MH3100 Real Analysis I

Tutorial 5 Solutions

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Overview

- **Lecture coverage:** Lectures 13–17 — limit points, isolated points and closures (L13–14); closed sets (L15); extrema of closed and bounded sets and (sequential) compactness (L16); the Nested Set Property (L17).
- **Questions:** 8 questions (5 main + 3 Additional) from Tutorial 5.
- **Notation:** $\mathbb{N} = \{1, 2, 3, \dots\}$ (0 is not a natural number in this course); closure of A is $\overline{A} = \overline{A}$. **This course never uses open sets, open covers, or the Heine–Borel Theorem**: compactness is always *(sequentially) compact* in the sense of the course's own definition (every sequence in the set has a subsequence converging to a limit in the set).

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Final Answers **32**

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Question 1 (Limit points, isolated points, closures of five sets)

Problem

For each of the following sets, determine (i) the set of all its limit points; (ii) the set of all its isolated points; and (iii) its closure.

(a) $A = \left\{(-1)^n + \frac{2}{n} : n \in \mathbb{N}\right\}$.

(b) $B = \{x \in \mathbb{Q} : 0 < x < 1\}$.

(c) $C = \{x \in \mathbb{R} : x \neq 0\}$.

(d) $D = \left\{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$.

(e) $E = \left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} : n \in \mathbb{N}\right\}$.

(Tutorial 5, Question 1; Lectures 13–15, limit points / isolated points / closure.)

Solution

Method 1: Sequential Criterion for Limit Points

For a set $S \subseteq \mathbb{R}$ write $L(S)$ for the set of limit points of S. By the definition of isolated point, the isolated points of S are exactly the elements of $S \setminus L(S)$. By the definition of closure, $\overline{S} = S \cup L(S)$. Four of the five sets are the range of a sequence, so we first prove a lemma that turns "limit point of the range" into "limit of a subsequence".

Lemma (range of an injective sequence). Let $\{a_n\}$ be a sequence with $a_m \neq a_n$ whenever $m \neq n$, and let $S := \{a_n : n \in \mathbb{N}\}$. Then for every $x \in \mathbb{R}$,

$$x \in L(S) \iff \text{some subsequence of } \{a_n\} \text{ converges to } x.$$

Proof of Lemma. ($\Rightarrow$) Suppose $x \in L(S)$. By the Sequential Criterion for Limit Points there is a sequence $\{y_j\}$ contained in $S \setminus \{x\}$ with $x = \lim y_j$. For each $j \in \mathbb{N}$ we have $y_j \in S$, so $y_j = a_{k_j}$ for an index $k_j \in \mathbb{N}$, and this index is unique because $\{a_n\}$ is injective. *Claim 1: for every $M \in \mathbb{N}$ there is $j \in \mathbb{N}$ with $k_j > M$.* Suppose instead that some $M \in \mathbb{N}$ satisfies $k_j \leq M$ for every $j \in \mathbb{N}$. Then every y_j lies in the finite set $F := \{a_k : 1 \leq k \leq M,\ a_k \neq x\}$, which is nonempty because it contains y_1. Let

$$\delta := \min\{|f - x| : f \in F\}.$$

Each $|f - x|$ with $f \in F$ is positive, and the minimum of finitely many positive numbers is positive, so $\delta > 0$. Hence $|y_j - x| \geq \delta$ for every $j \in \mathbb{N}$. But $x = \lim y_j$, so there is $J \in \mathbb{N}$ with $|y_J - x| < \delta$ (def. of limit), a contradiction. Therefore Claim 1 holds.
Claim 2: for every $J, M \in \mathbb{N}$ there is $j \geq J$ with $k_j > M$. Let $J, M \in \mathbb{N}$, and let

$$M' := \max\{M, k_1, \ldots, k_{J-1}\} \qquad (\text{with } M' := M \text{ if } J = 1).$$

Claim 1 gives $j \in \mathbb{N}$ with $k_j > M'$. Since $k_i \le M'$ for every $i < J$, this j satisfies $j \ge J$, and $k_j > M' \ge M$.

Now choose indices recursively: $j_1 := 1$, and, given j_i, Claim 2 with $J = j_i + 1$ and $M = k_{j_i}$ gives $j_{i+1} > j_i$ with $k_{j_{i+1}} > k_{j_i}$. Put $n_i := k_{j_i}$. Then $n_1 < n_2 < \cdots$, so $\{a_{n_i}\}$ is a subsequence of $\{a_n\}$. Also $a_{n_i} = y_{j_i}$ with $j_1 < j_2 < \cdots$, so $\{a_{n_i}\}$ is a subsequence of $\{y_j\}$. Therefore $x = \lim a_{n_i}$ (Subsequential Limits).

($\Leftarrow$) Suppose $x = \lim a_{n_i}$ for a subsequence with $n_1 < n_2 < \cdots$. The indices n_i are distinct and $\{a_n\}$ is injective, so the terms a_{n_i} are pairwise distinct; hence $a_{n_i} = x$ for at most one i. Let i_0 be that i if it exists, and $i_0 := 0$ otherwise, and put $z_i := a_{n_{i+i_0}}$ for $i \in \mathbb{N}$. Then $z_i \in S \setminus \{x\}$ for every $i \in \mathbb{N}$. The sequence $\{z_i\}$ is a subsequence of $\{a_{n_i}\}$, so $x = \lim z_i$ (Subsequential Limits). Therefore $x \in L(S)$ (Sequential Criterion for Limit Points). □

(a) $A = \{(-1)^n + 2/n : n \in \mathbb{N}\}$.

Proof. Let $a_n := (-1)^n + \dfrac{2}{n}$, so that $A = \{a_n : n \in \mathbb{N}\}$. For $k \in \mathbb{N}$ the odd- and even-indexed terms are

$$a_{2k-1} = -1 + \frac{2}{2k-1}, \qquad a_{2k} = 1 + \frac{2}{2k} = 1 + \frac{1}{k}.$$

Injectivity. If n is odd, then $0 < \frac{2}{n} \le 2$, so $-1 < a_n \le 1$. If n is even, then $a_n = 1 + \frac{2}{n} > 1$. Hence an odd-indexed term never equals an even-indexed term. If $m < n$ have the same parity, then $\frac{2}{m} > \frac{2}{n}$, so $a_m > a_n$. Therefore $a_m \ne a_n$ whenever $m \ne n$, and the Lemma applies with $S = A$.

Every limit point is ± 1. Let $x \in L(A)$. By the Lemma, $x = \lim a_{n_i}$ for some subsequence $\{a_{n_i}\}$. Since $\mathbb{N} = \{i : n_i \text{ odd}\} \cup \{i : n_i \text{ even}\}$ is infinite, at least one of these two index sets is infinite.

Suppose $\{i : n_i \text{ odd}\}$ is infinite, and list it as $i_1 < i_2 < \cdots$. Put $m_l := n_{i_l}$; then m_l is odd and $m_l \ge i_l \ge l$ (Subsequence Index Bound), so

$$0 < \frac{2}{m_l} \le \frac{2}{l} \quad \text{for every } l \in \mathbb{N}.$$

Since $\frac{2}{l} \to 0$ (Archimedean), the Squeeze Theorem gives $\frac{2}{m_l} \to 0$. Hence

$$a_{m_l} = -1 + \frac{2}{m_l} \to -1 + 0 = -1 \quad \text{(ALT)}.$$

The sequence $\{a_{m_l}\}$ is a subsequence of $\{a_{n_i}\}$, so it also converges to x (Subsequential Limits). Therefore $x = -1$ (Uniqueness of Limits).

If instead $\{i : n_i \text{ even}\}$ is infinite, the same steps with $a_{m_l} = 1 + \frac{2}{m_l} \to 1$ give $x = 1$. Therefore $L(A) \subseteq \{-1, 1\}$.

Both ± 1 *are limit points.* Since $2k - 1 \ge k$,

$$0 < \frac{2}{2k-1} \le \frac{2}{k} \quad \text{for every } k \in \mathbb{N},$$

so $\frac{2}{2k-1} \to 0$ (Squeeze), and hence $a_{2k-1} = -1 + \frac{2}{2k-1} \to -1$ (ALT). Also $\frac{1}{k} \to 0$ (Archimedean), so $a_{2k} = 1 + \frac{1}{k} \to 1$ (ALT). Both $\{a_{2k-1}\}$ and $\{a_{2k}\}$ are subsequences of $\{a_n\}$, so $-1 \in L(A)$ and $1 \in L(A)$ (Lemma). Therefore $L(A) = \{-1, 1\}$.

Membership. $a_1 = -1 + 2 = 1$, so $1 \in A$. Every odd-indexed term satisfies $a_n > -1$ and every even-indexed term satisfies $a_n > 1$, so $-1 \notin A$.

Therefore the isolated points are $A \setminus L(A) = A \setminus \{-1, 1\} = A \setminus \{1\}$, and $\overline{A} = A \cup \{-1, 1\} = A \cup \{-1\}$. □

(a) Limit points $\{-1, 1\}$; isolated points $A \setminus \{1\}$; $\overline{A} = A \cup \{-1\}$.

Density Lemma. If $0 \le a < b$, then there is $q \in \mathbb{Q}$ with $a < q < b$.

