon}{2}$. Let $n \geq N$. Every element of T_n is some a_k with $k \geq n \geq N$, so it lies in $(a - \frac{\varepsilon}{2}, a + \frac{\varepsilon}{2})$. Therefore $a + \frac{\varepsilon}{2}$ is an upper bound of T_n, so $u_n \leq a + \frac{\varepsilon}{2}$ (def. of sup); and $a - \frac{\varepsilon}{2}$ is a lower bound of T_n, so $v_n \geq a - \frac{\varepsilon}{2}$ (def. of inf). Together with $v_n \leq u_n$,

$$a - \frac{\varepsilon}{2} \leq v_n \leq u_n \leq a + \frac{\varepsilon}{2}.$$

Hence $|u_n - a| \leq \frac{\varepsilon}{2} < \varepsilon$ and $|v_n - a| \leq \frac{\varepsilon}{2} < \varepsilon$ for all $n \geq N$. Therefore $a = \lim u_n$ and $a = \lim v_n$ (def. of limit). By Uniqueness of Limits, $\limsup a_n = U = a$ and $\liminf a_n = V = a$. Hence $\liminf a_n = \limsup a_n = \lim a_n$. □

$$\boxed{\begin{array}{c}\{a_n\} \text{ converges} \iff \liminf a_n = \limsup a_n, \\ \text{and then } \liminf a_n = \limsup a_n = \lim a_n.\end{array}}$$

Method 2: Divergence Criterion and the Order Limit Theorem

Proof ("if"). Suppose $V = U$, and call this common value L. The sequence $\{a_n\}$ is bounded by M. Let $\{a_{n_k}\}$ be a subsequence with $L' = \lim a_{n_k}$. For every $k \in \mathbb{N}$,

$$v_{n_k} \leq a_{n_k} \leq u_{n_k}.$$

Since $\{v_{n_k}\}_k$ and $\{u_{n_k}\}_k$ are subsequences of $\{v_n\}$ and $\{u_n\}$, they converge to $V = L$ and $U = L$ (Subsequential Limits). By the Order Limit Theorem, $v_{n_k} \leq a_{n_k}$ for all k gives $L \leq L'$, and $a_{n_k} \leq u_{n_k}$ for all k gives $L' \leq L$. Therefore $L' = L$. Hence every convergent subsequence of $\{a_n\}$ has limit L, so $\{a_n\}$ has no two subsequences converging to different limits. Since $\{a_n\}$ is bounded, the Divergence Criterion shows that $\{a_n\}$ is not divergent; hence $\{a_n\}$ converges. As $\{a_n\}$ is a convergent subsequence of itself, the previous paragraph gives $\lim a_n = L = \liminf a_n = \limsup a_n$. □

Proof ("only if"). Suppose $a = \lim a_n$, and let $\varepsilon > 0$ be given. There is $N \in \mathbb{N}$ such that for all $k \geq N$, $a - \varepsilon < a_k < a + \varepsilon$. For $n \geq N$, every element of T_n lies in $(a - \varepsilon, a + \varepsilon)$; hence $u_n \leq a + \varepsilon$ (def. of sup) and $v_n \geq a - \varepsilon$ (def. of inf). The tails $\{u_n\}_{n \geq N}$ and $\{v_n\}_{n \geq N}$ converge to U and V (Tail-Shift Remark). The Order Limit Theorem, applied to these tails and the constant sequences $a + \varepsilon$ and $a - \varepsilon$, gives

$$U \leq a + \varepsilon, \qquad V \geq a - \varepsilon.$$

Since also $V \leq U$, we obtain $a - \varepsilon \leq V \leq U \leq a + \varepsilon$, and this holds for every $\varepsilon > 0$. If $U \neq a$, then $\varepsilon := \frac{1}{2}|U - a| > 0$ would give $|U - a| \leq \frac{1}{2}|U - a|$, which is false; therefore $U = a$. In the same way $V = a$. Hence $\liminf a_n = \limsup a_n = a = \lim a_n$. □

$$\boxed{\begin{array}{c}\{a_n\} \text{ converges} \iff \liminf a_n = \limsup a_n, \\ \text{and then } \liminf a_n = \limsup a_n = \lim a_n.\end{array}}$$

