MH3100 Real Analysis I

Tutorial 4 Solutions

Academic Year 2026/2027, Semester 1

Quantitative Research Society @NTU

October 12, 2026

Overview

- **Lecture coverage:** Lectures 10–12 — series, the Cauchy Criterion for Series, the Alternating Series Test, the Comparison Test, the Absolute Convergence Test, absolute vs. conditional convergence, and rearrangements.
- **Questions:** 8 questions (Questions 1–5, Additional Questions A1–A3).
- **Notation:** $\mathbb{N} = \{1, 2, 3, \ldots\}$ ($0 \notin \mathbb{N}$ in this course); series $\sum a_n$ with partial sum $s_n := a_1 + \cdots + a_n$; results cited by slide name.
- **Running lemmas:** geometric series, $\sum 1/n^2$, and divergence of $\sum 1/n$ are proved once in Question 1 and reused throughout.
- **Question themes:** Q1 comparison with geometric and $\sum 1/n^2$ series; Q2 examples for products and sums of series; Q3 proof or counterexample for $\sum a_n^2$, $\sum a_n b_n$, $\sum n^2 a_n$; Q4 Cauchy estimate with a bounded multiplier; Q5 regrouping as a subsequence of partial sums; A1 summation by parts (Abel's Test); A2 infinite products versus series; A3 the Alternating Series Test from nested intervals.

Contents

Question 1 (Convergence of three series) 3

Question 2 (True/false with examples) 12

Question 3 (Proofs or counterexamples) 16

Question 4 (Cauchy partial sums without the named tests) 20

Question 5 (Regrouping preserves the sum) 22

Question A1 (Abel's Test) 24

For academic reference only. This document is provided for educational purposes and does not constitute an official question paper, answer key, or any formally authorised academic material. Its content is not guaranteed to be complete or accurate.

Question A2 (Infinite products) **28**

Question A3 (Alternating Series Test via NIP) **35**

Method Map **38**

Final Answers **38**

Unofficial – QRS@NTU – qrsntu.org

Question 1 (Convergence of three series)

Problem

Determine if each of the following series is convergent or divergent.

(a) $\sum \dfrac{1}{2^n + n}$.

(b) $\sum \dfrac{\sin(n)}{n^2}$.

(c) $1 - \dfrac{1}{2^2} + \dfrac{1}{3} - \dfrac{1}{4^2} + \dfrac{1}{5} - \dfrac{1}{6^2} + \dfrac{1}{7} - \dfrac{1}{8^2} + \cdots$.

(Tutorial 4, Question 1.)

Official hint. Answers only: (a) convergent; (b) convergent; (c) divergent. No method, no comparison series, and no explanation of the pattern in (c) are given.

Solution

Running lemmas

Three standard series are used by every method below and by later questions. Each is proved once here.

Lemma A. *Let $0 < r < 1$. Then $\sum r^n$ converges, with sum $\dfrac{r}{1-r}$.*

Proof. Let $0 < r < 1$ and, for $n \in \mathbb{N}$, let $x_n := r^n$ and $s_n := \sum_{k=1}^{n} r^k$.

First, $\{x_n\}$ converges to 0. For every $n \in \mathbb{N}$, $x_n > 0$ and

$$x_{n+1} = r\, x_n < x_n.$$

Hence $\{x_n\}$ is decreasing and bounded below by 0. Therefore $\{x_n\}$ converges to some $L \in \mathbb{R}$ (MCT). The sequence $\{x_{n+1}\}$ is a subsequence of $\{x_n\}$, so $x_{n+1} \to L$ (Subsequential Limits); also $x_{n+1} = r\, x_n \to rL$ (ALT). Hence

$$L = rL \implies (1-r)L = 0 \implies L = 0 \qquad \text{(Uniqueness of Limits; } 1 - r \neq 0\text{)}.$$

Next, the partial sums. For every $n \in \mathbb{N}$,

$$\begin{aligned}(1-r)s_n &= (r + r^2 + \cdots + r^n) - (r^2 + r^3 + \cdots + r^{n+1}) \\ &= r - r^{n+1}, \\ \therefore\ s_n &= \frac{r - r\, x_n}{1-r}.\end{aligned}$$

Since $x_n \to 0$,

$$s_n \to \frac{r - r \cdot 0}{1-r} = \frac{r}{1-r} \qquad \text{(ALT)}.$$

$\therefore$ $\sum r^n$ converges, with sum $\dfrac{r}{1-r}$ (def. of convergence of a series). □

Lemma B. *$\sum 1/n^2$ converges, and its partial sums $t_N := \sum_{n=1}^{N} \frac{1}{n^2}$ satisfy $t_N \leq 2 - \frac{1}{N}$ for every $N \in \mathbb{N}$.*

Proof. Let $n \geq 2$. Then $0 < n-1 < n$, so $0 < n(n-1) < n^2$, and therefore

$$0 < \frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{n-(n-1)}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

For $N = 1$: $t_1 = 1 = 2 - \frac{1}{1}$. For $N \geq 2$, sum the last display over $n = 2, \ldots, N$:

$$\begin{aligned} t_N &= 1 + \sum_{n=2}^{N} \frac{1}{n^2} < 1 + \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n} \right) \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) \\ &= 1 + 1 - \frac{1}{N} = 2 - \frac{1}{N}. \end{aligned}$$

Hence $t_N \leq 2$ for every $N \in \mathbb{N}$. Also $t_{N+1} - t_N = \frac{1}{(N+1)^2} > 0$ for every $N \in \mathbb{N}$, so $\{t_N\}$ is increasing.

$\therefore$ $\{t_N\}$ is increasing and bounded above by 2 $\therefore$ $\{t_N\}$ converges (MCT) $\therefore$ $\sum 1/n^2$ converges. □

Lemma C. *$\sum 1/n$ diverges: its partial sums $H_N := \sum_{n=1}^{N} \frac{1}{n}$ satisfy $H_{2^m} \geq 1 + \frac{m}{2}$ for every $m \in \mathbb{N}_0$, and $\{H_N\}$ is unbounded.*

Proof. Induction on m.

$m = 0$: $H_1 = 1 = 1 + \frac{0}{2}$.

Suppose $H_{2^m} \geq 1 + \frac{m}{2}$ for some $m \in \mathbb{N}_0$. The indices n with $2^m < n \leq 2^{m+1}$ are $2^m + 1, \ldots, 2^{m+1}$; there are $2^{m+1} - 2^m = 2^m$ of them, and each satisfies $n \leq 2^{m+1}$, so $\frac{1}{n} \geq \frac{1}{2^{m+1}}$. Therefore

$$\begin{aligned} H_{2^{m+1}} &= H_{2^m} + \sum_{n=2^m+1}^{2^{m+1}} \frac{1}{n} \\ &\geq 1 + \frac{m}{2} + 2^m \cdot \frac{1}{2^{m+1}} = 1 + \frac{m+1}{2}. \end{aligned}$$

Hence $H_{2^m} \geq 1 + \frac{m}{2}$ for every $m \in \mathbb{N}_0$.

Let $K \in \mathbb{R}$. There is $m \in \mathbb{N}$ with $m > 2K$ (Archimedean). Then $H_{2^m} \geq 1 + \frac{m}{2} > 1 + K > K$. $\therefore$ $\{H_N\}$ is not bounded above.

$\therefore$ $\{H_N\}$ diverges (Boundedness of Convergent Sequences) $\therefore$ $\sum 1/n$ diverges. □

Method 1: Comparison Test and unbounded partial sums

Part (a).

Proof. Let $n \in \mathbb{N}$. Since $n > 0$, $2^n + n > 2^n > 0$. Therefore

$$0 < \frac{1}{2^n + n} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n.$$

$\sum (1/2)^n$ converges (Lemma A with $r = \frac{1}{2}$; its sum is $\frac{1/2}{1-1/2} = 1$).

$\therefore 0 \le \dfrac{1}{2^n+n} \le \left(\dfrac{1}{2}\right)^n$ for all $n \in \mathbb{N}$ and $\sum(1/2)^n$ converges $\implies \sum \dfrac{1}{2^n+n}$ converges (Comparison Test). □

(a) $\displaystyle\sum \frac{1}{2^n+n}$ is convergent.

Part (b).

Proof. Let $n \in \mathbb{N}$. Then $|\sin(n)| \le 1$ and $n^2 > 0$, so

$$0 \le \left|\frac{\sin(n)}{n^2}\right| = \frac{|\sin(n)|}{n^2} \le \frac{1}{n^2}.$$

$\sum 1/n^2$ converges (Lemma B).

$\therefore \sum |\sin(n)/n^2|$ converges (Comparison Test).

$\therefore \sum \dfrac{\sin(n)}{n^2}$ converges (Absolute Convergence Test). □

(b) $\displaystyle\sum \frac{\sin(n)}{n^2}$ is convergent (in fact, absolutely).

Part (c).

Proof. The n-th term is $a_n = \frac{1}{n}$ for odd n and $a_n = -\frac{1}{n^2}$ for even n. Let $S_N := \sum_{n=1}^{N} a_n$ and, for $k \in \mathbb{N}$,

$$b_k := \frac{1}{2k-1}, \qquad c_k := \frac{1}{(2k)^2} = \frac{1}{4k^2},$$

so that $a_{2k-1} = b_k$ and $a_{2k} = -c_k$ for every $k \in \mathbb{N}$.

Let $m \in \mathbb{N}$. The first $2m$ terms are $a_1, a_3, \ldots, a_{2m-1}$ (odd indices) and $a_2, a_4, \ldots, a_{2m}$ (even indices), so

$$S_{2m} = \sum_{k=1}^{m} b_k - \sum_{k=1}^{m} c_k.$$

Upper bound for the negative part, from Lemma B:

$$\sum_{k=1}^{m} c_k = \frac{1}{4}\sum_{k=1}^{m} \frac{1}{k^2} = \frac{1}{4}\,t_m \le \frac{1}{4}\cdot 2 = \frac{1}{2}.$$

Lower bound for the positive part: for every $k \in \mathbb{N}$, $0 < 2k-1 < 2k$, so $b_k = \frac{1}{2k-1} > \frac{1}{2k}$, and hence

$$\sum_{k=1}^{m} b_k \ge \sum_{k=1}^{m} \frac{1}{2k} = \frac{1}{2}H_m.$$

Combining the two bounds,

$$S_{2m} \ge \frac{1}{2}H_m - \frac{1}{2} \qquad \text{for every } m \in \mathbb{N}.$$

Take $m = 2^j$ with $j \in \mathbb{N}$. By Lemma C, $H_{2^j} \ge 1 + \frac{j}{2}$, so

$$S_{2^{j+1}} \ge \frac{1}{2}\left(1 + \frac{j}{2}\right) - \frac{1}{2} = \frac{j}{4}.$$

Let $K \in \mathbb{R}$. There is $j \in \mathbb{N}$ with $j > 4K$ (Archimedean), and then $S_{2^{j+1}} \geq \frac{j}{4} > K$. $\therefore$ $\{S_N\}$ is not bounded above.

$\therefore$ $\{S_N\}$ diverges (Boundedness of Convergent Sequences) $\therefore$ the series diverges. $\square$

(c) $1 - \dfrac{1}{2^2} + \dfrac{1}{3} - \dfrac{1}{4^2} + \dfrac{1}{5} - \dfrac{1}{6^2} + \cdots$ is divergent.

Unofficial - QRS@NTU - qrsntu.org

Method 2: Cauchy Criterion for Series with an explicit N

Recall (Cauchy Criterion for Series): $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $n > m \geq N \implies |a_{m+1} + \cdots + a_n| < \varepsilon$.

Part (a).