Proof of Density Lemma. Since $b - a > 0$, the Archimedean Property gives $m \in \mathbb{N}$ with

$$m > \frac{1}{b-a}, \quad \text{hence} \quad m(b-a) > 1.$$

The set $\{j \in \mathbb{N} : j > ma\}$ is nonempty (Archimedean), so it has a least element k (well-ordering of $\mathbb{N}$). Then $k > ma$. Also $k - 1 \le ma$: if $k = 1$, then $k - 1 = 0 \le ma$ because $a \ge 0$; if $k \ge 2$, then $k - 1 \in \mathbb{N}$ is not in the set, so $k - 1 \le ma$. Hence

$$ma < k \le ma + 1 < ma + m(b-a) = mb.$$

Dividing by $m > 0$ gives $a < \frac{k}{m} < b$, and $q := \frac{k}{m} \in \mathbb{Q}$. □

(b) $B = \mathbb{Q} \cap (0, 1)$**.**

Proof. *Every $x \in [0, 1]$ is a limit point.* Let $x \in [0, 1]$ and $n \in \mathbb{N}$.

If $x < 1$, then $0 \le x < \min\{x + \frac{1}{n}, 1\}$, so the Density Lemma gives $q_n \in \mathbb{Q}$ with

$$x < q_n < \min\Big\{x + \frac{1}{n},\ 1\Big\}.$$

Then $0 \le x < q_n < 1$, so $q_n \in B$, $q_n \ne x$, and $0 < q_n - x < \frac{1}{n}$.

If $x = 1$, then $0 \le \max\{1 - \frac{1}{n}, 0\} < 1$, so the Density Lemma gives $q_n \in \mathbb{Q}$ with

$$\max\Big\{1 - \frac{1}{n},\ 0\Big\} < q_n < 1.$$

Then $0 < q_n < 1$, so $q_n \in B$, $q_n \ne 1$, and $0 < 1 - q_n < \frac{1}{n}$.

In both cases $\{q_n\}$ is contained in $B \setminus \{x\}$ and $|q_n - x| < \frac{1}{n}$ for every $n \in \mathbb{N}$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$ (Archimedean). Then for every $n \ge N$,

$$|q_n - x| < \frac{1}{n} \le \frac{1}{N} < \varepsilon.$$

Hence $x = \lim q_n$ (def. of limit), and therefore $x \in L(B)$ (Sequential Criterion for Limit Points).

No $x \notin [0, 1]$ is a limit point. Let $x < 0$ and suppose $x \in L(B)$. The Sequential Criterion for Limit Points gives a sequence $\{y_n\}$ in $B \setminus \{x\}$ with $x = \lim y_n$. Every y_n satisfies $y_n > 0$, so $x = \lim y_n \ge 0$ (Order Limit Theorem), which contradicts $x < 0$. Let $x > 1$ and suppose $x \in L(B)$; the same criterion gives $\{y_n\}$ in $B \setminus \{x\}$ with $x = \lim y_n$, and $y_n < 1$ for every n gives $x \le 1$ (Order Limit Theorem), a contradiction.

Therefore $L(B) = [0, 1]$. Since $B \subseteq [0, 1]$, the isolated points are $B \setminus [0, 1] = \emptyset$, and $\overline{B} = B \cup [0, 1] = [0, 1]$. □

(b) Limit points $[0, 1]$; isolated points $\emptyset$; $\overline{B} = [0, 1]$.

(c) $C = \mathbb{R} \setminus \{0\}$.

Proof. Let $x \in \mathbb{R}$. Define

$$y_n := \begin{cases} x + \dfrac{1}{n}, & x \geq 0, \\ x - \dfrac{1}{n}, & x < 0, \end{cases} \qquad n \in \mathbb{N}.$$

If $x \geq 0$, then $y_n > x \geq 0$; if $x < 0$, then $y_n < x < 0$. In both cases $y_n \neq 0$ and $y_n \neq x$, so $\{y_n\}$ is contained in $C \setminus \{x\}$. Also $|y_n - x| = \frac{1}{n} \to 0$ (Archimedean), so $x = \lim y_n$. Therefore $x \in L(C)$ (Sequential Criterion for Limit Points). Since $x \in \mathbb{R}$ was arbitrary, $L(C) = \mathbb{R}$. Hence the isolated points are $C \setminus \mathbb{R} = \emptyset$, and $\overline{C} = C \cup \mathbb{R} = \mathbb{R}$. □

(c) Limit points $\mathbb{R}$; isolated points $\emptyset$; $\overline{C} = \mathbb{R}$.

(d) $D = \{1 + \frac{1}{4} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\}$.

Proof. Let $s_n := \sum_{k=1}^{n} \frac{1}{k^2}$, so that $D = \{s_n : n \in \mathbb{N}\}$.
Monotone and injective. $s_{n+1} - s_n = \frac{1}{(n+1)^2} > 0$, so $\{s_n\}$ is strictly increasing; hence $m < n$ implies $s_m < s_n$, and $\{s_n\}$ is injective.
Bounded above. For $k \geq 2$ we have $k^2 > k(k-1)$, so

$$\frac{1}{k^2} < \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}.$$

Summing from $k = 2$ to n (for $n \geq 2$) and telescoping,

$$\begin{aligned} s_n &< 1 + \sum_{k=2}^{n} \Big(\frac{1}{k-1} - \frac{1}{k} \Big) \\ &= 1 + \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \cdots + \Big(\frac{1}{n-1} - \frac{1}{n}\Big) \\ &= 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n} < 2, \end{aligned}$$

and $s_1 = 1 < 2$. Hence $\{s_n\}$ is increasing and bounded above by 2, so $s := \lim s_n = \sup\{s_n : n \in \mathbb{N}\}$ exists (MCT); by the definition of the sum of a series, $s = \sum_{n=1}^{\infty} 1/n^2$.
Limit points. Every subsequence of $\{s_n\}$ converges to s (Subsequential Limits). By the Lemma, $s \in L(D)$; and if $x \in L(D)$, then x is the limit of a subsequence, so $x = s$ (Uniqueness of Limits). Therefore $L(D) = \{s\}$.
$s \notin D$. For every $n \in \mathbb{N}$, s is an upper bound of D and $s_{n+1} > s_n$, so

$$s \geq s_{n+1} > s_n.$$

Hence $s \neq s_n$ for every $n \in \mathbb{N}$, i.e. $s \notin D$.

Therefore the isolated points are $D \setminus \{s\} = D$, and $\overline{D} = D \cup \{s\}$. □

(d) Limit points $\{s\}$, $s = \sum_{n=1}^{\infty} 1/n^2$; isolated points D; $\overline{D} = D \cup \{s\}$.

(e) $E = \{1 + \frac{1}{2} + \cdots + \frac{1}{n} : n \in \mathbb{N}\}$.

Proof. Let $t_n := \sum_{k=1}^{n} \frac{1}{k}$, so that $E = \{t_n : n \in \mathbb{N}\}$. $t_{n+1} - t_n = \frac{1}{n+1} > 0$, so $\{t_n\}$ is strictly increasing and hence injective.
Growth along powers of 2. Let $N \in \mathbb{N}$. For $N+1 \le k \le 2N$ we have $\frac{1}{k} \ge \frac{1}{2N}$, and there are N such k, so

$$t_{2N} - t_N = \sum_{k=N+1}^{2N} \frac{1}{k} \ge N \cdot \frac{1}{2N} = \frac{1}{2}.$$

We show by induction that $t_{2^m} \ge 1 + \frac{m}{2}$ for every $m \in \mathbb{N}_0$. For $m = 0$: $t_1 = 1$. If $t_{2^m} \ge 1 + \frac{m}{2}$, then, applying the display with $N = 2^m$,

$$t_{2^{m+1}} = t_{2^m} + \left(t_{2.2^m} - t_{2^m}\right) \ge 1 + \frac{m}{2} + \frac{1}{2} = 1 + \frac{m+1}{2}.$$

Every subsequence diverges. Let $\{t_{n_i}\}$ be a subsequence and let $M > 0$. Choose $m \in \mathbb{N}$ with $m > 2M$ (Archimedean) and put $i := 2^m$. Then $n_i \ge i$ (Subsequence Index Bound), and $\{t_n\}$ is increasing, so

$$t_{n_i} \ge t_i = t_{2^m} \ge 1 + \frac{m}{2} > M.$$

Since $M > 0$ was arbitrary, $\{t_{n_i}\}$ is unbounded, so it diverges (Boundedness of Convergent Sequences). Hence no subsequence of $\{t_n\}$ converges, and the Lemma gives $L(E) = \emptyset$.
Therefore the isolated points are $E \setminus \emptyset = E$, and $\overline{E} = E \cup \emptyset = E$. □

(e) Limit points $\emptyset$; isolated points E; $\overline{E} = E$.

Set	**Limit points**	**Isolated points**	**Closure**
A	$\{-1, 1\}$	$A \setminus \{1\}$	$A \cup \{-1\}$
B	$[0, 1]$	$\emptyset$	$[0, 1]$
C	$\mathbb{R}$	$\emptyset$	$\mathbb{R}$
D	$\{s\}$	D	$D \cup \{s\}$
E	$\emptyset$	E	E

Unofficial Solutions

Method 2: ε-neighbourhood definition of limit point

Here every limit point is certified by producing, for each $\varepsilon > 0$, an explicit point of the set within ε; every non-limit point c is certified by producing one explicit $\varepsilon > 0$ for which no point of the set other than c lies within ε of c (def. of limit point). As in Method 1, $L(S)$ is the set of limit points of S, the isolated points are $S \setminus L(S)$, and $\overline{S} = S \cup L(S)$.

(a) A, with $a_n = (-1)^n + \frac{2}{n}$.