Method 3: Subsequences realising $\limsup$ and $\liminf$

Proof ("if"). Suppose $V = U =: L$, and let $\varepsilon > 0$ be given. Since $L = \lim u_n$, there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|u_n - L| < \varepsilon$. Since $L = \lim v_n$, there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|v_n - L| < \varepsilon$. Put $N := \max\{N_1, N_2\}$. For all $n \geq N$,

$$L - \varepsilon < v_n \leq a_n \leq u_n < L + \varepsilon,$$

so $|a_n - L| < \varepsilon$. Therefore $L = \lim a_n$ (def. of limit), and $\lim a_n = \liminf a_n = \limsup a_n$. □

Proof ("only if"). *Step 1: a subsequence converging to U.* By the ε-Criterion for Suprema applied to $u_1 = \sup T_1$ with $\varepsilon = 1$, there is $k_1 \geq 1$ with $a_{k_1} > u_1 - 1$. Suppose k_j has been chosen. By the ε-Criterion for Suprema applied to $u_{k_j+1} = \sup T_{k_j+1}$ with $\varepsilon = \frac{1}{j+1}$, there is $k_{j+1} \geq k_j + 1$ with $a_{k_{j+1}} > u_{k_j+1} - \frac{1}{j+1}$. Then $k_{j+1} > k_j$, and $a_{k_{j+1}} \in T_{k_j+1}$ gives $a_{k_{j+1}} \leq u_{k_j+1}$. Hence, for every $j \in \mathbb{N}$,

$$u_{k_j+1} - \frac{1}{j+1} < a_{k_{j+1}} \leq u_{k_j+1}.$$

The indices $k_j + 1$ are strictly increasing, so $\left\{u_{k_j+1}\right\}_j$ is a subsequence of $\{u_n\}$ and converges to U (Subsequential Limits). Since $\frac{1}{j+1} \to 0$, the Algebraic Limit Theorem gives $u_{k_j+1} - \frac{1}{j+1} \to U$. By the Squeeze Theorem, $\lim_j a_{k_{j+1}} = U$, and $\left\{a_{k_{j+1}}\right\}_j$ is a subsequence of $\{a_n\}$.

Step 2: a subsequence converging to V. In the same way, the ε-Criterion for Infima gives $h_1 \geq 1$ with $a_{h_1} < v_1 + 1$ and, recursively, $h_{j+1} \geq h_j + 1$ with $a_{h_{j+1}} < v_{h_j+1} + \frac{1}{j+1}$. Since $a_{h_{j+1}} \in T_{h_j+1}$,

$$v_{h_j+1} \leq a_{h_{j+1}} < v_{h_j+1} + \frac{1}{j+1} \qquad \text{for every } j \in \mathbb{N},$$

and the Squeeze Theorem (with Subsequential Limits and the Algebraic Limit Theorem as in Step 1) gives $\lim_j a_{h_{j+1}} = V$.

Step 3. Suppose $a = \lim a_n$. By Subsequential Limits, $a = \lim_j a_{k_{j+1}}$ and $a = \lim_j a_{h_{j+1}}$. By Steps 1–2 and Uniqueness of Limits, $U = a$ and $V = a$. Hence $\liminf a_n = \limsup a_n = \lim a_n$. □

$$\boxed{\begin{array}{c}\{a_n\} \text{ converges } \iff \liminf a_n = \limsup a_n, \\ \text{and then } \liminf a_n = \limsup a_n = \lim a_n.\end{array}}$$

Question A2 (The Monotone Convergence Theorem via the Divergence Criterion)

Problem

Another way to prove the Monotone Convergence Theorem is via the Divergence Criterion.

(a) Prove that if $\{a_{n_k}\}$ and $\{a_{m_k}\}$ are two convergent subsequences of a monotone sequence $\{a_n\}$, then $\lim_{k\to\infty} a_{n_k} \geq \lim_{k\to\infty} a_{m_k}$.

(b) Hence use the Divergence Criterion to deduce that if, in addition, $\{a_n\}$ is bounded, then it is convergent.