Proof. First, $2^m > m$ for every $m \in \mathbb{N}$: $2^1 = 2 > 1$, and if $2^m > m$ then $2^{m+1} = 2 \cdot 2^m > 2m \geq m + 1$.

Let $n > m \geq 1$. For every $k \in \mathbb{N}$, $0 < \frac{1}{2^k+k} < \frac{1}{2^k}$ (as $2^k + k > 2^k > 0$). The partial-sum formula of Lemma A with $r = \frac{1}{2}$ and $n-m$ terms gives $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-m}} = \frac{1/2 - 1/2^{n-m+1}}{1/2} = 1 - \frac{1}{2^{n-m}}$. Therefore

$$\begin{aligned} 0 < \sum_{k=m+1}^{n} \frac{1}{2^k + k} &< \sum_{k=m+1}^{n} \frac{1}{2^k} = \frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \cdots + \frac{1}{2^n} \\ &= \frac{1}{2^m}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-m}}\right) \\ &= \frac{1}{2^m}\left(1 - \frac{1}{2^{n-m}}\right) < \frac{1}{2^m} < \frac{1}{m}. \end{aligned}$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$ (Archimedean). For all $n > m \geq N$:

$$\left| \sum_{k=m+1}^{n} \frac{1}{2^k + k} \right| < \frac{1}{m} \leq \frac{1}{N} < \varepsilon.$$

$\therefore$ for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ with $n > m \geq N \implies \left|\sum_{k=m+1}^{n} \frac{1}{2^k+k}\right| < \varepsilon$.

$\therefore \sum \frac{1}{2^n + n}$ converges (Cauchy Criterion for Series). □

$$\boxed{\text{(a) } \sum \frac{1}{2^n + n} \text{ is convergent.}}$$

Part (b).

Proof. Let $n > m \geq 1$. For every $k \geq m + 1 \geq 2$,

$$\left|\frac{\sin(k)}{k^2}\right| \leq \frac{1}{k^2} < \frac{1}{k-1} - \frac{1}{k} \qquad (|\sin(k)| \leq 1;\ \text{Lemma B, proof}).$$

By the triangle inequality and telescoping,

$$\begin{aligned} \left| \sum_{k=m+1}^{n} \frac{\sin(k)}{k^2} \right| \leq \sum_{k=m+1}^{n} \left|\frac{\sin(k)}{k^2}\right| &< \sum_{k=m+1}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right) \\ &= \left(\frac{1}{m} - \frac{1}{m+1}\right) + \left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \cdots \\ &\quad + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= \frac{1}{m} - \frac{1}{n} < \frac{1}{m}. \end{aligned}$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$ (Archimedean). For all $n > m \geq N$:

$$\left| \sum_{k=m+1}^{n} \frac{\sin(k)}{k^2} \right| < \frac{1}{m} \leq \frac{1}{N} < \varepsilon.$$

$\therefore \sum \frac{\sin(n)}{n^2}$ converges (Cauchy Criterion for Series). The same estimate without the outer absolute value, $\sum_{k=m+1}^{n} \left| \frac{\sin(k)}{k^2} \right| < \frac{1}{m}$, shows that $\sum |\sin(n)/n^2|$ also converges (Cauchy Criterion for Series). □

(b) $\sum \frac{\sin(n)}{n^2}$ is convergent (in fact, absolutely).

Part (c).

Proof. Let a_n, S_N, $b_k = \frac{1}{2k-1}$ and $c_k = \frac{1}{4k^2}$ be as in Method 1, so $a_{2k-1} = b_k$ and $a_{2k} = -c_k$.

Let $j \in \mathbb{N}$. The terms $a_{2j+1}, a_{2j+2}, \ldots, a_{4j}$ are b_k and $-c_k$ for $k = j+1, \ldots, 2j$ (since $2j+1 = 2(j+1)-1$ and $4j = 2 \cdot 2j$). Hence

$$S_{4j} - S_{2j} = \sum_{k=j+1}^{2j} b_k - \sum_{k=j+1}^{2j} c_k.$$

For $j+1 \leq k \leq 2j$: $b_k = \frac{1}{2k-1} > \frac{1}{2k} \geq \frac{1}{4j}$, and there are j such k, so

$$\sum_{k=j+1}^{2j} b_k > j \cdot \frac{1}{4j} = \frac{1}{4}.$$

For $k \geq j+1 \geq 2$: $c_k = \frac{1}{4} \cdot \frac{1}{k^2} < \frac{1}{4}\left(\frac{1}{k-1} - \frac{1}{k}\right)$ (Lemma B, proof). Telescoping,

$$\begin{aligned} \sum_{k=j+1}^{2j} c_k &< \frac{1}{4}\left[\left(\frac{1}{j} - \frac{1}{j+1}\right) + \cdots + \left(\frac{1}{2j-1} - \frac{1}{2j}\right)\right] \\ &= \frac{1}{4}\left(\frac{1}{j} - \frac{1}{2j}\right) = \frac{1}{8j} \leq \frac{1}{8}. \end{aligned}$$

Therefore

$$S_{4j} - S_{2j} > \frac{1}{4} - \frac{1}{8} = \frac{1}{8} \qquad \text{for every } j \in \mathbb{N}.$$

Let $\varepsilon := \frac{1}{8}$ and let $N \in \mathbb{N}$. Put $m := 2N$ and $n := 4N$. Then $n > m \geq N$ and

$$|a_{m+1} + \cdots + a_n| = |S_{4N} - S_{2N}| > \frac{1}{8} = \varepsilon.$$

$\therefore$ no $N \in \mathbb{N}$ satisfies the condition of the Cauchy Criterion for Series with $\varepsilon = \frac{1}{8}$ $\therefore$ the series diverges (Cauchy Criterion for Series). □

(c) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \cdots$ is divergent.

Method 3: Monotone Convergence Theorem on bounded partial sums, for parts (a) and (b)

Part (a).

Proof. Let $A_N := \sum_{n=1}^{N} \frac{1}{2^n+n}$. For every $N \in \mathbb{N}$,

$$A_{N+1} - A_N = \frac{1}{2^{N+1}+N+1} > 0,$$

so $\{A_N\}$ is increasing. Since $0 < \frac{1}{2^n+n} < \frac{1}{2^n}$ for every $n \in \mathbb{N}$, and by the partial-sum formula of Lemma A with $r = \frac{1}{2}$,

$$A_N < \sum_{n=1}^{N} \frac{1}{2^n} = \frac{\frac{1}{2} - \frac{1}{2^{N+1}}}{1-\frac{1}{2}} = 1 - \frac{1}{2^N} < 1.$$

$\therefore \{A_N\}$ is increasing and bounded above by 1 $\therefore \{A_N\}$ converges (MCT) $\therefore \sum \frac{1}{2^n+n}$ converges. □

$$\boxed{\text{(a) } \sum \frac{1}{2^n+n} \text{ is convergent.}}$$

Part (b).

Proof. Let $B_N := \sum_{n=1}^{N} \left|\frac{\sin(n)}{n^2}\right|$. For every $N \in \mathbb{N}$, $B_{N+1} - B_N = \left|\frac{\sin(N+1)}{(N+1)^2}\right| \geq 0$, so $\{B_N\}$ is increasing. Since $\left|\frac{\sin(n)}{n^2}\right| \leq \frac{1}{n^2}$ for every $n \in \mathbb{N}$,

$$B_N \leq \sum_{n=1}^{N} \frac{1}{n^2} = t_N \leq 2 \qquad \text{(Lemma B)}.$$

$\therefore \{B_N\}$ is increasing and bounded above by 2.
$\therefore \{B_N\}$ converges (MCT), i.e. $\sum |\sin(n)/n^2|$ converges.
$\therefore \sum \frac{\sin(n)}{n^2}$ converges (Absolute Convergence Test). □

$$\boxed{\text{(b) } \sum \frac{\sin(n)}{n^2} \text{ is convergent (in fact, absolutely).}}$$

Method 4: Contradiction via the Algebraic Limit Theorem for Series, for part (c)

Proof. The n-th term is $a_n = \frac{1}{n}$ for odd n and $a_n = -\frac{1}{n^2}$ for even n. Define, for $n \in \mathbb{N}$,

$$d_n := \begin{cases} -\frac{1}{n^2}, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \qquad e_n := a_n - d_n = \begin{cases} 0, & n \text{ even}, \\ \frac{1}{n}, & n \text{ odd}. \end{cases}$$

For every $n \in \mathbb{N}$, $0 \le |d_n| \le \frac{1}{n^2}$. $\therefore$ $\sum |d_n|$ converges (Lemma B, Comparison Test) $\therefore$ $\sum d_n$ converges (Absolute Convergence Test).

Suppose, for contradiction, that $\sum a_n$ converges. Since $e_n = a_n + (-1)d_n$, $\sum e_n$ converges (ALT for Series). Let $E_N := \sum_{n=1}^{N} e_n$. For $m \in \mathbb{N}$, the odd indices in $\{1, \dots, 2m\}$ are $2k-1$ with $k = 1, \dots, m$, so

$$E_{2m} = \sum_{k=1}^{m} \frac{1}{2k-1} > \sum_{k=1}^{m} \frac{1}{2k} = \frac{1}{2} H_m.$$

With $m = 2^j$ ($j \in \mathbb{N}$), Lemma C gives $E_{2^{j+1}} > \frac{1}{2}\left(1 + \frac{j}{2}\right) > \frac{j}{4}$. For $K \in \mathbb{R}$ choose $j \in \mathbb{N}$ with $j > 4K$ (Archimedean); then $E_{2^{j+1}} > K$. $\therefore$ $\{E_N\}$ is unbounded, so $\sum e_n$ diverges (Boundedness of Convergent Sequences). This contradicts the convergence of $\sum e_n$.

$\therefore$ $\sum a_n$ diverges. □

(c) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \cdots$ is divergent.

Unofficial QRS@NTU - qrs.ntu.edu.sg

Method 5: Ratio Test, for part (a)

Note: uses the Ratio Test, which is not in Lectures 1–17 (beyond the midterm scope).

Recall (Ratio Test): if $a_n > 0$ for every $n \in \mathbb{N}$ and $\lim \frac{a_{n+1}}{a_n} = L$ with $L < 1$, then $\sum a_n$ converges.

Let $a_n := \frac{1}{2^n+n} > 0$. Dividing numerator and denominator by 2^n,

$$\frac{a_{n+1}}{a_n} = \frac{2^n + n}{2^{n+1} + n + 1} = \frac{1 + \dfrac{n}{2^n}}{2 + \dfrac{n+1}{2^n}}.$$

First, $\frac{n}{2^n} \to 0$. For $n \geq 2$, the binomial theorem gives

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} \geq \binom{n}{2} = \frac{n(n-1)}{2}, \qquad \therefore\ 0 < \frac{n}{2^n} \leq \frac{2}{n-1}.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ with $N > 1 + \frac{2}{\varepsilon}$ (Archimedean); then $N \geq 2$, and for all $n \geq N$, $n - 1 > \frac{2}{\varepsilon}$, so

$$\left| \frac{n}{2^n} - 0 \right| \leq \frac{2}{n-1} < \varepsilon.$$

Hence $\frac{n}{2^n} \to 0$ (def. of limit). Also $\frac{1}{2^n} \to 0$ (Lemma A, proof, with $r = \frac{1}{2}$), so

$$\frac{n+1}{2^n} = \frac{n}{2^n} + \frac{1}{2^n} \to 0 + 0 = 0 \qquad \text{(ALT)}.$$

Therefore, since the denominator tends to $2 \neq 0$,

$$\frac{a_{n+1}}{a_n} \to \frac{1+0}{2+0} = \frac{1}{2} \qquad \text{(ALT)}.$$

$L = \frac{1}{2} < 1 \therefore\ \sum \dfrac{1}{2^n + n}$ converges (Ratio Test).

$$\boxed{\text{(a) } \sum \frac{1}{2^n + n} \text{ is convergent.}}$$

Question 2 (True/false with examples)

Problem

Determine if each of the following statements is true.