Proof. $1 \in L(A)$. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ with $k > \frac{1}{\varepsilon}$ (Archimedean). Let $x := a_{2k} = 1 + \frac{1}{k} \in A$. Then $x \neq 1$ and

$$|x - 1| = \frac{1}{k} < \varepsilon.$$

$-1 \in L(A)$. Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ with $k > \frac{2}{\varepsilon}$ (Archimedean). Let $x := a_{2k-1} = -1 + \frac{2}{2k-1} \in A$. Then $x \neq -1$ because $\frac{2}{2k-1} > 0$, and, since $2k - 1 \geq k$,

$$|x - (-1)| = \frac{2}{2k-1} \leq \frac{2}{k} < \varepsilon.$$

No other c is a limit point. Let $c \notin \{-1, 1\}$ and put $d := \min\{|c-1|, |c+1|\} > 0$. Choose $N \in \mathbb{N}$ with $N > \frac{4}{d}$ (Archimedean); then $\frac{2}{n} \leq \frac{2}{N} < \frac{d}{2}$ for every $n \geq N$. For $n \geq N$ write $a_n = \sigma_n + \frac{2}{n}$ with $\sigma_n := (-1)^n \in \{-1, 1\}$, so $|\sigma_n - c| \geq d$. By the Reverse Triangle Inequality,

$$|a_n - c| = \left|(\sigma_n - c) + \frac{2}{n}\right| \geq |\sigma_n - c| - \frac{2}{n} > d - \frac{d}{2} = \frac{d}{2} \qquad (n \geq N).$$

Let

$$\varepsilon := \min\Big(\Big\{\frac{d}{2}\Big\} \cup \big\{|a_n - c| : 1 \leq n \leq N-1,\ a_n \neq c\big\}\Big).$$

This is the minimum of a finite nonempty set of positive numbers, so $\varepsilon > 0$. Let $x \in A \setminus \{c\}$; then $x = a_n$ for some n with $a_n \neq c$. If $n \geq N$, then $|x - c| > \frac{d}{2} \geq \varepsilon$; if $n \leq N-1$, then $|x - c| \geq \varepsilon$ by the choice of ε. Hence no $x \in A \setminus \{c\}$ satisfies $|x - c| < \varepsilon$, so $c \notin L(A)$.

Therefore $L(A) = \{-1, 1\}$. Since $a_1 = 1 \in A$ and $-1 \notin A$ (odd-indexed terms exceed -1, even-indexed terms exceed 1), the isolated points are $A \setminus \{1\}$ and $\overline{A} = A \cup \{-1\}$. □

(a) Limit points $\{-1, 1\}$; isolated points $A \setminus \{1\}$; $\overline{A} = A \cup \{-1\}$.

(b) $B = \mathbb{Q} \cap (0, 1)$.

Proof. *Every $x \in [0, 1]$ is a limit point.* Let $x \in [0, 1]$ and $\varepsilon > 0$. If $x < 1$, then $0 \leq x < \min\{x + \varepsilon, 1\}$, and the Density Lemma (proved in Method 1) gives $q \in \mathbb{Q}$ with

$$x < q < \min\{x + \varepsilon,\ 1\}.$$

Then $q \in B$, $q \neq x$ and $|q - x| = q - x < \varepsilon$. If $x = 1$, then $0 \leq \max\{1 - \varepsilon, 0\} < 1$, and the Density Lemma gives $q \in \mathbb{Q}$ with

$$\max\{1 - \varepsilon,\ 0\} < q < 1.$$

Then $q \in B$, $q \neq 1$ and $|q - 1| = 1 - q < \varepsilon$. Hence $x \in L(B)$.

No $x \notin [0,1]$ is a limit point. If $x < 0$, let $\varepsilon := -x > 0$; every $y \in B$ satisfies $y > 0$, so

$$|y - x| = y - x > -x = \varepsilon.$$

If $x > 1$, let $\varepsilon := x - 1 > 0$; every $y \in B$ satisfies $y < 1$, so

$$|y - x| = x - y > x - 1 = \varepsilon.$$

In both cases no $y \in B \setminus \{x\}$ satisfies $|y - x| < \varepsilon$, so $x \notin L(B)$.

Therefore $L(B) = [0,1]$; since $B \subseteq [0,1]$, the isolated points are $\emptyset$ and $\overline{B} = [0,1]$. □

(b) Limit points $[0,1]$; isolated points $\emptyset$; $\overline{B} = [0,1]$.

(c) $C = \mathbb{R} \setminus \{0\}$**.**

Proof. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. Put $y := x + \frac{\varepsilon}{2}$ if $x \geq 0$ and $y := x - \frac{\varepsilon}{2}$ if $x < 0$. If $x \geq 0$, then $y > x \geq 0$; if $x < 0$, then $y < x < 0$. Hence $y \neq 0$ and $y \neq x$, so $y \in C \setminus \{x\}$, and

$$|y - x| = \frac{\varepsilon}{2} < \varepsilon.$$

Therefore every $x \in \mathbb{R}$ is a limit point of C, i.e. $L(C) = \mathbb{R}$. Hence the isolated points are $C \setminus \mathbb{R} = \emptyset$, and $\overline{C} = \mathbb{R}$. □

(c) Limit points $\mathbb{R}$; isolated points $\emptyset$; $\overline{C} = \mathbb{R}$.

(d) D**, with** $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$**.**

Proof. *The supremum.* For $n \geq 2$, $\frac{1}{k^2} < \frac{1}{k-1} - \frac{1}{k}$ for $2 \leq k \leq n$, so telescoping gives

$$s_n < 1 + \sum_{k=2}^{n} \Big(\frac{1}{k-1} - \frac{1}{k} \Big) = 1 + \Big(1 - \frac{1}{n} \Big) < 2,$$

and $s_1 = 1 < 2$. So D is nonempty and bounded above by 2, and $s := \sup D$ exists (AoC). Since $s_{n+1} - s_n = \frac{1}{(n+1)^2} > 0$, the sequence is strictly increasing, and $s \geq s_{n+1} > s_n$ for every $n \in \mathbb{N}$; hence $s \notin D$. The increasing bounded sequence $\{s_n\}$ converges to $\sup D$ (MCT), so $s = \lim s_n = \sum_{n=1}^{\infty} 1/n^2$.

$s \in L(D)$. Let $\varepsilon > 0$. Since $s - \varepsilon$ is not an upper bound of D, there is $N \in \mathbb{N}$ with $s - \varepsilon < s_N$ (ε-Criterion for Suprema). Also $s_N \leq s$ and $s_N \neq s$ (as $s \notin D$), so $x := s_N \in D \setminus \{s\}$ satisfies

$$|x - s| = s - s_N < \varepsilon.$$

No $c \neq s$ is a limit point. Let $c \neq s$ and $d := |c - s| > 0$. The ε-Criterion for Suprema with $\varepsilon = \frac{d}{2}$ gives $N \in \mathbb{N}$ with $s - \frac{d}{2} < s_N$. For every $n \geq N$, $s - \frac{d}{2} < s_N \leq s_n \leq s$, so $|s_n - s| < \frac{d}{2}$, and by the Reverse Triangle Inequality

$$|s_n - c| \geq |c - s| - |s_n - s| > d - \frac{d}{2} = \frac{d}{2} \qquad (n \geq N).$$

Let

$$\varepsilon := \min\Big(\Big\{ \frac{d}{2} \Big\} \cup \big\{ |s_n - c| : 1 \leq n \leq N-1,\ s_n \neq c \big\} \Big) > 0,$$

the minimum of a finite nonempty set of positive numbers. Every $x \in D \setminus \{c\}$ equals some $s_n \neq c$; if $n \geq N$ then $|x - c| > \frac{d}{2} \geq \varepsilon$, and if $n \leq N - 1$ then $|x - c| \geq \varepsilon$. Hence $c \notin L(D)$.

Therefore $L(D) = \{s\}$, the isolated points are $D \setminus \{s\} = D$, and $\overline{D} = D \cup \{s\}$. □

(d) Limit points $\{s\}$, $s = \sum_{n=1}^{\infty} 1/n^2$; isolated points D; $\overline{D} = D \cup \{s\}$.

(e) E, with $t_n = \sum_{k=1}^{n} \frac{1}{k}$.

Proof. The sequence $\{t_n\}$ is strictly increasing ($t_{n+1} - t_n = \frac{1}{n+1} > 0$). For every $N \in \mathbb{N}$, each of the N terms $\frac{1}{k}$ with $N+1 \le k \le 2N$ is at least $\frac{1}{2N}$, so

$$t_{2N} - t_N \ge N \cdot \frac{1}{2N} = \frac{1}{2},$$

and induction on m gives $t_{2^m} \ge 1 + \frac{m}{2}$ for every $m \in \mathbb{N}_0$: the base case is $t_{2^0} = t_1 = 1$, and if $t_{2^m} \ge 1 + \frac{m}{2}$, then the display with $N = 2^m$ gives

$$t_{2^{m+1}} = t_{2^m} + \left(t_{2 \cdot 2^m} - t_{2^m}\right) \ge 1 + \frac{m}{2} + \frac{1}{2} = 1 + \frac{m+1}{2}.$$

Let $c \in \mathbb{R}$. Choose $m \in \mathbb{N}$ with $m > 2\,|c|$ (Archimedean) and put $N := 2^m$. Then

$$t_N \ge 1 + \frac{m}{2} > 1 + |c| \ge c + 1,$$

so for every $n \ge N$, $t_n \ge t_N > c + 1$, i.e. $|t_n - c| = t_n - c > 1$. Let

$$\varepsilon := \min\left(\{1\} \cup \{|t_n - c| : 1 \le n \le N-1,\ t_n \ne c\}\right) > 0,$$

the minimum of a finite nonempty set of positive numbers. Every $x \in E \setminus \{c\}$ equals some $t_n \ne c$; if $n \ge N$ then $|x - c| > 1 \ge \varepsilon$, and if $n \le N - 1$ then $|x - c| \ge \varepsilon$. Hence no $x \in E \setminus \{c\}$ satisfies $|x - c| < \varepsilon$, so $c \notin L(E)$.