(Tutorial 3, Additional Question 2; Lectures 7 and 8. Do not use the Monotone Convergence Theorem.)

Solution

Method 1: Leapfrogging one subsequence past the other

Part (a).

Proof. Let $A := \lim a_{n_k}$ and $B := \lim a_{m_k}$. First suppose $\{a_n\}$ is increasing. Since $a_n \leq a_{n+1}$ for every n, induction gives $a_p \leq a_q$ for all $p, q \in \mathbb{N}$ with $p \leq q$. Put $i(0) := 0$ and, for $k \in \mathbb{N}$, $i(k) := \max\{m_k, i(k-1)+1\}$. Then $i(k) \geq i(k-1)+1 > i(k-1)$, so $i(1) < i(2) < \cdots$ are natural numbers, and $i(k) \geq m_k$. By the Subsequence Index Bound, $n_{i(k)} \geq i(k) \geq m_k$; therefore, since $\{a_n\}$ is increasing,

$$a_{n_{i(k)}} \geq a_{m_k} \qquad \text{for every } k \in \mathbb{N}.$$

Since $\{i(k)\}$ is strictly increasing, $\left\{a_{n_{i(k)}}\right\}_k$ is a subsequence of $\{a_{n_k}\}_k$; hence $A = \lim_k a_{n_{i(k)}}$ (Subsequential Limits). By the Order Limit Theorem, $A \geq B$. Exchanging the roles of $\{a_{n_k}\}$ and $\{a_{m_k}\}$ in this argument gives $B \geq A$; therefore $A = B$.

Now suppose $\{a_n\}$ is decreasing. Then $\{-a_n\}$ is increasing, and $\{-a_{n_k}\} \to -A$, $\{-a_{m_k}\} \to -B$ (Algebraic Limit Theorem). By the increasing case, $-A = -B$; therefore $A = B$. Hence, in both cases, $\lim a_{n_k} = \lim a_{m_k}$; in particular $\lim a_{n_k} \geq \lim a_{m_k}$. □

Part (b).

Proof. Let $\{a_n\}$ be monotone and bounded, and suppose, for contradiction, that $\{a_n\}$ diverges. Since $\{a_n\}$ is bounded and divergent, the Divergence Criterion gives two subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ with $\lim a_{n_k} \neq \lim a_{m_k}$. But $\{a_n\}$ is monotone, so part (a) gives $\lim a_{n_k} = \lim a_{m_k}$: a contradiction. Therefore $\{a_n\}$ is not divergent; hence $\{a_n\}$ is convergent. □

(a) $\lim a_{n_k} \geq \lim a_{m_k}$, with equality by the labelling symmetry.
(b) monotone + bounded $\implies$ convergent.

Method 2: Contradiction from the definition of limit

Part (a).

Proof. Let $A := \lim a_{n_k}$ and $B := \lim a_{m_k}$. First suppose $\{a_n\}$ is increasing, so that $a_p \le a_q$ whenever $p \le q$. Suppose, for contradiction, that $A < B$, and put $\varepsilon := \dfrac{B-A}{2} > 0$; then $A + \varepsilon = B - \varepsilon = \dfrac{A+B}{2}$. Since $B = \lim a_{m_k}$, there is $K \in \mathbb{N}$ such that for all $k \ge K$, $|a_{m_k} - B| < \varepsilon$; in particular

$$a_{m_K} > B - \varepsilon = \frac{A+B}{2}.$$

Since $A = \lim a_{n_j}$, there is $J \in \mathbb{N}$ such that for all $j \ge J$, $\left|a_{n_j} - A\right| < \varepsilon$, so $a_{n_j} < A + \varepsilon = \dfrac{A+B}{2}$. Let $j := \max\{J, m_K\}$. Then $n_j \ge j \ge m_K$ (Subsequence Index Bound), so, since $\{a_n\}$ is increasing,

$$\frac{A+B}{2} > a_{n_j} \ge a_{m_K} > \frac{A+B}{2},$$

which is impossible. Therefore $A \ge B$. Exchanging the roles of the two subsequences gives $B \ge A$; therefore $A = B$. If $\{a_n\}$ is decreasing, apply the increasing case to $\{-a_n\}$, whose subsequences $\{-a_{n_k}\}$, $\{-a_{m_k}\}$ converge to $-A$, $-B$ (Algebraic Limit Theorem); this gives $-A = -B$, so $A = B$. Hence $\lim a_{n_k} = \lim a_{m_k}$; in particular $\lim a_{n_k} \ge \lim a_{m_k}$. □

Part (b).