(a) There are two divergent series $\sum a_n$ and $\sum b_n$ where $\sum a_n b_n$ converges.

(b) There is a convergent series $\sum a_n$ and a bounded sequence $\{b_n\}$ such that $\sum a_n b_n$ diverges.

(c) There are two sequences $\{a_n\}$ and $\{b_n\}$ where $\sum a_n$ and $\sum(a_n + b_n)$ both converge but $\sum b_n$ diverges.

(d) There is a sequence $\{a_n\}$ satisfying $0 \leq a_n \leq \frac{1}{n}$ where $\sum(-1)^n a_n$ diverges.

(Tutorial 4, Question 2; part (a) is also Sample Midterm Question 2(a).)

Official hint. Answers only: (a) True; (b) True; (c) False; (d) True. No justification, examples, or proofs are given.

Solution

Method 1: Explicit examples; Algebraic Limit Theorem for Series

Part (a).

Proof. Let $a_n = b_n := \dfrac{1}{n}$ for $n \in \mathbb{N}$.

$\sum a_n = \sum b_n = \sum \dfrac{1}{n}$ diverges (Lemma C).

For every $n \in \mathbb{N}$, $a_n b_n = \dfrac{1}{n} \cdot \dfrac{1}{n} = \dfrac{1}{n^2}$ $\therefore$ $\sum a_n b_n = \sum \dfrac{1}{n^2}$ converges (Lemma B).

$\therefore \sum a_n$ and $\sum b_n$ are divergent and $\sum a_n b_n$ is convergent. □

$$\boxed{\text{(a) True, e.g. } a_n = b_n = \tfrac{1}{n}.}$$

Part (b).

Proof. Let $a_n := \dfrac{(-1)^n}{n}$ and $b_n := (-1)^n$ for $n \in \mathbb{N}$.

$\sum a_n$ *converges.* Let $\beta_n := \frac{1}{n}$. For every $n \in \mathbb{N}$, $n < n+1$, so $\beta_{n+1} = \frac{1}{n+1} < \frac{1}{n} = \beta_n$; hence $\{\beta_n\}$ is decreasing. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$ (Archimedean). For all $n \geq N$, $|\beta_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$. Hence $0 = \lim \beta_n$ (def. of limit).

$\therefore$ $\sum(-1)^{n-1}\dfrac{1}{n}$ converges, say to L (AST).

For every $n \in \mathbb{N}$, $a_n = (-1) \cdot \dfrac{(-1)^{n-1}}{n}$ $\therefore$ $\sum a_n$ converges, to $-L$ (ALT for Series, with $c = -1$).

$\{b_n\}$ *is bounded.* $|b_n| = 1 \leq 1$ for every $n \in \mathbb{N}$.

$\sum a_n b_n$ *diverges.* For every $n \in \mathbb{N}$,

$$a_n b_n = \frac{(-1)^n(-1)^n}{n} = \frac{\left((-1)^2\right)^n}{n} = \frac{1}{n}.$$

$\therefore\ \sum a_n b_n = \sum \dfrac{1}{n}$ diverges (Lemma C). □

(b) True, e.g. $a_n = \frac{(-1)^n}{n},\ b_n = (-1)^n$.

Part (c).

Proof. Suppose $\sum a_n$ converges to s and $\sum(a_n + b_n)$ converges to t. For every $n \in \mathbb{N}$,

$$b_n = (a_n + b_n) + (-1)\, a_n.$$

$\sum(-1)a_n$ converges, to $-s$ (ALT for Series, part (1), $c = -1$).

$\therefore\ \sum\big[(a_n + b_n) + (-1)a_n\big] = \sum b_n$ converges, to $t - s$ (ALT for Series, part (2)).

$\therefore$ whenever $\sum a_n$ and $\sum(a_n + b_n)$ converge, $\sum b_n$ converges; so no sequences with the stated properties exist. □

(c) False: $\sum a_n,\ \sum(a_n + b_n)$ convergent $\implies \sum b_n$ convergent.

Part (d).

Proof. Let $a_n := \dfrac{1}{n}$ for even n and $a_n := 0$ for odd n. Then $0 \le a_n \le \dfrac{1}{n}$ for every $n \in \mathbb{N}$.

Let $T_N := \sum_{n=1}^{N}(-1)^n a_n$. For odd n, $(-1)^n a_n = 0$; for even $n = 2k$, $(-1)^{2k} a_{2k} = \frac{1}{2k}$. Hence, for every $m \in \mathbb{N}$,

$$T_{2m} = \sum_{k=1}^{m} \frac{1}{2k} = \frac{1}{2} H_m.$$

With $m = 2^j$ $(j \in \mathbb{N})$, Lemma C gives

$$T_{2^{j+1}} = \frac{1}{2} H_{2^j} \ge \frac{1}{2}\left(1 + \frac{j}{2}\right) > \frac{j}{4}.$$

Let $K \in \mathbb{R}$. There is $j \in \mathbb{N}$ with $j > 4K$ (Archimedean), and then $T_{2^{j+1}} > K$. $\therefore \{T_N\}$ is unbounded.

$\therefore\ \sum(-1)^n a_n$ diverges (Boundedness of Convergent Sequences). □

(d) True, e.g. $a_n = 1/n$ for even n, $a_n = 0$ for odd n.

Method 2: Disjoint-support and indicator examples; Cauchy Criterion for Series

Part (a).

Proof. Let

$$a_n := \begin{cases} 1, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \qquad b_n := \begin{cases} 0, & n \text{ odd}, \\ 1, & n \text{ even}. \end{cases}$$

Let $A_N := \sum_{n=1}^{N} a_n$ and $B_N := \sum_{n=1}^{N} b_n$. Among $1, \ldots, 2m$ there are exactly m odd and m even indices, so

$$A_{2m} = m, \qquad B_{2m} = m \qquad (m \in \mathbb{N}).$$

For $K \in \mathbb{R}$ choose $m \in \mathbb{N}$ with $m > K$ (Archimedean); then $A_{2m} = B_{2m} = m > K$. $\therefore$ $\{A_N\}$ and $\{B_N\}$ are unbounded $\therefore$ $\sum a_n$ and $\sum b_n$ diverge (Boundedness of Convergent Sequences).

For every $n \in \mathbb{N}$ one of a_n, b_n equals 0, so $a_n b_n = 0$. $\therefore$ every partial sum of $\sum a_n b_n$ equals 0 $\therefore$ $\sum a_n b_n$ converges, to 0. □

(a) True, e.g. $a_n = 1,\ b_n = 0$ for odd n; $a_n = 0,\ b_n = 1$ for even n.

Part (b).

Proof. Let $a_n := \dfrac{(-1)^{n-1}}{n}$ and let $b_n := 1$ for odd n, $b_n := 0$ for even n.

$\{1/n\}$ is decreasing with $0 = \lim \frac{1}{n}$ (Method 1, part (b)) $\therefore$ $\sum a_n$ converges (AST). $|b_n| \leq 1$ for every $n \in \mathbb{N}$, so $\{b_n\}$ is bounded.

For odd n, $(-1)^{n-1} = 1$, so $a_n b_n = \frac{1}{n}$; for even n, $a_n b_n = 0$. Let $U_N := \sum_{n=1}^{N} a_n b_n$. For $m \in \mathbb{N}$,

$$U_{2m} = \sum_{k=1}^{m} \frac{1}{2k-1} > \sum_{k=1}^{m} \frac{1}{2k} = \frac{1}{2} H_m.$$

With $m = 2^j$ ($j \in \mathbb{N}$), Lemma C gives $U_{2^{j+1}} > \frac{1}{2}\left(1 + \frac{j}{2}\right) > \frac{j}{4}$. For $K \in \mathbb{R}$ choose $j \in \mathbb{N}$ with $j > 4K$ (Archimedean); then $U_{2^{j+1}} > K$.

$\therefore$ $\{U_N\}$ is unbounded $\therefore$ $\sum a_n b_n$ diverges (Boundedness of Convergent Sequences). □

(b) True, e.g. $a_n = \frac{(-1)^{n-1}}{n},\ b_n = 1$ for odd $n,\ b_n = 0$ for even n.

Part (c).

Proof (contradiction). Suppose $\sum a_n$ and $\sum (a_n + b_n)$ converge and $\sum b_n$ diverges. Let $\varepsilon > 0$.

$\sum a_n$ converges $\therefore$ there is $N_1 \in \mathbb{N}$ such that

$$n > m \geq N_1 \implies \left| \sum_{k=m+1}^{n} a_k \right| < \frac{\varepsilon}{2} \qquad \text{(Cauchy Criterion for Series)}.$$

$\sum (a_n + b_n)$ converges $\therefore$ there is $N_2 \in \mathbb{N}$ such that

$$n > m \geq N_2 \implies \left| \sum_{k=m+1}^{n} (a_k + b_k) \right| < \frac{\varepsilon}{2} \qquad \text{(Cauchy Criterion for Series)}.$$

Let $N := \max\{N_1, N_2\}$ and $n > m \geq N$. By the triangle inequality,

$$\left|\sum_{k=m+1}^{n} b_k\right| = \left|\sum_{k=m+1}^{n} (a_k + b_k) - \sum_{k=m+1}^{n} a_k\right|$$
$$\leq \left|\sum_{k=m+1}^{n} (a_k + b_k)\right| + \left|\sum_{k=m+1}^{n} a_k\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore \sum b_n$ converges (Cauchy Criterion for Series), contradicting the assumption that $\sum b_n$ diverges.

$\therefore$ no such sequences exist. □

(c) False: $\sum a_n$, $\sum(a_n + b_n)$ convergent $\implies$ $\sum b_n$ convergent.

Part (d).

Proof. Let $a_n := \dfrac{1}{n}$ for even n and $a_n := 0$ for odd n, so $0 \leq a_n \leq \frac{1}{n}$, and let $T_N := \sum_{n=1}^{N}(-1)^n a_n$.

Let $N \in \mathbb{N}$. In the block $n = 2N+1, \ldots, 4N$ the odd n contribute 0 and the even $n = 2k$, $k = N+1, \ldots, 2N$, contribute $\frac{1}{2k} \geq \frac{1}{4N}$. There are N such k, so

$$T_{4N} - T_{2N} = \sum_{k=N+1}^{2N} \frac{1}{2k} \geq N \cdot \frac{1}{4N} = \frac{1}{4}.$$

Let $\varepsilon := \frac{1}{4}$. For every $N \in \mathbb{N}$, $n := 4N$ and $m := 2N$ satisfy $n > m \geq N$ and $|T_n - T_m| \geq \varepsilon$. $\therefore$ the condition of the Cauchy Criterion for Series fails for $\varepsilon = \frac{1}{4}$.

$\therefore$ $\sum(-1)^n a_n$ diverges (Cauchy Criterion for Series). □

(d) True, e.g. $a_n = 1/n$ for even n, $a_n = 0$ for odd n.

Question 3 (Proofs or counterexamples)

Problem

For each of the following statements, provide either a short proof if it is true or a counterexample if it is false.

(a) If a series $\sum a_n$ converges absolutely, then the series $\sum a_n^2$ also converges absolutely.

(b) If a series $\sum a_n$ converges and a sequence $\{b_n\}$ converges, then the series $\sum a_n b_n$ converges.

(c) If a series $\sum a_n$ converges conditionally, then the series $\sum n^2 a_n$ diverges.

(Tutorial 4, Question 3.)