Since $c \in \mathbb{R}$ was arbitrary, $L(E) = \emptyset$; the isolated points are E, and $\overline{E} = E$. □

(e) Limit points $\emptyset$; isolated points E; $\overline{E} = E$.

Set	Limit points	Isolated points	Closure
A	$\{-1, 1\}$	$A \setminus \{1\}$	$A \cup \{-1\}$
B	$[0, 1]$	$\emptyset$	$[0, 1]$
C	$\mathbb{R}$	$\emptyset$	$\mathbb{R}$
D	$\{s\}$	D	$D \cup \{s\}$
E	$\emptyset$	E	E

Question 2 ((Sequential) compactness of five sets)

Problem

Determine if each of the following sets is (sequentially) compact.

(a) $A = [0,1] \cup [5,6]$.

(b) $B = \mathbb{R}_0^+$.

(c) $C = \{x \in \mathbb{R} : 0 \leq x \leq 1 \text{ and } x \text{ is irrational}\}$.

(d) $D = \left\{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$.

(e) $E = \left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} : n \in \mathbb{N}\right\}$.

(Tutorial 5, Question 2; Lectures 16–17, Sequential Compactness / Extrema of Closed and Bounded Sets; builds on Question 1.)

Solution

Method 1: Sequential Compactness of Closed and Bounded Sets

The Sequential Compactness of Closed and Bounded Sets states that a set $K \subseteq \mathbb{R}$ is (sequentially) compact if and only if K is closed and bounded. So each set is compact exactly when it is both closed and bounded; to show a set is not compact it suffices to show it is unbounded, or that it is not closed.

(a) $A = [0,1] \cup [5,6]$**.**

Proof. *Bounded.* Every $a \in A$ satisfies $0 \leq a \leq 6$, so $|a| \leq 6$ for every $a \in A$.
Closed. Let $\{a_n\}$ be a convergent sequence contained in A, with $a := \lim a_n$. Let $I_1 := \{n \in \mathbb{N} : a_n \in [0,1]\}$ and $I_2 := \{n \in \mathbb{N} : a_n \in [5,6]\}$. Then $I_1 \cup I_2 = \mathbb{N}$ is infinite, so at least one of I_1, I_2 is infinite.

Suppose I_1 is infinite, and list it as $n_1 < n_2 < \cdots$. Then $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, so $a = \lim a_{n_k}$ (Subsequential Limits). Every term satisfies $0 \leq a_{n_k} \leq 1$, so

$$0 \leq \lim a_{n_k} = a \leq 1 \qquad \text{(Order Limit Theorem)},$$

hence $a \in [0,1] \subseteq A$. If instead I_2 is infinite, the same steps with $5 \leq a_{n_k} \leq 6$ give $5 \leq a \leq 6$, hence $a \in [5,6] \subseteq A$. In both cases $a \in A$, so A is closed (Sequential Criterion for Closed Sets).

Therefore A is closed and bounded, and so A is compact (Sequential Compactness of Closed and Bounded Sets). □

(a) A is compact.

(b) $B = \mathbb{R}_0^+ = [0, \infty)$**.**

Proof. Suppose B is bounded, so there is $M > 0$ with $|b| \leq M$ for every $b \in B$. There is $n \in \mathbb{N}$ with $n > M$ (Archimedean). Then $n \in B$ and $|n| = n > M$, a contradiction. Hence B is unbounded. A compact set is bounded (Sequential Compactness of Closed and Bounded Sets), so B is not compact. □

(b) B is not compact.

(c) $C = \{x \in [0,1] : x$ **irrational**$\}$**.**

Proof. Let $x_n := \dfrac{\sqrt{2}}{n+2}$ for $n \in \mathbb{N}$.
Each $x_n \in C$. Suppose $x_n = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ and $q \neq 0$. Then

$$\sqrt{2} = \frac{p(n+2)}{q} \in \mathbb{Q},$$

which contradicts the irrationality of $\sqrt{2}$ (Lecture 1). Hence x_n is irrational. Also $0 < \sqrt{2} < 3 \leq n+2$, so $0 < x_n < 1$. Therefore $x_n \in C$ for every $n \in \mathbb{N}$.
$x_n \to 0$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ with $N > \frac{2}{\varepsilon}$ (Archimedean). For every $n \geq N$, using $\sqrt{2} < 2$,

$$|x_n - 0| = \frac{\sqrt{2}}{n+2} < \frac{2}{n} \leq \frac{2}{N} < \varepsilon.$$

Hence $0 = \lim x_n$ (def. of limit).
Not closed. $\{x_n\}$ is a convergent sequence contained in C, and its limit 0 is rational, so $0 \notin C$. Hence C is not closed (Sequential Criterion for Closed Sets). A compact set is closed (Sequential Compactness of Closed and Bounded Sets), so C is not compact. □

(c) C is not compact.

(d) D**, with** $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$**.**

Proof. In Question 1(d) we showed the following. The sequence $\{s_n\}$ is strictly increasing, since $s_{n+1} - s_n = \frac{1}{(n+1)^2} > 0$. It is bounded above, since telescoping gives $s_n \leq 2 - \frac{1}{n} < 2$. Hence $s := \lim s_n = \sup D$ exists (MCT). Finally $s \geq s_{n+1} > s_n$ for every $n \in \mathbb{N}$, so $s \notin D$.

So $\{s_n\}$ is a convergent sequence contained in D whose limit s is not in D. Hence D is not closed (Sequential Criterion for Closed Sets). A compact set is closed (Sequential Compactness of Closed and Bounded Sets), so D is not compact. □

(d) D is not compact.

(e) E**, with** $t_n = \sum_{k=1}^{n} \frac{1}{k}$**.**

Proof. For every $N \in \mathbb{N}$,

$$t_{2N} - t_N = \sum_{k=N+1}^{2N} \frac{1}{k} \geq N \cdot \frac{1}{2N} = \frac{1}{2},$$

and induction on m (base $t_1 = 1$; step $t_{2^{m+1}} = t_{2^m} + (t_{2 \cdot 2^m} - t_{2^m}) \geq t_{2^m} + \frac{1}{2}$) gives

$$t_{2^m} \geq 1 + \frac{m}{2} \qquad \text{for every } m \in \mathbb{N}_0.$$

Suppose E is bounded, so there is $M > 0$ with $|t_n| \leq M$ for every $n \in \mathbb{N}$. Choose $m \in \mathbb{N}$ with $m > 2M$ (Archimedean). Then $t_{2^m} \geq 1 + \frac{m}{2} > M$, a contradiction. Hence E is unbounded. A compact set is bounded (Sequential Compactness of Closed and Bounded Sets), so E is not compact. □

(e) E is not compact.

Set	**(Sequentially) compact?**
$A = [0,1] \cup [5,6]$	Yes
$B = \mathbb{R}_0^+$	No (unbounded)
C	No (not closed: irrationals $\to 0 \notin C$)
D	No (not closed: partial sums $\to s \notin D$)
E	No (unbounded)

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Method 2: Definition of (sequential) compactness with witnessing sequences

Here we work with the definition directly: K is compact if every sequence contained in K has a subsequence converging to a limit in K. For (a) we take an arbitrary sequence and produce such a subsequence; for (b)–(e) we exhibit one sequence in the set that has no subsequence converging to a limit in the set.

(a) $A = [0,1] \cup [5,6]$**.**

Proof. Let $\{a_n\}$ be any sequence contained in A. Let $I_1 := \{n : a_n \in [0,1]\}$ and $I_2 := \{n : a_n \in [5,6]\}$; since $I_1 \cup I_2 = \mathbb{N}$, at least one of them is infinite.

Suppose I_1 is infinite, and list it as $n_1 < n_2 < \cdots$. The subsequence $\{a_{n_k}\}$ satisfies $0 \le a_{n_k} \le 1$ for every k, so it is bounded. By the Bolzano–Weierstrass Theorem it has a convergent subsequence $\left\{a_{n_{k_j}}\right\}$; let $a := \lim a_{n_{k_j}}$. Since $0 \le a_{n_{k_j}} \le 1$ for every j,

$$0 \le a \le 1 \qquad \text{(Order Limit Theorem)},$$

so $a \in [0,1] \subseteq A$. The indices $n_{k_1} < n_{k_2} < \cdots$ are strictly increasing, so $\left\{a_{n_{k_j}}\right\}$ is a subsequence of $\{a_n\}$ converging to $a \in A$.

If instead I_2 is infinite, the same steps with $5 \le a_{n_k} \le 6$ give a subsequence of $\{a_n\}$ converging to some $a \in [5,6] \subseteq A$.

Since $\{a_n\}$ was arbitrary, every sequence in A has a subsequence converging to a limit in A, so A is compact. □

(a) A is compact.