Proof (contrapositive of the Divergence Criterion). Let $\{a_n\}$ be monotone and bounded. By part (a), any two convergent subsequences of $\{a_n\}$ have the same limit; therefore $\{a_n\}$ does not have two subsequences converging to different limits. Since $\{a_n\}$ is bounded, the Divergence Criterion states that $\{a_n\}$ is divergent if and only if it has two such subsequences. Therefore $\{a_n\}$ is not divergent; hence $\{a_n\}$ is convergent. □

> (a) $\lim a_{n_k} \ge \lim a_{m_k}$, with equality by the labelling symmetry.
> (b) monotone + bounded $\implies$ convergent.

Method 3: Monotone Convergence Lemma

This method should be used only if the course allows this interpretation (uses the Monotone Convergence Lemma, Lecture 9).

Part (a).

Proof. Let $A := \lim a_{n_k}$ and $B := \lim a_{m_k}$. Since $\{a_n\}$ is monotone (in particular eventually monotone) and has the convergent subsequence $\{a_{n_k}\}$, the Monotone Convergence Lemma implies that $\{a_n\}$ converges; let $L := \lim a_n$. By Subsequential Limits, $A = L$ and $B = L$. Therefore $A = B$; in particular $\lim a_{n_k} \geq \lim a_{m_k}$. □

Part (b).

Proof. Let $\{a_n\}$ be monotone and bounded, and suppose, for contradiction, that $\{a_n\}$ diverges. By the Divergence Criterion, $\{a_n\}$ has subsequences $\{a_{n_k}\}$, $\{a_{m_k}\}$ with $\lim a_{n_k} \neq \lim a_{m_k}$; by part (a) these limits are equal: a contradiction. Therefore $\{a_n\}$ is convergent. □

(a) $\lim a_{n_k} \geq \lim a_{m_k}$, with equality by the labelling symmetry.
(b) monotone + bounded $\implies$ convergent.

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Question A3 (The Bolzano-Weierstrass Theorem via Peak Terms)

Problem

Another way to prove the Bolzano-Weierstrass Theorem is via the Monotone Convergence Theorem. A useful notion for this is a *peak term*: a term a_m of a sequence $\{a_n\}$ is called a peak term if it is an upper bound for $\{a_n : n \in \mathbb{N} \text{ and } n \geq m\}$.

(a) Show that if a sequence $\{a_n\}$ has infinitely many peak terms, then the peak terms form a decreasing subsequence.

(b) Show that if a sequence $\{a_n\}$ does not have infinitely many peak terms, then it has an increasing subsequence.

The above show that every sequence $\{a_n\}$ contains a monotone subsequence. Use this, and the Monotone Convergence Theorem, to deduce the Bolzano-Weierstrass Theorem.

[Remark: The above two questions, together with the proof of the Divergence Criterion via the Bolzano-Weierstrass Theorem, show that the Divergence Criterion, the Bolzano-Weierstrass Theorem, and the Monotone Convergence Theorem are equivalent to each other for ordered fields.] *(Tutorial 3, Additional Question 3; Lectures 8 and 9. Do not use the Bolzano-Weierstrass Theorem.)*

Solution

Method 1: Peak terms, then the Monotone Convergence Theorem

Part (a).