Official hint. (a) Apply the Comparison Test using the Terms of Convergent Series, the Boundedness of Convergent Sequences, and the Algebraic Limit Theorem. (b) Find a conditionally convergent series written as $\sum a_n b_n$ with $\sum a_n$ conditionally convergent and $\{b_n\}$ convergent. (c) Prove the contrapositive, applying the Terms of Convergent Series to $\sum n^2 a_n$, then the Comparison Test with a specific absolutely convergent series. No inequality chains, comparison series, or worked examples are given.

Solution

Method 1: Comparison Test with a global bound; explicit counterexample; contrapositive

Part (a) – true.

Proof. Suppose $\sum |a_n|$ converges.

$\therefore\ 0 = \lim |a_n|$ (Terms of Convergent Series, applied to $\sum |a_n|$).

$\therefore\ \{|a_n|\}$ is bounded: there is $M > 0$ with $|a_n| \leq M$ for all $n \in \mathbb{N}$ (Boundedness of Convergent Sequences). Hence, for every $n \in \mathbb{N}$,

$$0 \leq a_n^2 = |a_n| \cdot |a_n| \leq M\,|a_n|\,.$$

$\sum M\,|a_n|$ converges, to $M \sum |a_n|$ (ALT for Series, $c = M$).

$\therefore\ \sum a_n^2$ converges (Comparison Test).

For every $n \in \mathbb{N}$, $|a_n^2| = a_n^2$ $\therefore\ \sum |a_n^2|$ converges, i.e. $\sum a_n^2$ converges absolutely. □

(a) True: $\sum a_n$ absolutely convergent $\implies$ $\sum a_n^2$ absolutely convergent.

Part (b) – false.

Proof. Let $a_n = b_n := \dfrac{(-1)^n}{\sqrt{n}}$ for $n \in \mathbb{N}$, and $\beta_n := \dfrac{1}{\sqrt{n}}$.

$\{\beta_n\}$ *is decreasing.* Let $n \in \mathbb{N}$. If $\sqrt{n+1} \leq \sqrt{n}$, then squaring gives $n + 1 \leq n$, which is false. Hence $\sqrt{n} < \sqrt{n+1}$, so $\beta_{n+1} < \beta_n$.

$0 = \lim \beta_n$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon^2}$ (Archimedean). Let $n \geq N$. If $\sqrt{n} \leq \frac{1}{\varepsilon}$, then squaring gives $n \leq \frac{1}{\varepsilon^2} < N$, which is false. Hence $\sqrt{n} > \frac{1}{\varepsilon}$, so $|\beta_n - 0| = \frac{1}{\sqrt{n}} < \varepsilon$ (def. of limit).

$\therefore$ $\sum(-1)^{n-1}\beta_n$ converges (AST) $\therefore$ $\sum a_n = \sum(-1)\cdot(-1)^{n-1}\beta_n$ converges (ALT for Series, $c=-1$).

$|b_n - 0| = \beta_n$ for every $n \in \mathbb{N}$ and $0 = \lim \beta_n$ $\therefore$ $\{b_n\}$ converges, to 0 (def. of limit).

For every $n \in \mathbb{N}$,

$$a_n b_n = \frac{(-1)^n(-1)^n}{\sqrt{n}\,\sqrt{n}} = \frac{\left((-1)^2\right)^n}{n} = \frac{1}{n}.$$

$\therefore$ $\sum a_n b_n = \sum \dfrac{1}{n}$ diverges (Lemma C). □

(b) False, e.g. $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$.

Part (c) – true.

Proof (contrapositive). Suppose $\sum n^2 a_n$ converges.

$\therefore$ $0 = \lim n^2 a_n$ (Terms of Convergent Series).

$\therefore$ there is $M > 0$ with $|n^2 a_n| \leq M$ for all $n \in \mathbb{N}$ (Boundedness of Convergent Sequences). Dividing by $n^2 > 0$,

$$0 \leq |a_n| \leq \frac{M}{n^2} \qquad \text{for all } n \in \mathbb{N}.$$

$\sum \frac{M}{n^2}$ converges (Lemma B; ALT for Series, $c = M$).

$\therefore$ $\sum |a_n|$ converges (Comparison Test), i.e. $\sum a_n$ converges absolutely, so $\sum a_n$ does not converge conditionally.

$\therefore$ (contrapositive) $\sum a_n$ conditionally convergent $\implies$ $\sum n^2 a_n$ diverges. □

(c) True: $\sum a_n$ conditionally convergent $\implies$ $\sum n^2 a_n$ divergent.

Unofficial Solutions by QRS@NTU

Method 2: Cauchy-tail estimates, a second counterexample, and contradiction via Question 4

Part (a) – true.

Proof. Suppose $\sum |a_n|$ converges.

Square of a tail. For $n > m$ and the nonnegative numbers $x_k := |a_k|$, $k = m+1, \dots, n$,

$$\left(\sum_{k=m+1}^{n} x_k\right)^2 = \sum_{k=m+1}^{n}\sum_{l=m+1}^{n} x_k x_l = \sum_{k=m+1}^{n} x_k^2 + \sum_{\substack{k,l=m+1 \\ k\neq l}}^{n} x_k x_l \geq \sum_{k=m+1}^{n} x_k^2.$$

Since $x_k^2 = a_k^2$, this reads $\sum_{k=m+1}^{n} a_k^2 \leq \left(\sum_{k=m+1}^{n} |a_k|\right)^2$.

Choice of N. Let $\varepsilon > 0$ and $\eta := \min\{1, \varepsilon\} > 0$. $\sum |a_n|$ converges $\therefore$ there is $N \in \mathbb{N}$ with $n > m \geq N \implies 0 \leq \sum_{k=m+1}^{n} |a_k| < \eta$ (Cauchy Criterion for Series). For $n > m \geq N$, write $u := \sum_{k=m+1}^{n} |a_k|$; then $0 \leq u < \eta \leq 1$, so $u^2 \leq u$, and

$$\left|\sum_{k=m+1}^{n} a_k^2\right| = \sum_{k=m+1}^{n} a_k^2 \leq u^2 \leq u < \eta \leq \varepsilon.$$

$\therefore$ $\sum a_n^2$ converges (Cauchy Criterion for Series). Since $|a_n^2| = a_n^2$, $\sum |a_n^2|$ is the same series, so $\sum a_n^2$ converges absolutely. □

(a) True: $\sum a_n$ absolutely convergent $\implies$ $\sum a_n^2$ absolutely convergent.

Part (b) – false.

Proof. Let $a_n := \dfrac{(-1)^n}{\sqrt{n}}$ and $b_n := 1 + \dfrac{(-1)^n}{\sqrt{n}}$ for $n \in \mathbb{N}$.

$\sum a_n$ converges (Method 1, part (b)). $|b_n - 1| = \frac{1}{\sqrt{n}}$ and $0 = \lim \frac{1}{\sqrt{n}}$ (Method 1, part (b)) $\therefore$ $\{b_n\}$ converges, to 1 (def. of limit). For every $n \in \mathbb{N}$,

$$a_n b_n = \frac{(-1)^n}{\sqrt{n}} + \frac{(-1)^n(-1)^n}{\sqrt{n}\,\sqrt{n}} = a_n + \frac{1}{n}.$$

Suppose, for contradiction, that $\sum a_n b_n$ converges. Then $\sum\big(a_n b_n + (-1)a_n\big) = \sum \frac{1}{n}$ converges (ALT for Series), contradicting Lemma C.

$\therefore$ $\sum a_n b_n$ diverges, although $\sum a_n$ and $\{b_n\}$ converge. □

(b) False, e.g. $a_n = \frac{(-1)^n}{\sqrt{n}}$, $b_n = 1 + \frac{(-1)^n}{\sqrt{n}}$.

Part (c) – true.

Proof (contradiction). Suppose $\sum a_n$ converges conditionally and $\sum n^2 a_n$ converges. Let $c_n := n^2 a_n$.

$0 = \lim c_n$ (Terms of Convergent Series) $\therefore$ there is $M > 0$ with $|c_n| \leq M$ for all $n \in \mathbb{N}$ (Boundedness of Convergent Sequences); hence $\{|c_n|\}$ is bounded.

$\sum \left|\frac{1}{n^2}\right| = \sum \frac{1}{n^2}$ converges (Lemma B) $\therefore$ its partial sums form a convergent, hence Cauchy, sequence (Cauchy Criterion).

Apply Question 4 to the series $\sum \frac{1}{n^2}$ and the bounded sequence $\{|c_n|\}$: the series $\sum \frac{1}{n^2}|c_n|$ has a Cauchy sequence of partial sums. For every $n \in \mathbb{N}$,

$$\frac{1}{n^2}|c_n| = \frac{|n^2 a_n|}{n^2} = |a_n|\,.$$

$\therefore \sum |a_n|$ has Cauchy partial sums $\therefore \sum |a_n|$ converges (Cauchy Criterion for Series), i.e. $\sum a_n$ converges absolutely. This contradicts the conditional convergence of $\sum a_n$.

$\therefore \sum a_n$ conditionally convergent $\implies \sum n^2 a_n$ diverges. □

(c) True: $\sum a_n$ conditionally convergent $\implies \sum n^2 a_n$ divergent.

Unofficial QRS@NTU - qrsntu.org

Question 4 (Cauchy partial sums without the named tests)

Problem

Prove that if the series $\sum |a_n|$ has a Cauchy sequence of partial sums and the sequence $\{b_n\}$ is bounded, then the series $\sum a_n b_n$ has a Cauchy sequence of partial sums. **Do not use the Cauchy Criterion, the Comparison Test, or the Absolute Convergence Test.** *(Tutorial 4, Question 4.)*

Official hint. (a) Write down the definitions for the sequences of partial sums of both $\sum |a_n|$ and $\sum a_n b_n$ to be Cauchy, and use the boundedness of $\{b_n\}$ to compare them. (b) Follow the proof of the Absolute Convergence Test via the Cauchy Criterion. No explicit inequality or ε-bookkeeping is given.

Solution

Method 1: Triangle inequality on partial-sum differences

Proof. Let $s_n := \sum_{k=1}^{n} |a_k|$ and $t_n := \sum_{k=1}^{n} a_k b_k$ for $n \in \mathbb{N}$. $\{b_n\}$ is bounded, so there is $M_0 \geq 0$ with $|b_n| \leq M_0$ for all $n \in \mathbb{N}$; let $M := M_0 + 1 > 0$, so $|b_n| \leq M$ for all $n \in \mathbb{N}$.

Step 1: differences of partial sums. For $n > m$,

$$t_n - t_m = (a_1 b_1 + \cdots + a_n b_n) - (a_1 b_1 + \cdots + a_m b_m) = \sum_{k=m+1}^{n} a_k b_k,$$

$$s_n - s_m = (|a_1| + \cdots + |a_n|) - (|a_1| + \cdots + |a_m|) = \sum_{k=m+1}^{n} |a_k| \geq 0.$$

Step 2: the estimate. For $n > m$,

$$\begin{aligned} |t_n - t_m| = \left| \sum_{k=m+1}^{n} a_k b_k \right| &\leq \sum_{k=m+1}^{n} |a_k|\,|b_k| && \text{(triangle inequality)} \\ &\leq \sum_{k=m+1}^{n} M\,|a_k| = M(s_n - s_m) && (|b_k| \leq M). \end{aligned}$$

Step 3: choice of N. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{M} > 0$, and $\{s_n\}$ is Cauchy, so there is $N \in \mathbb{N}$ with $|s_n - s_m| < \frac{\varepsilon}{M}$ for all $n, m \geq N$ (def. of Cauchy sequence). Let $n, m \geq N$.