(b) $B = \mathbb{R}_0^+$**.**

Proof. Let $b_n := n$, so $b_n \in B$ for every $n \in \mathbb{N}$. Let $\{b_{n_k}\}$ be any subsequence and let $M > 0$. Choose $k \in \mathbb{N}$ with $k > M$ (Archimedean). Then $n_k \ge k$ (Subsequence Index Bound), so

$$b_{n_k} = n_k \ge k > M.$$

Since $M > 0$ was arbitrary, $\{b_{n_k}\}$ is unbounded, and hence it diverges (Boundedness of Convergent Sequences). Therefore $\{b_n\}$ is a sequence in B with no convergent subsequence, so B is not compact. □

(b) B is not compact.

(c) $C = \{x \in [0,1] : x$ **irrational**$\}$**.**

Proof. Let $x_n := \dfrac{\sqrt{2}}{n+2}$. If $x_n = \frac{p}{q}$ with $p, q \in \mathbb{Z}$, $q \neq 0$, then $\sqrt{2} = \frac{p(n+2)}{q} \in \mathbb{Q}$, which is impossible; so x_n is irrational. Also $0 < x_n < 1$ because $\sqrt{2} < 3 \le n+2$. Hence $\{x_n\}$ is contained in C. For $\varepsilon > 0$ choose $N \in \mathbb{N}$ with $N > \frac{2}{\varepsilon}$ (Archimedean); then for $n \ge N$,

$$|x_n - 0| = \frac{\sqrt{2}}{n+2} < \frac{2}{n} \le \frac{2}{N} < \varepsilon,$$

so $0 = \lim x_n$. Every subsequence of $\{x_n\}$ also converges to 0 (Subsequential Limits), and $0 \notin C$ because 0 is rational. Therefore $\{x_n\}$ is a sequence in C with no subsequence converging to a limit in C, so C is not compact. □

(c) C is not compact.

(d) D, with $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$.

Proof. The sequence $\{s_n\}$ is contained in D. It is strictly increasing ($s_{n+1} - s_n = \frac{1}{(n+1)^2} > 0$) and bounded above by 2 (telescoping, $s_n \leq 2 - \frac{1}{n}$), so $s := \lim s_n = \sup D$ exists (MCT). For every $n \in \mathbb{N}$, $s \geq s_{n+1} > s_n$, so $s \notin D$. Every subsequence of $\{s_n\}$ converges to s (Subsequential Limits), and $s \notin D$. Therefore $\{s_n\}$ is a sequence in D with no subsequence converging to a limit in D, so D is not compact. □

(d) D is not compact.

(e) E, with $t_n = \sum_{k=1}^{n} \frac{1}{k}$.

Proof. The sequence $\{t_n\}$ is contained in E and is strictly increasing, and $t_{2^m} \geq 1 + \frac{m}{2}$ for every $m \in \mathbb{N}_0$ (the bound $t_{2N} - t_N \geq \frac{1}{2}$ and induction, as in Method 1). Let $\{t_{n_i}\}$ be any subsequence and $M > 0$. Choose $m \in \mathbb{N}$ with $m > 2M$ (Archimedean) and put $i := 2^m$. Then $n_i \geq i$ (Subsequence Index Bound), and $\{t_n\}$ is increasing, so

$$t_{n_i} \geq t_{2^m} \geq 1 + \frac{m}{2} > M.$$

Hence $\{t_{n_i}\}$ is unbounded, so it diverges (Boundedness of Convergent Sequences). Therefore $\{t_n\}$ is a sequence in E with no convergent subsequence, so E is not compact. □

(e) E is not compact.

Set	**(Sequentially) compact?**
$A = [0,1] \cup [5,6]$	Yes
$B = \mathbb{R}_0^+$	No (unbounded)
C	No (not closed: irrationals $\to 0 \notin C$)
D	No (not closed: partial sums $\to s \notin D$)
E	No (unbounded)

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Method 3: Extrema of Closed and Bounded Sets (parts (b)–(e))

If K is nonempty and compact, then K is closed and bounded (Sequential Compactness of Closed and Bounded Sets), so K has a maximum and a minimum (Extrema of Closed and Bounded Sets). Hence a nonempty set with no maximum, or with no minimum, is not compact. This route decides (b)–(e); it cannot decide (a), because having a maximum and a minimum does not imply compactness.

(b) $B = \mathbb{R}_0^+$**.**

Proof. Let $b \in B$. Then $b+1 \in B$ and $b+1 > b$, so b is not an upper bound of B. Hence no element of B is an upper bound of B, so B has no maximum. B is nonempty ($0 \in B$), so B is not compact. □

(b) B is not compact.

(c) $C = \{x \in [0,1] : x$ **irrational**$\}$**.**

Proof. Let $x \in C$. Then $x \geq 0$ and $x \neq 0$ (as 0 is rational), so $x > 0$. Choose $n \in \mathbb{N}$ with $n > \max\{\frac{\sqrt{2}}{x}, 2\}$ (Archimedean) and put $y := \frac{\sqrt{2}}{n}$. Then y is irrational (if $y = \frac{p}{q}$ then $\sqrt{2} = \frac{pn}{q} \in \mathbb{Q}$), and $0 < y < \frac{\sqrt{2}}{2} < 1$, so $y \in C$. Also $n > \frac{\sqrt{2}}{x}$ gives $y = \frac{\sqrt{2}}{n} < x$. Hence x is not a lower bound of C. Since $x \in C$ was arbitrary, C has no minimum. C is nonempty ($\frac{\sqrt{2}}{2} \in C$), so C is not compact. □

(c) C is not compact.

(d) D**, with** $s_n = \sum_{k=1}^{n} \frac{1}{k^2}$**.**

Proof. Let $x \in D$, so $x = s_n$ for some $n \in \mathbb{N}$. Then

$$s_{n+1} - s_n = \frac{1}{(n+1)^2} > 0,$$

so $s_{n+1} \in D$ exceeds x, and x is not an upper bound of D. Hence D has no maximum; D is nonempty, so D is not compact. □

(d) D is not compact.

(e) E**, with** $t_n = \sum_{k=1}^{n} \frac{1}{k}$**.**

Proof. Let $x \in E$, so $x = t_n$ for some $n \in \mathbb{N}$. Then

$$t_{n+1} - t_n = \frac{1}{n+1} > 0,$$

so $t_{n+1} \in E$ exceeds x, and x is not an upper bound of E. Hence E has no maximum; E is nonempty, so E is not compact. □

(e) E is not compact.

Question 3 ($\sup A \in \overline{A}$ via the ε-Criterion for Closures)

Problem

Use the ε-Criterion for Closures to prove that if $A \subseteq \mathbb{R}$ is nonempty and bounded above, then $\sup A \in \overline{A}$. *(Tutorial 5, Question 3; Lecture 14, ε-Criterion for Closures; Lecture 1–3, ε-Criterion for Suprema.)*

Solution

Method 1: ε-Criterion for Suprema and for Closures

Proof. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Then $s := \sup A$ exists (AoC). Let $\varepsilon > 0$. Since s is the least upper bound of A and $s - \varepsilon < s$, the number $s - \varepsilon$ is not an upper bound of A (ε-Criterion for Suprema). Hence there is $x \in A$ with $s - \varepsilon < x$. Since s is an upper bound of A and $x \in A$, also $x \leq s$. Therefore $s - x \geq 0$, so

$$|x - s| = s - x < s - (s - \varepsilon) = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, for every $\varepsilon > 0$ there is $x \in A$ with $|x - s| < \varepsilon$. Therefore $s \in \overline{A}$ (ε-Criterion for Closures). □

$$\boxed{\sup A \in \overline{A} \text{ for every nonempty, bounded-above } A \subseteq \mathbb{R}.}$$

Method 2: Sequential Criterion for Closures

This method should be used only if the course allows this interpretation (the question prescribes the ε-Criterion for Closures).

Proof. Let $s := \sup A$, which exists because A is nonempty and bounded above (AoC). Let $n \in \mathbb{N}$. Applying the ε-Criterion for Suprema with $\varepsilon = \frac{1}{n}$, the number $s - \frac{1}{n}$ is not an upper bound of A, so there is $a_n \in A$ with

$$s - \frac{1}{n} < a_n \leq s,$$

where $a_n \leq s$ holds because s is an upper bound of A. This defines a sequence $\{a_n\}$ contained in A. Since $\frac{1}{n} \to 0$ (Archimedean), the Algebraic Limit Theorem gives $s - \frac{1}{n} \to s$, and the constant sequence s converges to s. Hence $a_n \to s$ (Squeeze Theorem). Therefore s is the limit of a sequence contained in A, so $s \in \overline{A}$ (Sequential Criterion for Closures). □

$$\boxed{\sup A \in \overline{A} \text{ for every nonempty, bounded-above } A \subseteq \mathbb{R}.}$$

Method 3: Contradiction from the least-upper-bound property

This method should be used only if the course allows this interpretation (the question prescribes the ε-Criterion for Closures).

Proof. Let $s := \sup A$ (AoC), and suppose $s \notin \overline{A}$. By the definition of closure, $s \notin A$ and s is not a limit point of A. Since s is not a limit point of A, there is $\varepsilon > 0$ such that no $x \in A \setminus \{s\}$ satisfies $|x - s| < \varepsilon$ (negation of the def. of limit point). Since $s \notin A$, we have $A \setminus \{s\} = A$, so $|x - s| \geq \varepsilon$ for every $x \in A$.

Let $x \in A$. Then $x \leq s$ because s is an upper bound of A, so $|x - s| = s - x$, and hence

$$s - x \geq \varepsilon \implies x \leq s - \varepsilon.$$

Since $x \in A$ was arbitrary, $s - \varepsilon$ is an upper bound of A. But $s - \varepsilon < s$, which contradicts that s is the least upper bound of A (def. of supremum). Therefore $s \in \overline{A}$. □

$\sup A \in \overline{A}$ for every nonempty, bounded-above $A \subseteq \mathbb{R}$.