Proof. Let $P := \{m \in \mathbb{N} : a_m \text{ is a peak term}\}$, and suppose P is infinite. Put $m_1 := \min P$ (Well-Ordering). Having chosen $m_j \in P$, the set $P \cap \{m \in \mathbb{N} : m > m_j\}$ is nonempty (otherwise $P \subseteq \{1, \ldots, m_j\}$ would be finite), so put $m_{j+1} := \min\big(P \cap \{m > m_j\}\big)$. This lists the peak-term indices as $m_1 < m_2 < m_3 < \cdots$, so $\big\{a_{m_j}\big\}_j$ is a subsequence of $\{a_n\}$. Let $j \in \mathbb{N}$. Since a_{m_j} is a peak term, $a_{m_j} \geq a_n$ for every $n \geq m_j$. Since $m_{j+1} > m_j$, this gives $a_{m_j} \geq a_{m_{j+1}}$. Therefore $a_{m_1} \geq a_{m_2} \geq a_{m_3} \geq \cdots$; that is, the peak terms form a decreasing subsequence. □

Part (b).

Proof. Suppose P is finite. Let $M_0 := \max P$ if $P \neq \emptyset$, and $M_0 := 0$ if $P = \emptyset$. Then no $m > M_0$ belongs to P. Let $m \in \mathbb{N}$ with $m > M_0$. Since a_m is not a peak term, a_m is not an upper bound of $\{a_n : n \geq m\}$; hence there is $n \geq m$ with $a_n > a_m$. Since $a_m > a_m$ is false, $n \neq m$, and therefore $n > m$. So: *for every $m > M_0$ there is $n > m$ with $a_n > a_m$.* Put $n_1 := M_0 + 1$. Having chosen $n_k > M_0$, apply the statement in italics with $m = n_k$ to choose $n_{k+1} > n_k$ with $a_{n_{k+1}} > a_{n_k}$; then $n_{k+1} > n_k > M_0$, so the recursion continues. Therefore $n_1 < n_2 < n_3 < \cdots$ and $a_{n_1} < a_{n_2} < a_{n_3} < \cdots$; that is, $\{a_{n_k}\}$ is an increasing subsequence of $\{a_n\}$. □

Deducing Bolzano-Weierstrass.

Proof. Let $\{a_n\}$ be any sequence. Either P is infinite, and part (a) gives a decreasing subsequence, or P is finite, and part (b) gives an increasing subsequence. Therefore every sequence has a monotone subsequence. Now let $\{a_n\}$ be bounded, with $|a_n| \leq M$ for all $n \in \mathbb{N}$, and let $\{a_{n_k}\}$ be a monotone subsequence. Then $|a_{n_k}| \leq M$ for all $k \in \mathbb{N}$. Therefore $\{a_{n_k}\}$ is monotone and bounded, so it converges (MCT). Here the MCT must be the version proved without Bolzano-Weierstrass: if $\{a_{n_k}\}$ is increasing, Question A4 (Axiom of Completeness only) gives its convergence; if it is decreasing, Question A4 applied to the increasing bounded sequence $\{-a_{n_k}\}$ gives a limit ℓ, and then $a_{n_k} \to -\ell$ (Algebraic Limit Theorem). Therefore the deduction is not circular. Hence every bounded sequence has a convergent subsequence; this is the Bolzano-Weierstrass Theorem. □

Every sequence has a monotone subsequence;
bounded + monotone $\xRightarrow{\text{MCT}}$ convergent, proving Bolzano-Weierstrass.

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Question A4 (The Monotone Convergence Theorem via the Axiom of Completeness)

Problem

Yet another way to prove the Monotone Convergence Theorem is directly from the Axiom of Completeness. To this end, relate a sequence $\{a_n\}$ with the set $A := \{a_n : n \in \mathbb{N}\}$. Prove that if $\{a_n\}$ is increasing and bounded, then the supremum a of the set A is the limit of $\{a_n\}$. *(Tutorial 3, Additional Question 4; Lectures 2 and 9. Use only the Axiom of Completeness (and results proved directly from it); do not use the Bolzano-Weierstrass Theorem or the Monotone Convergence Theorem.)*