- If $n > m$: $|t_n - t_m| \leq M(s_n - s_m) = M\,|s_n - s_m| < M \cdot \frac{\varepsilon}{M} = \varepsilon$.
- If $n < m$: the same estimate with n, m exchanged gives $|t_m - t_n| < \varepsilon$, and $|t_n - t_m| = |t_m - t_n|$.
- If $n = m$: $|t_n - t_m| = 0 < \varepsilon$.

$\therefore$ for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $n, m \geq N \implies |t_n - t_m| < \varepsilon$.

$\therefore$ $\{t_n\}$ is Cauchy (def. of Cauchy sequence). $\square$

$$\boxed{\sum |a_n| \text{ Cauchy partial sums, } \{b_n\} \text{ bounded } \implies \sum a_n b_n \text{ has Cauchy partial sums.}}$$

Method 2: Contradiction with the negated Cauchy condition

Proof (contradiction). Let $s_n := \sum_{k=1}^{n} |a_k|$, $t_n := \sum_{k=1}^{n} a_k b_k$, and $M > 0$ with $|b_n| \leq M$ for all $n \in \mathbb{N}$ (take $M_0 + 1$ for any bound $M_0 \geq 0$). For $n > m$, the triangle inequality gives

$$|t_n - t_m| = \left| \sum_{k=m+1}^{n} a_k b_k \right| \leq \sum_{k=m+1}^{n} |a_k|\,|b_k| \leq M \sum_{k=m+1}^{n} |a_k| = M(s_n - s_m).$$

Suppose $\{t_n\}$ is not Cauchy. Negating the definition: there is $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there are $n, m \geq N$ with $|t_n - t_m| \geq \varepsilon_0$. For such n, m, $n \neq m$ (otherwise $|t_n - t_m| = 0 < \varepsilon_0$); since $|t_n - t_m| = |t_m - t_n|$, label them so that $n > m$. Then

$$\varepsilon_0 \leq |t_n - t_m| \leq M(s_n - s_m) \implies |s_n - s_m| \geq \frac{\varepsilon_0}{M}.$$

Let $\varepsilon := \frac{\varepsilon_0}{M} > 0$. Then for every $N \in \mathbb{N}$ there are $n > m \geq N$ with $|s_n - s_m| \geq \varepsilon$. $\therefore$ $\{s_n\}$ is not Cauchy, contradicting the hypothesis.

$\therefore$ $\{t_n\}$ is Cauchy. □

$$\boxed{\sum |a_n| \text{ Cauchy partial sums, } \{b_n\} \text{ bounded} \implies \sum a_n b_n \text{ has Cauchy partial sums.}}$$

Unofficial QRS@NTU Tutorial

Question 5 (Regrouping preserves the sum)

Problem

Given a series $\sum a_n$ and a strictly increasing sequence of natural numbers $\{n_k\}$, the regrouping of the terms

$$(a_1 + a_2 + \cdots + a_{n_1}) + (a_{n_1+1} + a_{n_1+2} + \cdots + a_{n_2}) + (a_{n_2+1} + a_{n_2+2} + \cdots + a_{n_3}) + \cdots$$

leads to a series $\sum b_k$ with the terms

$$b_k := a_{n_{k-1}+1} + a_{n_{k-1}+2} + \cdots + a_{n_k} \qquad \forall k \in \mathbb{N},$$

where n_0 denotes 0. Prove that if $\sum a_n$ sums to s, then $\sum b_k$ sums to s. *(Tutorial 4, Question 5.)*

Official hint. "Compare the sequence of partial sums of $\sum b_k$ with that of $\sum a_n$." No further detail is given.

Solution

Method 1: Partial sums of $\sum b_k$ form a subsequence; Subsequential Limits

Proof. Let $s_n := \sum_{i=1}^{n} a_i$ and $\sigma_k := \sum_{j=1}^{k} b_j$. Claim: $\sigma_k = s_{n_k}$ for every $k \in \mathbb{N}$. Induction on k.

$k = 1$: since $n_0 = 0$,

$$\sigma_1 = b_1 = a_{n_0+1} + \cdots + a_{n_1} = a_1 + \cdots + a_{n_1} = s_{n_1}.$$

Suppose $\sigma_k = s_{n_k}$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} \sigma_{k+1} &= \sigma_k + b_{k+1} = s_{n_k} + \left(a_{n_k+1} + \cdots + a_{n_{k+1}}\right) \\ &= \left(a_1 + \cdots + a_{n_k}\right) + \left(a_{n_k+1} + \cdots + a_{n_{k+1}}\right) = s_{n_{k+1}}. \end{aligned}$$

Hence $\sigma_k = s_{n_k}$ for every $k \in \mathbb{N}$.

$\{n_k\}$ is a strictly increasing sequence of natural numbers $\therefore$ $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$ (def. of subsequence).

$\sum a_n$ sums to s, i.e. $s = \lim s_n$ $\therefore$ $s = \lim s_{n_k}$ (Subsequential Limits).

$\sigma_k = s_{n_k}$ for all $k \in \mathbb{N}$ $\therefore$ $s = \lim \sigma_k$, i.e. $\sum b_k$ converges, with sum s (def. of convergence of a series). $\square$

$$\boxed{\sum a_n \text{ sums to } s \implies \sum b_k \text{ sums to } s.}$$

Method 2: Direct ε–N argument with the index bound $n_k \geq k$

Proof. Let $s_n := \sum_{i=1}^{n} a_i$ and $\sigma_k := \sum_{j=1}^{k} b_j$.

Partial sums. For $k \in \mathbb{N}$, the index sets $\{n_{j-1}+1, \ldots, n_j\}$, $j = 1, \ldots, k$, are disjoint (as $0 = n_0 < n_1 < \cdots < n_k$) and their union is $\{1, \ldots, n_k\}$. Hence

$$\sigma_k = \sum_{j=1}^{k} \sum_{i=n_{j-1}+1}^{n_j} a_i = \sum_{i=1}^{n_k} a_i = s_{n_k}.$$

Index bound. $n_k \geq k$ for every $k \in \mathbb{N}$: $n_1 \in \mathbb{N}$ gives $n_1 \geq 1$; if $n_k \geq k$, then $n_{k+1} > n_k \geq k$, and $n_{k+1} \in \mathbb{N}$, so $n_{k+1} \geq k+1$.

Limit. Let $\varepsilon > 0$. $s = \lim s_n$ $\therefore$ there is $N \in \mathbb{N}$ with $|s_n - s| < \varepsilon$ for all $n \geq N$ (def. of limit). Let $k \geq N$. Then $n_k \geq k \geq N$, so

$$|\sigma_k - s| = |s_{n_k} - s| < \varepsilon.$$

$\therefore$ for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ with $k \geq N \implies |\sigma_k - s| < \varepsilon$ $\therefore$ $s = \lim \sigma_k$ (def. of limit), i.e. $\sum b_k$ sums to s. □

$$\boxed{\sum a_n \text{ sums to } s \implies \sum b_k \text{ sums to } s.}$$

Unofficial - QRS@NTU - ntu.edu.sg

Question A1 (Abel's Test)

Problem

Use the Cauchy Criterion for Series to prove that if the series $\sum a_n$ is convergent and the sequence $\{b_n\}$ is nonnegative and decreasing, then the series $\sum a_n b_n$ is convergent. **Do not use the Monotone Convergence Theorem or the Algebraic Limit Theorem for Series.** *[Note: This is known as Abel's Test.] (Tutorial 4, Additional Question 1.)*

Official hint. (a) Apply the Cauchy Criterion to $\sum a_n$. (b) Abel's Lemma: for $n, m \in \mathbb{N}$ with $m > n$, $\sum_{k=n+1}^{m} a_k b_k = s_m b_m - s_n b_{n+1} + \sum_{k=n+1}^{m-1} s_k(b_k - b_{k+1})$, where s_k is the k-th partial sum of $\sum a_n$. (c) "What happens to Abel's Lemma when $\{s_n\}$ is constant?" Neither the proof of Abel's Lemma, the bounding argument, nor the purpose of (c) is given.

Solution

Method 1: Abel's Lemma with shifted partial sums; Cauchy Criterion for Series

Proof. Let $s_k := \sum_{i=1}^{k} a_i$ for $k \in \mathbb{N}$.

Step 1: summation by parts. Fix $n \in \mathbb{N}$ and put $d_k := s_k - s_n$ for $k \geq n$. Then $d_n = 0$, and for $k \geq n+1$, $d_k - d_{k-1} = s_k - s_{k-1} = a_k$. Let $m > n$. Replacing a_k by $d_k - d_{k-1}$ and shifting the index in the second sum ($k - 1 \mapsto k$),

$$\begin{aligned}
\sum_{k=n+1}^{m} a_k b_k &= \sum_{k=n+1}^{m} d_k b_k - \sum_{k=n+1}^{m} d_{k-1} b_k \\
&= \sum_{k=n+1}^{m} d_k b_k - \sum_{k=n}^{m-1} d_k b_{k+1} \\
&= d_m b_m + \sum_{k=n+1}^{m-1} d_k b_k - d_n b_{n+1} - \sum_{k=n+1}^{m-1} d_k b_{k+1} \\
&= d_m b_m - d_n b_{n+1} + \sum_{k=n+1}^{m-1} d_k (b_k - b_{k+1}) \\
&= (s_m - s_n) b_m + \sum_{k=n+1}^{m-1} (s_k - s_n)(b_k - b_{k+1}) \qquad (d_n = 0).
\end{aligned}$$

This is Abel's Lemma of the hint applied to the partial sums $d_k = s_k - s_n$, whose value at $k = n$ is 0; the term $-d_n b_{n+1}$ vanishes.

Step 2: bounds on $\{b_n\}$. $\{b_n\}$ is decreasing and $b_n \geq 0$ $\therefore$ for every $k \in \mathbb{N}$: $0 \leq b_k \leq b_1$ and $b_k - b_{k+1} \geq 0$.

Step 3: choice of N. Let $\varepsilon > 0$ and $\varepsilon' := \dfrac{\varepsilon}{b_1 + 1} > 0$. $\sum a_n$ converges $\therefore$ there is $N \in \mathbb{N}$ with $|s_k - s_j| < \varepsilon'$ for all $k > j \geq N$ (Cauchy Criterion for Series).

Step 4: the estimate. Let $m > n \geq N$. Then $|s_k - s_n| < \varepsilon'$ for $n < k \leq m$. By Step 1, the triangle inequality and Step 2,

$$\begin{aligned}\left|\sum_{k=n+1}^{m} a_k b_k\right| &\leq |s_m - s_n|\, b_m + \sum_{k=n+1}^{m-1} |s_k - s_n|\,(b_k - b_{k+1}) \\ &\leq \varepsilon' b_m + \varepsilon' \sum_{k=n+1}^{m-1} (b_k - b_{k+1}) \\ &= \varepsilon' b_m + \varepsilon'\big[(b_{n+1} - b_{n+2}) + \cdots + (b_{m-1} - b_m)\big] \\ &= \varepsilon' b_m + \varepsilon'(b_{n+1} - b_m) = \varepsilon' b_{n+1} \\ &\leq \varepsilon' b_1 = \frac{b_1}{b_1 + 1}\,\varepsilon < \varepsilon.\end{aligned}$$

(For $m = n + 1$ the sum over k is empty and the bound reads $\varepsilon' b_m = \varepsilon' b_{n+1}$.)

$\therefore$ for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ with $m > n \geq N \implies \left|\sum_{k=n+1}^{m} a_k b_k\right| < \varepsilon$.

$\therefore$ $\sum a_n b_n$ converges (Cauchy Criterion for Series). □

$$\boxed{\sum a_n \text{ convergent, } \{b_n\} \text{ nonnegative and decreasing } \implies \sum a_n b_n \text{ convergent.}}$$

Unofficial QRS@NTU

Method 2: Abel's Lemma with remainders $r_k = s_k - s$; definition of limit

Proof. Let $s_k := \sum_{i=1}^{k} a_i$ for $k \in \mathbb{N}$.