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Question 4 (Closed sets under union and arbitrary intersection)

Problem

Use the Sequential Criterion for Closed Sets to prove the following statements.

(a) If F_1 and F_2 are closed sets, then so is their union $F_1 \cup F_2$.

(b) If $\{F_t\}_{t\in T}$ is an arbitrary collection of closed sets, then so is their intersection

$$\bigcap_{t\in T} F_t := \{a : \forall t \in T \;\; a \in F_t\}.$$

(Tutorial 5, Question 4; Lecture 15, Sequential Criterion for Closed Sets; (a) reuses the "shuffled sequence" technique of Tutorial 3, Question 3.)

Solution

Method 1: Sequential Criterion for Closed Sets

(a) $F_1 \cup F_2$ is closed.

Proof. Let $\{c_n\}$ be a convergent sequence contained in $F_1 \cup F_2$, and let $c := \lim c_n$. Define the index sets

$$I_1 := \{n \in \mathbb{N} : c_n \in F_1\}, \qquad I_2 := \{n \in \mathbb{N} : c_n \in F_2\}.$$

Every c_n lies in F_1 or in F_2, so $I_1 \cup I_2 = \mathbb{N}$. If both I_1 and I_2 were finite, then $\mathbb{N} = I_1 \cup I_2$ would be finite, which is false. Hence at least one of I_1, I_2 is infinite.

Suppose I_1 is infinite, and list its elements in increasing order as $n_1 < n_2 < n_3 < \cdots$. Then $\{c_{n_k}\}$ is a subsequence of $\{c_n\}$, so $c = \lim c_{n_k}$ (Subsequential Limits). Also $c_{n_k} \in F_1$ for every $k \in \mathbb{N}$, so $\{c_{n_k}\}$ is a convergent sequence contained in F_1. Since F_1 is closed, its limit lies in F_1 (Sequential Criterion for Closed Sets), i.e. $c \in F_1 \subseteq F_1 \cup F_2$.

If instead I_2 is infinite, the same argument with F_2 in place of F_1 gives $c \in F_2 \subseteq F_1 \cup F_2$.

In both cases $c \in F_1 \cup F_2$. Since $\{c_n\}$ was an arbitrary convergent sequence contained in $F_1 \cup F_2$, the set $F_1 \cup F_2$ is closed (Sequential Criterion for Closed Sets). □

$$\boxed{F_1, F_2 \text{ closed} \implies F_1 \cup F_2 \text{ closed.}}$$

(b) $\bigcap_{t\in T} F_t$ is closed.

Proof. Let $\{a_n\}$ be a convergent sequence contained in $\bigcap_{t\in T} F_t$, and let $a := \lim a_n$. Let $t \in T$. For every $n \in \mathbb{N}$, $a_n \in \bigcap_{t'\in T} F_{t'}$, so $a_n \in F_t$ (def. of intersection). Hence $\{a_n\}$ is a convergent sequence contained in F_t. Since F_t is closed, $a \in F_t$ (Sequential Criterion for Closed Sets). Since $t \in T$ was arbitrary, $a \in F_t$ for every $t \in T$, i.e. $a \in \bigcap_{t\in T} F_t$. Since $\{a_n\}$ was arbitrary, the intersection is closed (Sequential Criterion for Closed Sets). □

$$\boxed{\{F_t\}_{t\in T} \text{ all closed} \implies \bigcap_{t\in T} F_t \text{ closed.}}$$

Method 2: Definition of closed set via ε-neighbourhoods of limit points

This method should be used only if the course allows this interpretation (the question prescribes the Sequential Criterion for Closed Sets).

Here "closed" is used in its defining form: F is closed if every limit point of F belongs to F.

(a) $F_1 \cup F_2$ is closed.

Proof. Let x be a limit point of $F_1 \cup F_2$, and suppose $x \notin F_1 \cup F_2$. Then $x \notin F_1$. Since F_1 is closed, x is not a limit point of F_1, so there is $\varepsilon_1 > 0$ such that no $y \in F_1 \setminus \{x\}$ satisfies $|y - x| < \varepsilon_1$ (negation of the def. of limit point). Likewise $x \notin F_2$ and F_2 is closed, so there is $\varepsilon_2 > 0$ such that no $y \in F_2 \setminus \{x\}$ satisfies $|y - x| < \varepsilon_2$. Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\} > 0$.

Since x is a limit point of $F_1 \cup F_2$, there is $y \in (F_1 \cup F_2) \setminus \{x\}$ with $|y - x| < \varepsilon$. If $y \in F_1$, then $y \in F_1 \setminus \{x\}$ and $|y - x| < \varepsilon \leq \varepsilon_1$, which contradicts the choice of ε_1. If $y \in F_2$, then $y \in F_2 \setminus \{x\}$ and $|y - x| < \varepsilon \leq \varepsilon_2$, which contradicts the choice of ε_2. Hence $x \in F_1 \cup F_2$.

Since x was an arbitrary limit point of $F_1 \cup F_2$, the set $F_1 \cup F_2$ contains all its limit points, so it is closed (def. of closed set). □

$$\boxed{F_1, F_2 \text{ closed} \implies F_1 \cup F_2 \text{ closed.}}$$

(b) $\bigcap_{t \in T} F_t$ is closed.

Proof. Let $G := \bigcap_{t \in T} F_t$, let x be a limit point of G, and let $t \in T$. Let $\varepsilon > 0$. Since x is a limit point of G, there is $y \in G \setminus \{x\}$ with $|y - x| < \varepsilon$. Since $G \subseteq F_t$, also $y \in F_t \setminus \{x\}$. Hence for every $\varepsilon > 0$ there is $y \in F_t \setminus \{x\}$ with $|y - x| < \varepsilon$, so x is a limit point of F_t (def. of limit point). Since F_t is closed, $x \in F_t$. Since $t \in T$ was arbitrary, $x \in G$. Therefore G contains all its limit points, so G is closed (def. of closed set). □

$$\boxed{\{F_t\}_{t \in T} \text{ all closed} \implies \bigcap_{t \in T} F_t \text{ closed.}}$$

Method 3: Complements of closed sets

This method should be used only if the course allows this interpretation (it uses complements and the notion of an open set, which this course does not define).

Definition (outside the course). Call $U \subseteq \mathbb{R}$ *open* if for every $x \in U$ there is $\varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \subseteq U$. Write $F^c := \mathbb{R} \setminus F$.

Lemma. A set $F \subseteq \mathbb{R}$ is closed if and only if F^c is open.

Proof of Lemma. ($\Rightarrow$) Let F be closed and let $x \in F^c$. Since $x \notin F$ and F contains all its limit points, x is not a limit point of F. Hence there is $\varepsilon > 0$ such that no $y \in F \setminus \{x\}$ satisfies $|y - x| < \varepsilon$. Since $x \notin F$, $F \setminus \{x\} = F$, so no $y \in F$ lies in $(x-\varepsilon, x+\varepsilon)$. Therefore $(x-\varepsilon, x+\varepsilon) \subseteq F^c$, and F^c is open.

($\Leftarrow$) Let F^c be open, let x be a limit point of F, and suppose $x \notin F$. Then $x \in F^c$, so there is $\varepsilon > 0$ with $(x-\varepsilon, x+\varepsilon) \subseteq F^c$. Hence no $y \in F$ satisfies $|y - x| < \varepsilon$, which contradicts that x is a limit point of F. Therefore $x \in F$, and F contains all its limit points, i.e. F is closed. □

(a) $F_1 \cup F_2$ is closed.

Proof. By De Morgan's law,

$$(F_1 \cup F_2)^c = F_1^c \cap F_2^c.$$

By the Lemma, F_1^c and F_2^c are open. Let $x \in F_1^c \cap F_2^c$. Then there are $\varepsilon_1, \varepsilon_2 > 0$ with $(x-\varepsilon_1, x+\varepsilon_1) \subseteq F_1^c$ and $(x-\varepsilon_2, x+\varepsilon_2) \subseteq F_2^c$. Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\} > 0$. Then

$$(x-\varepsilon, x+\varepsilon) \subseteq (x-\varepsilon_1, x+\varepsilon_1) \cap (x-\varepsilon_2, x+\varepsilon_2) \subseteq F_1^c \cap F_2^c.$$

Hence $(F_1 \cup F_2)^c$ is open, and therefore $F_1 \cup F_2$ is closed (Lemma). □

$$\boxed{F_1, F_2 \text{ closed} \implies F_1 \cup F_2 \text{ closed.}}$$

(b) $\bigcap_{t \in T} F_t$ is closed.