Solution

Method 1: The ε-Criterion for Suprema

Proof. The set $A := \{a_n : n \in \mathbb{N}\}$ is nonempty ($a_1 \in A$) and bounded above (since $\{a_n\}$ is bounded). Therefore $a := \sup A$ exists (AoC). Since $a_n \leq a_{n+1}$ for every n, induction gives $a_N \leq a_n$ whenever $N \leq n$. Let $\varepsilon > 0$ be given. Since $a = \sup A$, the ε-Criterion for Suprema gives an element of A greater than $a - \varepsilon$; that is, there is $N \in \mathbb{N}$ with $a - \varepsilon < a_N$. Let $n \in \mathbb{N}$ with $n \geq N$. Since $\{a_n\}$ is increasing, $a_N \leq a_n$. Since a is an upper bound of A and $a_n \in A$, $a_n \leq a$. Hence

$$a - \varepsilon < a_N \leq a_n \leq a < a + \varepsilon.$$

Therefore $|a_n - a| < \varepsilon$ for all $n \geq N$. Hence, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$; that is, $a = \lim a_n$ (def. of limit). □

$$\boxed{\{a_n\} \text{ increasing and bounded } \implies \{a_n\} \to \sup\{a_n : n \in \mathbb{N}\}.}$$

Method 2: Contradiction via the least-upper-bound property

Proof. The set $A := \{a_n : n \in \mathbb{N}\}$ is nonempty ($a_1 \in A$) and bounded above, so $a := \sup A$ exists (AoC). Since $a_n \leq a_{n+1}$ for every n, induction gives $a_m \leq a_n$ whenever $m \leq n$. Suppose, for contradiction, that a is not the limit of $\{a_n\}$. By the negation of the definition of limit, there is $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there is $n \geq N$ with $|a_n - a| \geq \varepsilon_0$. For every n, $a_n \leq a$ (as a is an upper bound of A), so $|a_n - a| = a - a_n$; hence for every $N \in \mathbb{N}$ there is $n \geq N$ with $a_n \leq a - \varepsilon_0$.

We claim $a - \varepsilon_0$ is an upper bound of A. Let $m \in \mathbb{N}$. Applying the previous sentence with $N := m$ gives $n \geq m$ with $a_n \leq a - \varepsilon_0$; since $\{a_n\}$ is increasing, $a_m \leq a_n \leq a - \varepsilon_0$. Therefore $a - \varepsilon_0$ is an upper bound of A with $a - \varepsilon_0 < a$. This contradicts the definition of $a = \sup A$ as the least upper bound of A. Hence $a = \lim a_n$. □

$$\boxed{\{a_n\} \text{ increasing and bounded } \implies \{a_n\} \to \sup\{a_n : n \in \mathbb{N}\}.}$$

Method 3: Cauchy Criterion and the Order Limit Theorem

Note: the Cauchy Criterion (Lecture 6) is proved from the Nested Interval Property, a consequence of the Axiom of Completeness, without Bolzano-Weierstrass or the Monotone Convergence Theorem.

Proof. Let $M > 0$ with $|a_n| \leq M$ for all $n \in \mathbb{N}$; since $\{a_n\}$ is increasing, $a_m \leq a_n$ whenever $m \leq n$.

Step 1: $\{a_n\}$ is Cauchy. Suppose not. Then there is $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there are $m, n \geq N$ with $|a_m - a_n| \geq \varepsilon_0$. Let $N \in \mathbb{N}$ and take such m, n; by relabelling, $n \leq m$. Then $a_N \leq a_n \leq a_m$, so

$$a_m - a_N \geq a_m - a_n = |a_m - a_n| \geq \varepsilon_0.$$

Hence for every $N \in \mathbb{N}$ there is $m \geq N$ with $a_m \geq a_N + \varepsilon_0$. Put $p_1 := 1$ and, having chosen p_j, apply the previous sentence with $N := p_j$ to choose $p_{j+1} \geq p_j$ with $a_{p_{j+1}} \geq a_{p_j} + \varepsilon_0$. By induction on j (for $j = 1$ there is equality, and $a_{p_{j+1}} \geq a_{p_j} + \varepsilon_0 \geq a_1 + j\varepsilon_0$),

$$a_{p_j} \geq a_1 + (j-1)\varepsilon_0 \qquad \text{for every } j \in \mathbb{N}.$$

By the Archimedean Property there is $j \in \mathbb{N}$ with $j - 1 > \dfrac{2M}{\varepsilon_0}$. For this j,

$$a_{p_j} \geq a_1 + (j-1)\varepsilon_0 > -M + 2M = M,$$

which contradicts $\left|a_{p_j}\right| \leq M$. Therefore $\{a_n\}$ is Cauchy.