Step 1: Abel's Lemma. Let $m > n \geq 1$. For $k \geq n+1 \geq 2$, $a_k = s_k - s_{k-1}$. Shifting the index in the second sum $(k-1 \mapsto k)$,

$$\begin{aligned}\sum_{k=n+1}^{m} a_k b_k &= \sum_{k=n+1}^{m} s_k b_k - \sum_{k=n+1}^{m} s_{k-1} b_k \\ &= \sum_{k=n+1}^{m} s_k b_k - \sum_{k=n}^{m-1} s_k b_{k+1} \\ &= s_m b_m + \sum_{k=n+1}^{m-1} s_k b_k - s_n b_{n+1} - \sum_{k=n+1}^{m-1} s_k b_{k+1} \\ &= s_m b_m - s_n b_{n+1} + \sum_{k=n+1}^{m-1} s_k (b_k - b_{k+1}).\end{aligned}$$

Step 2: remainders. $\sum a_n$ converges, so $s := \lim s_k$ exists (def. of convergence of a series). Let $r_k := s_k - s$. Substituting $s_k = s + r_k$ in Step 1,

$$\begin{aligned}\sum_{k=n+1}^{m} a_k b_k = s\Big[b_m - b_{n+1} + \sum_{k=n+1}^{m-1} (b_k - b_{k+1})\Big] \\ + r_m b_m - r_n b_{n+1} + \sum_{k=n+1}^{m-1} r_k (b_k - b_{k+1}).\end{aligned}$$

By telescoping, $\sum_{k=n+1}^{m-1} (b_k - b_{k+1}) = (b_{n+1} - b_{n+2}) + \cdots + (b_{m-1} - b_m) = b_{n+1} - b_m$, so the bracket equals $b_m - b_{n+1} + b_{n+1} - b_m = 0$. Hence

$$\sum_{k=n+1}^{m} a_k b_k = r_m b_m - r_n b_{n+1} + \sum_{k=n+1}^{m-1} r_k (b_k - b_{k+1}).$$

Step 3: choice of N. $\{b_n\}$ is decreasing and nonnegative, so $0 \leq b_k \leq b_1$ and $b_k - b_{k+1} \geq 0$ for every $k \in \mathbb{N}$. Let $\varepsilon > 0$ and $\varepsilon' := \dfrac{\varepsilon}{2b_1 + 1} > 0$. Since $s = \lim s_k$, there is $N \in \mathbb{N}$ with $|r_k| = |s_k - s| < \varepsilon'$ for all $k \geq N$ (def. of limit).

Step 4: the estimate. Let $m > n \geq N$. Every index k with $n \leq k \leq m$ satisfies $k \geq N$, so by the triangle inequality,

$$\begin{aligned}\left|\sum_{k=n+1}^{m} a_k b_k\right| &\leq |r_m|\, b_m + |r_n|\, b_{n+1} + \sum_{k=n+1}^{m-1} |r_k|\, (b_k - b_{k+1}) \\ &\leq \varepsilon' b_m + \varepsilon' b_{n+1} + \varepsilon'(b_{n+1} - b_m) \\ &= 2\varepsilon' b_{n+1} \leq 2\varepsilon' b_1 = \frac{2b_1}{2b_1 + 1}\,\varepsilon < \varepsilon.\end{aligned}$$

$\therefore$ for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ with $m > n \geq N \implies \left|\sum_{k=n+1}^{m} a_k b_k\right| < \varepsilon$.

$\therefore$ $\sum a_n b_n$ converges (Cauchy Criterion for Series). Neither the MCT nor the ALT for Series was used. $\square$

$$\boxed{\sum a_n \text{ convergent, } \{b_n\} \text{ nonnegative and decreasing } \implies \sum a_n b_n \text{ convergent.}}$$

Unofficial - QRS@NTU - qrsntu.org

Question A2 (Infinite products)

Problem

A close relative of the infinite series is the *infinite product*, which is a formal expression of the form

$$\prod b_n := \prod_{n=1}^{\infty} b_n := b_1 b_2 b_3 \cdots ,$$

where $\{b_n\}$ is a sequence of real numbers. This infinite product is understood in terms of its sequence of *partial products* $\{p_n\}$, where, for each $n \in \mathbb{N}$, $p_n := b_1 b_2 b_3 \cdots b_n$. An infinite product is said to be *convergent* if its sequence of partial products is convergent.

(a) Use the Cauchy Criterion to prove that if $\{b_n\}$ and $\{c_n\}$ are sequences for which $1 \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $\prod c_n$ is convergent, then $\prod b_n$ is also convergent. **Do not use the Monotone Convergence Theorem.**

For the rest of this question, we consider the special case of infinite products of the form

$$\prod (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots ,$$

where the sequence $\{a_n\}$ is nonnegative.

(b) Find an explicit formula for the partial products in the case where $a_n = \frac{1}{n}$. Is the infinite product convergent?

(c) Use the Cauchy Criterion to prove that the infinite product $\prod(1 + a_n)$ is convergent if and only if the series $\sum a_n$ is convergent. **Do not use the Monotone Convergence Theorem.**

(Tutorial 4, Additional Question 2.)

Official hint. (a) Follow the proof of the Comparison Test, using the Cauchy Criterion *for sequences* (not series). (b) Each partial product is a natural number; the infinite product is not convergent. (c) "If": use $1+x \leq e^x$. "Only if": use $(1+x)(1+y) \geq 1+x+y$. None of the actual estimates, the explicit formula, or the unboundedness argument in (b) is carried out.

Solution

Method 1: Cauchy Criterion on partial products

Lemma P. *Let $r \in \mathbb{N}$ and $1 \leq x_i \leq y_i$ for $i = 1, \ldots, r$. Then $1 \leq \prod_{i=1}^{r} x_i \leq \prod_{i=1}^{r} y_i$.*

Proof. Induction on r. For $r = 1$ this is the hypothesis $1 \leq x_1 \leq y_1$. Suppose the claim holds for r, and write $X_r := \prod_{i=1}^{r} x_i$, $Y_r := \prod_{i=1}^{r} y_i$, so $1 \leq X_r \leq Y_r$. Since $x_{r+1} \geq 1$ and $X_r > 0$,

$$X_{r+1} = X_r \, x_{r+1} \geq X_r \geq 1.$$

Since $X_r \leq Y_r$ and $x_{r+1} > 0$, and $x_{r+1} \leq y_{r+1}$ and $Y_r > 0$,

$$X_{r+1} = X_r \, x_{r+1} \leq Y_r \, x_{r+1} \leq Y_r \, y_{r+1} = Y_{r+1}.$$

$\therefore$ the claim holds for $r + 1$, hence for every $r \in \mathbb{N}$. $\square$

Part (a).

Proof. Let $p_n := b_1 \cdots b_n$ and $q_n := c_1 \cdots c_n$. By Lemma P, $1 \le p_m \le q_m$ for every $m \in \mathbb{N}$. Let $n > m$ and

$$B := b_{m+1} \cdots b_n, \qquad C := c_{m+1} \cdots c_n,$$

so $p_n = p_m B$, $q_n = q_m C$, and $1 \le B \le C$ (Lemma P). Then

$$\begin{aligned} 0 \le p_n - p_m &= p_m(B-1) && (B-1 \ge 0,\ p_m > 0) \\ &\le q_m(B-1) && (p_m \le q_m,\ B-1 \ge 0) \\ &\le q_m(C-1) && (B \le C,\ q_m > 0) \\ &= q_n - q_m. \end{aligned}$$

Hence $|p_n - p_m| \le |q_n - q_m|$ whenever $n > m$; by symmetry also when $n < m$, and both sides are 0 when $n = m$.

$\prod c_n$ converges, i.e. $\{q_n\}$ converges $\therefore$ $\{q_n\}$ is Cauchy (Cauchy Criterion). Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ with $|q_n - q_m| < \varepsilon$ for all $n, m \ge N$. $\therefore$ for all $n, m \ge N$,

$$|p_n - p_m| \le |q_n - q_m| < \varepsilon.$$

$\therefore$ $\{p_n\}$ is Cauchy $\therefore$ $\{p_n\}$ converges (Cauchy Criterion), i.e. $\prod b_n$ converges. □

(a) $1 \le b_n \le c_n\ \forall n$, $\prod c_n$ convergent $\implies$ $\prod b_n$ convergent.

Part (b).

Proof. For $k \in \mathbb{N}$, $1 + \frac{1}{k} = \frac{k+1}{k}$. Hence, for every $n \in \mathbb{N}$,

$$p_n = \prod_{k=1}^{n} \frac{k+1}{k} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n} = n+1,$$

since the numerator $k+1$ of the k-th factor cancels the denominator of the $(k+1)$-th factor $\frac{k+2}{k+1}$, leaving the last numerator $n+1$ over the first denominator 1.

Let $K \in \mathbb{R}$. There is $N \in \mathbb{N}$ with $N > K$ (Archimedean), and then for every $n \ge N$, $p_n = n + 1 > N > K$. $\therefore$ $p_n \to \infty$; in particular $\{p_n\}$ is unbounded $\therefore$ $\{p_n\}$ diverges (Boundedness of Convergent Sequences). □

(b) $p_n = n + 1$ for all $n \in \mathbb{N}$; the infinite product $\prod(1 + 1/n)$ diverges ($p_n \to \infty$).

Part (c). Let $s_n := \sum_{k=1}^{n} a_k$ and $p_n := \prod_{k=1}^{n}(1 + a_k)$. Since $a_k \ge 0$ for all k: $p_n \ge 1$ (Lemma P with $x_i = y_i = 1 + a_i \ge 1$); $p_{n+1} = p_n(1 + a_{n+1}) \ge p_n$ (as $p_n > 0$, $a_{n+1} \ge 0$); and for $n > m$,

$$\frac{p_n}{p_m} = \prod_{k=m+1}^{n} (1 + a_k), \qquad s_n - s_m = \sum_{k=m+1}^{n} a_k \ge 0.$$

Proof ("if"). Assume $\sum a_n$ converges.