Proof. By De Morgan's law,

$$\left(\bigcap_{t \in T} F_t\right)^c = \bigcup_{t \in T} F_t^c.$$

Let $x \in \bigcup_{t \in T} F_t^c$; then $x \in F_{t_0}^c$ for some $t_0 \in T$. By the Lemma, $F_{t_0}^c$ is open, so there is $\varepsilon > 0$ with

$$(x-\varepsilon, x+\varepsilon) \subseteq F_{t_0}^c \subseteq \bigcup_{t \in T} F_t^c.$$

Hence $\left(\bigcap_{t \in T} F_t\right)^c$ is open, and therefore $\bigcap_{t \in T} F_t$ is closed (Lemma). □

$$\boxed{\{F_t\}_{t \in T} \text{ all closed} \implies \bigcap_{t \in T} F_t \text{ closed.}}$$

Question 5 (A closed subset of a compact set is compact)

Problem

Use the definition of (sequential) compactness to prove that a closed subset of a (sequentially) compact set is (sequentially) compact; i.e., for any compact set $K \subseteq \mathbb{R}$ and any subset $A \subseteq K$, prove that A is compact if it is closed. **Do not use the Sequential Compactness of Closed and Bounded Sets or the Bolzano-Weierstrass Theorem.**

[Note: A proof without using the Sequential Compactness of Closed and Bounded Sets or the Bolzano-Weierstrass Theorem is applicable to compact sets in $\mathbb{Q}$ (or any ordered field) for which these properties do not hold.] *(Tutorial 5, Question 5; Lecture 16, definition of (sequential) compactness; Lecture 15, Sequential Criterion for Closed Sets.)*

Solution

Method 1: Definition of compactness with the Sequential Criterion for Closed Sets

Proof. Let $K \subseteq \mathbb{R}$ be compact and let $A \subseteq K$ be closed. Let $\{a_n\}$ be an arbitrary sequence contained in A. Since $a_n \in A \subseteq K$ for every $n \in \mathbb{N}$, the sequence $\{a_n\}$ is contained in K. Since K is compact, $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ that converges to some $x \in K$ (def. of compactness). Since $a_{n_k} \in A$ for every $k \in \mathbb{N}$, the sequence $\{a_{n_k}\}$ is a convergent sequence contained in A. Since A is closed, its limit lies in A (Sequential Criterion for Closed Sets), i.e. $x \in A$. Hence $\{a_n\}$ has a subsequence converging to a limit in A. Since $\{a_n\}$ was arbitrary, every sequence contained in A has a subsequence converging to a limit in A. Therefore A is compact (def. of compactness). □

$$\boxed{K \text{ compact and } A \subseteq K \text{ closed } \implies A \text{ compact.}}$$

Method 2: Contradiction via the limit-point definition of closed

Proof. Let K be compact, let $A \subseteq K$ be closed, and suppose A is not compact. Then there is a sequence $\{a_n\}$ contained in A such that no subsequence of $\{a_n\}$ converges to a limit in A (negation of the def. of compactness). Since $\{a_n\}$ is contained in $A \subseteq K$ and K is compact, some subsequence $\{a_{n_k}\}$ converges to some $x \in K$ (def. of compactness). By the choice of $\{a_n\}$, this limit satisfies $x \notin A$.

Since $a_{n_k} \in A$ and $x \notin A$, we have $a_{n_k} \neq x$ for every $k \in \mathbb{N}$. Hence $\{a_{n_k}\}$ is a sequence contained in $A \setminus \{x\}$ with $x = \lim a_{n_k}$. Therefore x is a limit point of A (Sequential Criterion for Limit Points). Since A is closed, it contains all its limit points (def. of closed set), so $x \in A$. This contradicts $x \notin A$. Therefore A is compact. □

$$\boxed{K \text{ compact and } A \subseteq K \text{ closed } \implies A \text{ compact.}}$$

Method 3: Closed and bounded characterisation

This method should be used only if the course allows this interpretation (the question forbids the Sequential Compactness of Closed and Bounded Sets, which this method uses; the method does not apply in $\mathbb{Q}$).

Proof. Let K be compact and let $A \subseteq K$ be closed. Since K is compact, K is bounded (Sequential Compactness of Closed and Bounded Sets), so there is $M > 0$ with $|x| \leq M$ for every $x \in K$. Since $A \subseteq K$, also $|a| \leq M$ for every $a \in A$, so A is bounded. Hence A is closed and bounded, and therefore A is compact (Sequential Compactness of Closed and Bounded Sets). □

$$\boxed{K \text{ compact and } A \subseteq K \text{ closed } \implies A \text{ compact.}}$$

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Question A1 (Closed sets via distance-zero points)

Problem

For a nonempty set $A \subseteq \mathbb{R}$ and a number $a \in \mathbb{R}$, we define the *distance of a from A* as the infimum of the set $\{|x-a| : x \in A\}$, which is nonempty and bounded below. Prove that a nonempty set $A \subseteq \mathbb{R}$ is closed if and only if every number that has distance 0 from A is in A. *(Tutorial 5, Additional Question 1; Lecture 14, ε-Criterion for Closures; Lecture 1–3, ε-Criterion for Infima; builds on Question 3.)*

Solution

Method 1: ε-Criterion for Infima and for Closures

Throughout, for $a \in \mathbb{R}$ write $S_a := \{|x-a| : x \in A\}$ and $d(a,A) := \inf S_a$. S_a is nonempty because A is nonempty, and 0 is a lower bound of S_a because absolute values are nonnegative; hence $d(a,A)$ exists (AoC) and $d(a,A) \geq 0$.

Proof. *Step 1: for every $a \in \mathbb{R}$, $d(a,A) = 0 \iff a \in \overline{A}$.*

($\Rightarrow$) Suppose $d(a,A) = 0$ and let $\varepsilon > 0$. Since $0 = \inf S_a$, the number $0+\varepsilon$ is not a lower bound of S_a (ε-Criterion for Infima). Hence there is $x \in A$ with $|x-a| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $a \in \overline{A}$ (ε-Criterion for Closures).

($\Leftarrow$) Suppose $a \in \overline{A}$. Then for every $\varepsilon > 0$ there is $x \in A$ with $|x-a| < \varepsilon$ (ε-Criterion for Closures), i.e. there is an element of S_a less than $0+\varepsilon$. Also 0 is a lower bound of S_a. Hence $0 = \inf S_a = d(a,A)$ (ε-Criterion for Infima).

Step 2: the equivalence.

($\Rightarrow$) Suppose A is closed, and let $a \in \mathbb{R}$ with $d(a,A) = 0$. By Step 1, $a \in \overline{A}$. Since A is closed, $\overline{A} \subseteq A$ (Closure Criterion for Closed Sets). Therefore $a \in A$.

($\Leftarrow$) Suppose every $a \in \mathbb{R}$ with $d(a,A) = 0$ lies in A. Let $a \in \overline{A}$. By Step 1, $d(a,A) = 0$, so $a \in A$ by the hypothesis. Since $a \in \overline{A}$ was arbitrary, $\overline{A} \subseteq A$. Therefore A is closed (Closure Criterion for Closed Sets). □

$$\boxed{A \text{ closed} \iff \big(d(a,A) = 0 \implies a \in A\big) \text{ for every } a \in \mathbb{R}.}$$

Method 2: Sequential Criterion for Closed Sets

Proof. ($\Rightarrow$) Suppose A is closed, and let $a \in \mathbb{R}$ with $d(a, A) = 0$. Let $n \in \mathbb{N}$. Since $0 = \inf S_a$, the number $\frac{1}{n}$ is not a lower bound of S_a (ε-Criterion for Infima with $\varepsilon = \frac{1}{n}$), so there is $x_n \in A$ with

$$|x_n - a| < \frac{1}{n}.$$

This defines a sequence $\{x_n\}$ contained in A. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$ (Archimedean); for every $n \geq N$,

$$|x_n - a| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence $a = \lim x_n$ (def. of limit). So $\{x_n\}$ is a convergent sequence contained in the closed set A, and therefore $a \in A$ (Sequential Criterion for Closed Sets).

($\Leftarrow$) Suppose every $a \in \mathbb{R}$ with $d(a, A) = 0$ lies in A. Let $\{x_n\}$ be a convergent sequence contained in A, with $a := \lim x_n$. For every $n \in \mathbb{N}$, $|x_n - a| \in S_a$, and $d(a, A)$ is a lower bound of S_a, so

$$0 \leq d(a, A) \leq |x_n - a|.$$

Since $x_n - a \to 0$ (ALT), we have $|x_n - a| \to 0$: for $\varepsilon > 0$ there is N with $||x_n - a| - 0| = |x_n - a| < \varepsilon$ for all $n \geq N$. The constant sequence $d(a, A)$ and the sequence $|x_n - a|$ satisfy $d(a, A) \leq |x_n - a|$ for every n, so

$$d(a, A) \leq \lim |x_n - a| = 0 \qquad \text{(Order Limit Theorem)}.$$

Combined with $d(a, A) \geq 0$, this gives $d(a, A) = 0$, so $a \in A$ by the hypothesis. Since $\{x_n\}$ was arbitrary, every convergent sequence contained in A has its limit in A. Therefore A is closed (Sequential Criterion for Closed Sets). □

$$\boxed{A \text{ closed} \iff \big(d(a, A) = 0 \implies a \in A\big) \text{ for every } a \in \mathbb{R}.}$$

Method 3: Limit-point definition of closed, with contrapositive

Proof. ($\Rightarrow$, by contrapositive) Suppose A is closed and let $a \notin A$; we show $d(a, A) > 0$. Since A contains all its limit points and $a \notin A$, the number a is not a limit point of A. Hence there is $\varepsilon > 0$ such that no $x \in A \setminus \{a\}$ satisfies $|x - a| < \varepsilon$ (negation of the def. of limit point). Since $a \notin A$, $A \setminus \{a\} = A$, so $|x - a| \geq \varepsilon$ for every $x \in A$. Thus ε is a lower bound of S_a, and since $d(a, A)$ is the greatest lower bound,

$$d(a, A) \geq \varepsilon > 0.$$

Hence $d(a, A) = 0$ implies $a \in A$.