Step 2: the limit exists. By the Cauchy Criterion, $\{a_n\}$ converges; let $L := \lim a_n$.

Step 3: $L = \sup A$. Let $A := \{a_n : n \in \mathbb{N}\}$. The set A is nonempty ($a_1 \in A$) and bounded above by M; therefore $a := \sup A$ exists (AoC). Fix $m \in \mathbb{N}$. For all $n \geq m$, $a_m \leq a_n$. The tail $\{a_n\}_{n \geq m}$ converges to L (Tail-Shift Remark), so the Order Limit Theorem gives $a_m \leq L$. Since m was arbitrary, L is an upper bound of A; therefore $a \leq L$ (def. of sup). On the other hand, $a_n \leq a$ for every $n \in \mathbb{N}$, so the Order Limit Theorem gives $L \leq a$. Therefore $L = a$; that is, $a = \lim a_n$. □

$$\boxed{\{a_n\} \text{ increasing and bounded } \implies \{a_n\} \to \sup\{a_n : n \in \mathbb{N}\}.}$$

Method Map

Q.	Core topic	Methods included
1	T/F: subsequences and convergence	(a) tail shift and the definition of limit; Divergence Criterion; Cauchy Criterion. (b) counterexample n (odd) / 0 (even); vacuous counterexample $a_n = n$. (c) contrapositive via Subsequential Limits; negation of the definition of limit.
2	T/F: existence of prescribed subsequences	(a) interleaving two sequences; enumerating the fractions j/m. (b) Bolzano-Weierstrass on the bounded subsequence; monotone subsequence (A3) and MCT. (c) case split on boundedness; Monotone Convergence Lemma.
3	Shuffled sequence, Divergence Criterion	Contrapositive of the (bounded) Divergence Criterion; direct ε-N from the definition of limit (outside the question's restriction).
4	Three overlapping subsequences	Common subsequences a_{6k}, a_{6k+3}, then: odd/even split and the Divergence Criterion; reduction to Question 3 and a tail shift; direct ε-N on even and odd indices.
5	Constructing a null subsequence	Recursive construction by Well-Ordering with $\varepsilon_k = 1/k$; unbounded index sets and the Squeeze Theorem.
A1	Limsup/liminf iff convergence	Tail sups/infs converge (MCT), then: Squeeze Theorem and sup/inf bounds; Divergence Criterion and Order Limit Theorem; subsequences realising lim sup and lim inf.
A2	MCT via Divergence Criterion	Leapfrogging indices $i(k)$ and the Order Limit Theorem; contradiction from the definition of limit; Monotone Convergence Lemma (Lecture 9).
A3	Bolzano-Weierstrass via peak terms	Peak terms, then MCT (in its Question A4 form).
A4	MCT via Axiom of Completeness	ε-Criterion for Suprema; contradiction via the least-upper-bound property; Cauchy Criterion and the Order Limit Theorem.

Final Answers

1. (a) True. (b) False. (c) True.
2. (a) True. (b) False. (c) False.
3. $\{c_n\}$ converges, to $L = \lim a_n = \lim b_n$.
4. $\{a_n\}$ converges, to $L = \lim a_{2k} = \lim a_{2k+1} = \lim a_{3k}$.
5. Such a subsequence exists; constructed via $\varepsilon_k = 1/k$, converging to 0.
6. $\{a_n\}$ converges $\iff \liminf a_n = \limsup a_n$, and then both equal $\lim a_n$.

7. (a) $\lim a_{n_k} \geq \lim a_{m_k}$ (equality by symmetry). (b) monotone + bounded $\implies$ convergent.

8. Every sequence has a monotone subsequence; bounded + monotone $\implies$ convergent (MCT), proving Bolzano-Weierstrass.

9. $\{a_n\}$ increasing and bounded $\implies$ $\{a_n\} \to \sup\{a_n : n \in \mathbb{N}\}$.

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