Step 1 (product bound). Claim: for $r \in \mathbb{N}$ and $x_1, \ldots, x_r \geq 0$ with $S_r := \sum_{i=1}^{r} x_i < 1$,

$$\prod_{i=1}^{r}(1+x_i) \leq \frac{1}{1-S_r}.$$

Induction on r. $r=1$: $1-x_1 > 0$ and $(1+x_1)(1-x_1) = 1-x_1^2 \leq 1$, so $1+x_1 \leq \frac{1}{1-x_1}$. Suppose the claim holds for r, and let $S_{r+1} < 1$. Then $S_r \leq S_{r+1} < 1$, so the claim for r applies. Write $x := x_{r+1}$. Then

$$\begin{aligned}(1+x)(1-S_{r+1}) &= (1+x)(1-S_r-x) = 1-S_r-xS_r-x^2 \\ &= (1-S_r) - xS_{r+1} \leq 1-S_r.\end{aligned}$$

Dividing by $(1-S_r)(1-S_{r+1}) > 0$ gives $\frac{1+x}{1-S_r} \leq \frac{1}{1-S_{r+1}}$, hence

$$\prod_{i=1}^{r+1}(1+x_i) \leq \frac{1+x_{r+1}}{1-S_r} \leq \frac{1}{1-S_{r+1}}.$$

Step 2 (a bound for $\{p_n\}$). There is $N_0 \in \mathbb{N}$ with $0 \leq s_n - s_m < \frac{1}{2}$ for all $n > m \geq N_0$ (Cauchy Criterion for Series, $\varepsilon = \frac{1}{2}$). For $n > m \geq N_0$, Step 1 with $x_i := a_{m+i}$ and $r := n-m$ gives

$$\frac{p_n}{p_m} \leq \frac{1}{1-(s_n-s_m)}.$$

With $m = N_0$ and $n > N_0$: $1-(s_n - s_{N_0}) > 1 - \frac{1}{2} = \frac{1}{2}$, so $\frac{p_n}{p_{N_0}} \leq \frac{1}{1-(s_n-s_{N_0})} < 2$, i.e. $p_n < 2p_{N_0}$. For $n \leq N_0$: $p_n \leq p_{N_0}$. Hence $p_n \leq K := 2p_{N_0}$ for every $n \in \mathbb{N}$. For $n > m \geq N_0$, with $y := s_n - s_m \in [0, \frac{1}{2})$, so $1-y > \frac{1}{2}$,

$$\begin{aligned}0 \leq p_n - p_m = p_m\left(\frac{p_n}{p_m} - 1\right) &\leq K\left(\frac{1}{1-y} - 1\right) \\ &= K\,\frac{y}{1-y} \leq 2Ky = 2K(s_n - s_m).\end{aligned}$$

Step 3 (choice of N). Let $\varepsilon > 0$. There is $N' \in \mathbb{N}$ with $0 \leq s_n - s_m < \frac{\varepsilon}{2K+1}$ for all $n > m \geq N'$ (Cauchy Criterion for Series); let $N_1 := \max\{N', N_0\}$, so Step 2 also applies for $n > m \geq N_1$. For $n > m \geq N_1$,

$$0 \leq p_n - p_m \leq 2K(s_n - s_m) < \frac{2K}{2K+1}\,\varepsilon < \varepsilon.$$

By symmetry ($n < m$) and triviality ($n = m$), $|p_n - p_m| < \varepsilon$ for all $n, m \geq N_1$. $\therefore$ $\{p_n\}$ is Cauchy $\therefore$ $\{p_n\}$ converges (Cauchy Criterion). $\square$

Proof ("only if"). Assume $\{p_n\}$ converges.

Step 1 (product–sum bound). Claim: for $r \in \mathbb{N}$ and $x_1, \ldots, x_r \geq 0$, $\prod_{i=1}^{r}(1+x_i) \geq 1 + \sum_{i=1}^{r} x_i$. Induction on r: equality for $r = 1$; if it holds for r, with $S_r := \sum_{i=1}^{r} x_i$,

$$\prod_{i=1}^{r+1}(1+x_i) \geq (1+S_r)(1+x_{r+1}) = 1 + S_r + x_{r+1} + S_r x_{r+1} \geq 1 + S_{r+1},$$

using $1 + x_{r+1} > 0$ and $S_r x_{r+1} \geq 0$.

Step 2. For $n > m$, Step 1 with $x_i := a_{m+i}$ gives $\frac{p_n}{p_m} \geq 1 + (s_n - s_m)$. Since $p_m \geq 1$ and $p_n - p_m \geq 0$,

$$0 \leq s_n - s_m \leq \frac{p_n - p_m}{p_m} \leq p_n - p_m.$$

Step 3. $\{p_n\}$ converges $\therefore$ $\{p_n\}$ is Cauchy (Cauchy Criterion). Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ with $|p_n - p_m| < \varepsilon$ for all $n, m \geq N$. For $n > m \geq N$: $|s_n - s_m| \leq p_n - p_m < \varepsilon$; by symmetry and for $n = m$, the same holds for all $n, m \geq N$.

$\therefore$ $\{s_n\}$ is Cauchy $\therefore$ $\sum a_n$ converges (Cauchy Criterion for Series). $\square$

$$\boxed{\text{(c) } \prod(1 + a_n) \text{ convergent } \iff \sum a_n \text{ convergent} \quad (a_n \geq 0).}$$

Unofficial QRS@NTU - qrs.ntu.org

Method 2: Bolzano–Weierstrass for increasing sequences; induction

This method should be used only if the course allows this interpretation: parts (a) and (c) below replace the requested Cauchy Criterion by the Bolzano–Weierstrass Theorem, and Lemma M is the increasing case of the MCT, reproved from BW, although the question forbids the MCT.

Lemma M. *If $\{x_n\}$ is increasing and bounded, then $\{x_n\}$ converges.*

Proof. $\{x_n\}$ is bounded $\therefore$ it has a subsequence $\{x_{n_k}\}$ converging to some $L \in \mathbb{R}$ (BW).

$x_n \le L$ for every $n \in \mathbb{N}$: fix n; for every $k \ge n$, $n_k \ge k \ge n$ (Subsequence Index Bound), so $x_{n_k} \ge x_n$ ($\{x_n\}$ increasing); hence $L = \lim_k x_{n_k} \ge x_n$ (OLT).

Let $\varepsilon > 0$. There is $K \in \mathbb{N}$ with $|x_{n_k} - L| < \varepsilon$ for all $k \ge K$. Let $N := n_K$. For every $n \ge N$,

$$L \ge x_n \ge x_{n_K} > L - \varepsilon \implies |x_n - L| < \varepsilon.$$

$\therefore$ $L = \lim x_n$ (def. of limit). □

Part (a).

Proof. Let $p_n := b_1 \cdots b_n$, $q_n := c_1 \cdots c_n$. For every $n \in \mathbb{N}$, $p_n \ge 1 > 0$ and $b_{n+1} \ge 1$, so $p_{n+1} = p_n b_{n+1} \ge p_n$: $\{p_n\}$ is increasing. $\{q_n\}$ converges $\therefore$ there is $Q > 0$ with $|q_n| \le Q$ for all n (Boundedness of Convergent Sequences). By Lemma P (Method 1),

$$1 \le p_n \le q_n \le Q \qquad (n \in \mathbb{N}).$$

$\therefore$ $\{p_n\}$ is increasing and bounded $\therefore$ converges (Lemma M). □

(a) $1 \le b_n \le c_n$ $\forall n$, $\prod c_n$ convergent $\implies$ $\prod b_n$ convergent.

Part (b).

Proof. Claim: $p_n = n + 1$ for every $n \in \mathbb{N}$, by induction. $p_1 = 1 + \frac{1}{1} = 2$. If $p_n = n + 1$, then

$$p_{n+1} = p_n \left(1 + \frac{1}{n+1}\right) = (n+1) \cdot \frac{n+2}{n+1} = n + 2.$$

Divergence: $a_n = \frac{1}{n} \ge 0$ and $\sum \frac{1}{n}$ diverges (Lemma C). If $\prod(1 + 1/n)$ converged, then $\sum \frac{1}{n}$ would converge by part (c) ("only if", Method 1); this is false. $\therefore$ $\prod(1 + 1/n)$ diverges. □

(b) $p_n = n + 1$ for all $n \in \mathbb{N}$; the infinite product $\prod(1 + 1/n)$ diverges ($p_n \to \infty$).

Part (c).

Proof. Let $s_n := \sum_{k=1}^{n} a_k$ and $p_n := \prod_{k=1}^{n}(1 + a_k)$; both sequences are increasing since $a_k \ge 0$.

"If": assume $\sum a_n$ converges. By Step 2 of the "if" proof in Method 1 (which uses only the product bound of Step 1 and the Cauchy Criterion for Series with $\varepsilon = \frac{1}{2}$), $1 \le p_n \le K$ for all $n \in \mathbb{N}$. $\therefore$ $\{p_n\}$ is increasing and bounded $\therefore$ converges (Lemma M).

"Only if": assume $\{p_n\}$ converges. There is $P > 0$ with $p_n \le P$ for all n (Boundedness of Convergent Sequences). By the product–sum bound (Method 1, "only if", Step 1, with $x_i := a_i$, $r := n$),

$$0 \le s_n \le p_n - 1 \le P - 1 \qquad (n \in \mathbb{N}).$$

$\therefore$ $\{s_n\}$ is increasing and bounded $\therefore$ converges (Lemma M), i.e. $\sum a_n$ converges. $\square$

(c) $\prod(1+a_n)$ convergent $\iff$ $\sum a_n$ convergent $\quad (a_n \geq 0)$.

Unofficial - QRS@NTU - qrsntu.org

Method 3: Exponential bound $1 + x \le e^x$, for part (c)

Note: uses the exponential function $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$ and $e^{x+y} = e^x e^y$, which are not course material in Lectures 1–17.

Proof ("if"). Assume $\sum a_n$ converges, with sum s. Let $s_n := \sum_{k=1}^{n} a_k$ and $p_n := \prod_{k=1}^{n}(1 + a_k)$.

Facts. For $x \ge 0$: $e^x = 1 + x + \frac{x^2}{2!} + \cdots \ge 1 + x$. For $j \in \mathbb{N}$, $j! = 1 \cdot 2 \cdots j \ge 2^{j-1}$, so $\sum_{j=1}^{\infty} \frac{1}{j!} \le \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} = 2$. Hence, for $0 \le y \le 1$ (so $y^{j-1} \le 1$):

$$e^y - 1 = \sum_{j=1}^{\infty} \frac{y^j}{j!} = y \sum_{j=1}^{\infty} \frac{y^{j-1}}{j!} \le y \sum_{j=1}^{\infty} \frac{1}{j!} \le 2y.$$

For $x \le y$: $e^y = e^x e^{y-x} \ge e^x\big(1 + (y - x)\big) \ge e^x$.

Estimate. For $n > m$, $1 \le 1 + a_k \le e^{a_k}$ for each k, so by Lemma P and $e^{x+y} = e^x e^y$,

$$\frac{p_n}{p_m} = \prod_{k=m+1}^{n} (1 + a_k) \le \prod_{k=m+1}^{n} e^{a_k} = e^{s_n - s_m}, \qquad p_m \le e^{s_m} \le e^s.$$

($s_m \le s$: for $j \ge m$, $s_j \ge s_m$, so $s = \lim s_j \ge s_m$ by the OLT.) Hence, when $0 \le s_n - s_m \le 1$,

$$0 \le p_n - p_m = p_m \left(\frac{p_n}{p_m} - 1 \right) \le e^s\big(e^{s_n - s_m} - 1\big) \le 2e^s(s_n - s_m).$$

Choice of N. Let $\varepsilon > 0$ and $\varepsilon' := \min\big\{1, \frac{\varepsilon}{2e^s + 1}\big\}$. There is $N \in \mathbb{N}$ with $0 \le s_n - s_m < \varepsilon'$ for all $n > m \ge N$ (Cauchy Criterion for Series). For $n > m \ge N$,

$$0 \le p_n - p_m \le 2e^s(s_n - s_m) < \frac{2e^s}{2e^s + 1}\,\varepsilon < \varepsilon.$$

By symmetry and for $n = m$, $|p_n - p_m| < \varepsilon$ for all $n, m \ge N$. $\therefore$ $\{p_n\}$ is Cauchy $\therefore$ converges (Cauchy Criterion). □

Proof ("only if"). Assume $\{p_n\}$ converges. For $n > m$, the product–sum bound (Method 1, "only if", Step 1) gives $\frac{p_n}{p_m} \ge 1 + (s_n - s_m)$, so $0 \le s_n - s_m \le \frac{p_n - p_m}{p_m} \le p_n - p_m$ (as $p_m \ge 1$). $\{p_n\}$ is Cauchy (Cauchy Criterion); given $\varepsilon > 0$, the N that works for $\{p_n\}$ gives $|s_n - s_m| < \varepsilon$ for all $n, m \ge N$. $\therefore$ $\{s_n\}$ is Cauchy $\therefore$ $\sum a_n$ converges (Cauchy Criterion for Series). □

$$\boxed{\text{(c) } \prod(1 + a_n) \text{ convergent } \iff \sum a_n \text{ convergent} \quad (a_n \ge 0).}$$

Question A3 (Alternating Series Test via NIP)

Problem

The Alternating Series Test can be proved directly from the Nested Interval Property.

(a) Given an alternating series $b_1 - b_2 + b_3 - b_4 + \cdots$ with the sequence $\{b_n\}$ decreasing and converging to 0, construct a nested sequence of closed and bounded intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, whose lengths converge to 0, such that for each $n, m \in \mathbb{N}$, $m \geq n \implies s_m \in I_n$, where s_m denotes the m-th partial sum of the alternating series.

(b) Hence use the Nested Interval Property to deduce a sum for the alternating series.

(Tutorial 4, Additional Question 3.)

Official hint. (a) Recall that the lecture proof of the Alternating Series Test establishes that for every $N, n \in \mathbb{N}$, $n > N \implies 0 \leq b_{N+1} - b_{N+2} + \cdots + (-1)^{n-N-1} b_n \leq b_{N+1}$. (b) The Nested Interval Property gives a real number s; show s is the sum via the definition of a limit, remembering $\{b_n\} \to 0$. No explicit interval, no nesting check, and no final ε–N argument are given.

Solution

Method 1: Nested intervals from consecutive partial sums

Part (a). Let $s_0 := 0$ and $s_m := \sum_{k=1}^{m} (-1)^{k-1} b_k$ for $m \in \mathbb{N}$. Define, for $n \in \mathbb{N}$,

$$I_n := \big[\min(s_{n-1}, s_n),\ \max(s_{n-1}, s_n)\big],$$

a closed and bounded interval with endpoints s_{n-1} and s_n.

Proof (signs of b_n). Fix $n \in \mathbb{N}$. For every $k \geq n$, $b_k \leq b_n$ ($\{b_n\}$ decreasing), and $0 = \lim_k b_k$. $\therefore$ $0 \leq b_n$ (OLT). Hence $b_n \geq 0$ for every $n \in \mathbb{N}$. □

Proof (containment). Fix $n \in \mathbb{N}$ and $m \geq n$. Recall the lecture inequality (AST proof): for $N, m \in \mathbb{N}$ with $m > N$,

$$0 \leq b_{N+1} - b_{N+2} + \cdots + (-1)^{m-N-1} b_m \leq b_{N+1}. \tag{LI}$$

Case $n \geq 2$. Apply (LI) with $N := n - 1 \in \mathbb{N}$ (so $m \geq n > N$):

$$0 \leq \sum_{k=n}^{m} (-1)^{k-n} b_k \leq b_n.$$

For every k, $(-1)^{k-1} = (-1)^{n-1}(-1)^{k-n}$, so with $\sigma := (-1)^{n-1} \in \{1, -1\}$,

$$s_m - s_{n-1} = \sum_{k=n}^{m} (-1)^{k-1} b_k = \sigma \sum_{k=n}^{m} (-1)^{k-n} b_k, \qquad s_n - s_{n-1} = \sigma b_n.$$

Since $\sigma^2 = 1$, the inequality above reads

$$0 \leq \sigma(s_m - s_{n-1}) \leq \sigma(s_n - s_{n-1}).$$

If $\sigma = 1$: $s_{n-1} \le s_m \le s_n$. If $\sigma = -1$: $s_n \le s_m \le s_{n-1}$. In both cases s_m lies between s_{n-1} and s_n, i.e. $s_m \in I_n$.

Case $n = 1$. I_1 has endpoints $s_0 = 0$ and $s_1 = b_1 \ge 0$, so $I_1 = [0, b_1]$. For $m = 1$: $s_1 = b_1 \in I_1$. For $m \ge 2$:

$$s_m = b_1 - \big(b_2 - b_3 + \cdots + (-1)^{m-2} b_m\big),$$

and the bracket lies in $[0, b_2]$ by (LI) with $N = 1$. Since $0 \le b_2 \le b_1$,

$$0 \le b_1 - b_2 \le s_m \le b_1,$$

i.e. $s_m \in I_1$.

$\therefore$ for all $n, m \in \mathbb{N}$, $m \ge n \implies s_m \in I_n$. □

Proof (nesting). Let $n \in \mathbb{N}$. By containment with $m = n$ and $m = n + 1$, $s_n \in I_n$ and $s_{n+1} \in I_n$. I_n is an interval, so it contains every number between s_n and s_{n+1}; I_{n+1} is exactly the set of those numbers. $\therefore$ $I_{n+1} \subseteq I_n$ for every $n \in \mathbb{N}$. □

Proof (lengths). For $n \in \mathbb{N}$, the length of I_n is

$$|I_n| = |s_n - s_{n-1}| = \big|(-1)^{n-1} b_n\big| = b_n \qquad (b_n \ge 0),$$

and $0 = \lim b_n$. $\therefore$ the lengths converge to 0. □

Part (b).

Proof. $\{I_n\}$ is a nested sequence of closed and bounded intervals $\therefore$ there is $s \in \mathbb{R}$ with $s \in I_n$ for every $n \in \mathbb{N}$ (NIP).

Uniqueness. If also $s' \in I_n$ for every n, then $0 \le |s - s'| \le |I_n| = b_n$ for every n; $0 = \lim b_n$ $\therefore$ $|s - s'| \le 0$ (OLT) $\therefore$ $s = s'$.

Limit. Let $\varepsilon > 0$. $0 = \lim b_n$ $\therefore$ there is $N \in \mathbb{N}$ with $b_N = |b_N - 0| < \varepsilon$ (def. of limit). For every $m \ge N$: $s_m \in I_N$ (part (a)) and $s \in I_N$, so

$$|s_m - s| \le |I_N| = b_N < \varepsilon.$$

$\therefore$ $s = \lim s_m$ (def. of limit), i.e. the series converges, with sum s. □

$$\boxed{b_1 - b_2 + b_3 - \cdots \text{ converges to the unique } s \in \bigcap_{n=1}^{\infty} I_n.}$$

Compiled by QRS@NTU

Method 2: Nested intervals from even and odd partial sums

Part (a). Let $s_m := \sum_{k=1}^{m}(-1)^{k-1}b_k$. As in Method 1, $b_n \geq 0$ for every $n \in \mathbb{N}$ (OLT).

Proof (monotonicity). For every $j \in \mathbb{N}$, since $\{b_n\}$ is decreasing and nonnegative,

$$\begin{aligned} s_{2j+2} - s_{2j} &= b_{2j+1} - b_{2j+2} \geq 0, \\ s_{2j+1} - s_{2j-1} &= -b_{2j} + b_{2j+1} = -(b_{2j} - b_{2j+1}) \leq 0, \\ s_{2j-1} - s_{2j} &= b_{2j} \geq 0. \end{aligned}$$

So the even partial sums increase, the odd partial sums decrease, and $s_{2j} \leq s_{2j-1}$. Hence, for $i \leq j$,

$$s_{2i} \leq s_{2j} \leq s_{2j-1} \leq s_{2i-1}. \qquad (*)$$

$\therefore$ $(*)$ holds for all $i, j \in \mathbb{N}$ with $i \leq j$. □

Define, for $j \in \mathbb{N}$,

$$I_{2j-1} := I_{2j} := [s_{2j},\ s_{2j-1}],$$

a closed and bounded interval (by $(*)$ with $i = j$).

Proof (containment, nesting, lengths). *Containment.* Let $n \in \mathbb{N}$ and $m \geq n$; write $n = 2j-1$ or $n = 2j$, so $m \geq 2j-1$. If $m = 2i$, then $2i \geq 2j-1$ gives $i \geq j$; if $m = 2i-1$, then $2i - 1 \geq 2j - 1$ gives $i \geq j$. In both cases $(*)$ gives $s_{2j} \leq s_{2i} \leq s_m \leq s_{2i-1} \leq s_{2j-1}$, i.e. $s_m \in I_n$.

Nesting. $I_{2j} = I_{2j-1}$, and by $(*)$ with $i = j,\ j+1$: $s_{2j} \leq s_{2j+2} \leq s_{2j+1} \leq s_{2j-1}$, so $I_{2j+1} = [s_{2j+2}, s_{2j+1}] \subseteq [s_{2j}, s_{2j-1}] = I_{2j}$.

Lengths. $|I_{2j-1}| = |I_{2j}| = s_{2j-1} - s_{2j} = b_{2j}$. For $n \in \{2j-1, 2j\}$, $2j \geq n$, so $0 \leq |I_n| = b_{2j} \leq b_n$; since $0 = \lim b_n$, $|I_n| \to 0$ (Squeeze Theorem). □

Part (b).

Proof. $\{I_n\}$ is a nested sequence of closed and bounded intervals $\therefore$ there is $s \in \mathbb{R}$ with $s \in I_n$ for every $n \in \mathbb{N}$ (NIP). If also $s' \in I_n$ for every n, then $0 \leq |s - s'| \leq |I_n| \leq b_n$ for every n, so $s = s'$ (OLT).

Let $\varepsilon > 0$. There is $N \in \mathbb{N}$ with $b_N < \varepsilon$ (def. of limit). For every $m \geq N$: $s_m \in I_N$ (containment) and $s \in I_N$, so

$$|s_m - s| \leq |I_N| \leq b_N < \varepsilon.$$

$\therefore$ $s = \lim s_m$ (def. of limit), i.e. the series converges, with sum s. □

$$\boxed{b_1 - b_2 + b_3 - \cdots \text{ converges to the unique } s \in \bigcap_{n=1}^{\infty} I_n.}$$

Method Map

Q.	Core topic	Methods included
1	Convergence of three series	M1 Comparison Test and unbounded partial sums; M2 Cauchy Criterion for Series with an explicit N; M3 MCT on bounded partial sums ((a), (b)); M4 contradiction via the ALT for Series ((c)); M5 Ratio Test ((a), beyond scope).
2	True/false with examples	M1 explicit examples, ALT for Series; M2 disjoint-support and indicator examples, Cauchy Criterion for Series.
3	Proof or counterexample	M1 Comparison Test with a global bound, explicit counterexample, contrapositive; M2 Cauchy-tail estimates, a second counterexample, contradiction via Question 4.
4	Cauchy partial sums	M1 triangle inequality on partial-sum differences; M2 contradiction with the negated Cauchy condition.
5	Regrouping preserves the sum	M1 subsequence of partial sums, Subsequential Limits; M2 direct ε–N argument with $n_k \geq k$.
A1	Abel's Test	M1 Abel's Lemma with shifted partial sums, Cauchy Criterion for Series; M2 Abel's Lemma with remainders $s_k - s$.
A2	Infinite products	M1 Cauchy Criterion on partial products, telescoping, product bounds $\prod(1+x_i) \leq \frac{1}{1-\sum x_i}$ and $\prod(1+x_i) \geq 1+\sum x_i$; M2 Bolzano–Weierstrass for increasing sequences, induction; M3 exponential bound $1+x \leq e^x$ (beyond scope).
A3	Alternating Series Test via NIP	M1 nested intervals from consecutive partial sums; M2 nested intervals from even and odd partial sums.

Final Answers

1. (a) Convergent. (b) Convergent. (c) Divergent.
2. (a) True. (b) True. (c) False. (d) True.
3. (a) True. (b) False. (c) True.
4. $\sum |a_n|$ Cauchy partial sums, $\{b_n\}$ bounded $\implies \sum a_n b_n$ Cauchy partial sums.
5. $\sigma_k = s_{n_k}$, so $\sum b_k$ sums to s.

A1. $\sum a_n b_n$ converges (Abel's Test).

A2. (a) $\prod b_n$ converges. (b) $p_n = n+1$; divergent. (c) $\prod(1+a_n)$ converges iff $\sum a_n$ converges.

A3. $I_n = [\min(s_{n-1}, s_n), \max(s_{n-1}, s_n)]$; the series converges to the unique $s \in \bigcap_n I_n$.