($\Leftarrow$) Suppose every $a \in \mathbb{R}$ with $d(a, A) = 0$ lies in A, and let a be a limit point of A. Suppose $d(a, A) > 0$ and put $\varepsilon := d(a, A)$. Since a is a limit point of A, there is $x \in A \setminus \{a\}$ with $|x - a| < \varepsilon = d(a, A)$. But $|x - a| \in S_a$ and $d(a, A)$ is a lower bound of S_a, so $|x - a| \geq d(a, A)$, a contradiction. Hence $d(a, A) \leq 0$, and with $d(a, A) \geq 0$ this gives $d(a, A) = 0$, so $a \in A$ by the hypothesis. Therefore A contains all its limit points, i.e. A is closed (def. of closed set). □

$$\boxed{A \text{ closed} \iff \big(d(a, A) = 0 \implies a \in A\big) \text{ for every } a \in \mathbb{R}.}$$

Question A2 (Disjoint closed sets can have distance 0)

Problem

For two sets $A, B \subseteq \mathbb{R}$, we define the *distance* between them as the infimum of the set $\{|a-b| : a \in A, b \in B\}$. If two closed sets A and B are disjoint (i.e., $A \cap B = \emptyset$), must the distance between them necessarily be positive? *(Tutorial 5, Additional Question 2; Lecture 13–15, closed sets and distance to a set; contrasts with Additional Question 3, the compact case.)*

Solution

Method 1: Explicit counterexample, closedness by the Sequential Criterion for Closed Sets

Let $A := \mathbb{N}$ and $B := \{n + \frac{1}{n} : n \in \mathbb{N},\ n \geq 2\}$. Both sets are spread out with a uniform gap between distinct points; the lemma below turns such a gap into closedness.

Lemma (uniform gap). Let $\delta > 0$ and let $S \subseteq \mathbb{R}$ satisfy $|s - s'| \geq \delta$ for all distinct $s, s' \in S$. Then S is closed.

Proof of Lemma. Let $\{x_k\}$ be a convergent sequence contained in S, with $x := \lim x_k$. There is $K \in \mathbb{N}$ with $|x_k - x| < \frac{\delta}{2}$ for every $k \geq K$ (def. of limit, $\varepsilon = \frac{\delta}{2}$). For $k, l \geq K$ the Triangle Inequality gives

$$|x_k - x_l| \leq |x_k - x| + |x - x_l| < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Since distinct points of S are at least δ apart, $x_k = x_l$ for all $k, l \geq K$. Hence $x_k = x_K$ for every $k \geq K$, so the tail $\{x_k\}_{k \geq K}$ is the constant sequence x_K, whose limit is x_K. The tail is a subsequence of $\{x_k\}$, so its limit is also x (Subsequential Limits); therefore $x = x_K \in S$ (Uniqueness of Limits). Hence S is closed (Sequential Criterion for Closed Sets). □

Proof. $A \cap B = \emptyset$. For $n \geq 2$ we have $0 < \frac{1}{n} \leq \frac{1}{2} < 1$, so

$$n < n + \frac{1}{n} < n + 1.$$

Hence no element of B is an integer, so no element of B lies in $A = \mathbb{N}$.
A is closed. Distinct natural numbers $m \neq n$ satisfy $|m - n| \geq 1$, so the Lemma with $\delta = 1$ applies.
B is closed. Two distinct elements of B are $m + \frac{1}{m}$ and $n + \frac{1}{n}$ with $m \neq n$; label them so that $m > n \geq 2$. Then $m - n \geq 1$ and $0 < \frac{1}{n} - \frac{1}{m} < \frac{1}{n} \leq \frac{1}{2}$, so

$$\left(m + \frac{1}{m}\right) - \left(n + \frac{1}{n}\right) = (m - n) - \left(\frac{1}{n} - \frac{1}{m}\right) > 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence the Lemma with $\delta = \frac{1}{2}$ applies.
Distance 0. Let $S := \{|a - b| : a \in A,\ b \in B\}$; 0 is a lower bound of S. For $n \geq 2$, $n \in A$ and $n + \frac{1}{n} \in B$, and

$$\left|n - \left(n + \frac{1}{n}\right)\right| = \frac{1}{n} \in S.$$

Let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ with $n > \max\{\frac{1}{\varepsilon}, 2\}$ (Archimedean); then $\frac{1}{n} \in S$ and $\frac{1}{n} < 0 + \varepsilon$. Hence $\inf S = 0$ (ε-Criterion for Infima).

Therefore A and B are disjoint closed sets whose distance is 0, so the distance need not be positive. □

No: $A = \mathbb{N}$ and $B = \{n + 1/n : n \geq 2\}$ are disjoint, closed, and at distance 0.

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Method 2: Same counterexample, closedness by having no limit points

Here closedness is checked from the definition: a set with no limit points contains all of its (zero) limit points, so it is closed.

Proof. Let $A := \mathbb{N}$, $B := \{n + \frac{1}{n} : n \in \mathbb{N},\ n \geq 2\}$, and let S be either of them, with uniform gap δ ($\delta = 1$ for A, $\delta = \frac{1}{2}$ for B, as computed in Method 1): $|s - s'| \geq \delta$ for all distinct $s, s' \in S$.
S has no limit points. Let $c \in \mathbb{R}$. At most one $s \in S$ satisfies $|s - c| < \frac{\delta}{2}$: two distinct such points s, s' would satisfy $|s - s'| \leq |s - c| + |c - s'| < \delta$ (Triangle Inequality). If no point of $S \setminus \{c\}$ lies within $\frac{\delta}{2}$ of c, let $\varepsilon := \frac{\delta}{2}$. Otherwise let s_0 be the unique point of S within $\frac{\delta}{2}$ of c; it satisfies $s_0 \neq c$, and we let $\varepsilon := |s_0 - c| > 0$. In both cases $\varepsilon \leq \frac{\delta}{2}$, and no $s \in S \setminus \{c\}$ satisfies $|s - c| < \varepsilon$: the only candidate within $\frac{\delta}{2}$ is s_0, and $|s_0 - c| = \varepsilon \not< \varepsilon$. Hence c is not a limit point of S (def. of limit point). Since $c \in \mathbb{R}$ was arbitrary, S has no limit points, so S contains all its limit points and is closed (def. of closed set).
$A \cap B = \emptyset$. For $n \geq 2$, $n < n + \frac{1}{n} < n + 1$, so no element of B is an integer.
Distance 0, via a sequence. Let $d := \inf\{|a - b| : a \in A,\ b \in B\}$, which exists because the set is nonempty and bounded below by 0 (AoC); so $d \geq 0$. For every $n \geq 2$, $\left|n - (n + \frac{1}{n})\right| = \frac{1}{n}$ belongs to this set, so $d \leq \frac{1}{n}$. Hence $0 \leq d \leq \frac{1}{n}$ for every $n \geq 2$, and $\frac{1}{n} \to 0$ (Archimedean), so

$$0 \leq d \leq \lim \frac{1}{n} = 0 \qquad \text{(Order Limit Theorem)},$$

i.e. $d = 0$.

Therefore A and B are disjoint closed sets whose distance is 0. □

No: $A = \mathbb{N}$ and $B = \{n + 1/n : n \geq 2\}$ are disjoint, closed, and at distance 0.

Question A3 (Disjoint compact sets have positive, attained distance)

Problem

Recall that for two sets $A, B \subseteq \mathbb{R}$, we define the *distance* between them as the infimum of the set $\{|a-b| : a \in A, b \in B\}$.

Prove that if two (sequentially) compact sets K_1 and K_2 are disjoint, then the distance between them is positive and is attained by two numbers $a_1 \in K_1$ and $a_2 \in K_2$; i.e., if the sets K_1 and K_2 are (sequentially) compact and have an empty intersection, then there are two numbers $a_1 \in K_1$ and $a_2 \in K_2$ for which the distance between K_1 and K_2 is $|a_1 - a_2|$, which is positive. *(Tutorial 5, Additional Question 3; Lecture 16, Sequential Compactness and Extrema of Closed and Bounded Sets; Lecture 4–6, triangle inequality in the limit sense; contrasts with Additional Question 2.)*

Solution

Method 1: Compactness of the distance set and Extrema of Closed and Bounded Sets

Throughout, K_1 and K_2 are compact and disjoint, and nonempty (the distance is an infimum, which is defined only for a nonempty set), and

$$S := \{|a-b| : a \in K_1,\ b \in K_2\},$$

which is nonempty; $\operatorname{dist}(K_1, K_2) = \inf S$.

Lemma (limits of absolute values). If $u = \lim u_j$, then $|u| = \lim |u_j|$.

Proof of Lemma. Let $\varepsilon > 0$. There is $J \in \mathbb{N}$ with $|u_j - u| < \varepsilon$ for every $j \geq J$ (def. of limit). For every $j \geq J$, the Reverse Triangle Inequality gives

$$\big||u_j| - |u|\big| \leq |u_j - u| < \varepsilon.$$

Therefore $|u| = \lim |u_j|$ (def. of limit). □

Proof. *Step 1: S is compact.* Let $\{x_n\}$ be any sequence contained in S. For each $n \in \mathbb{N}$ choose $a_n \in K_1$ and $b_n \in K_2$ with $x_n = |a_n - b_n|$. Since $\{a_n\}$ is contained in the compact set K_1, it has a subsequence $\{a_{n_k}\}$ with $a := \lim a_{n_k} \in K_1$ (def. of compactness). Since $\{b_{n_k}\}$ is contained in the compact set K_2, it has a subsequence $\left\{b_{n_{k_j}}\right\}$ with $b := \lim b_{n_{k_j}} \in K_2$ (def. of compactness). Put $m_j := n_{k_j}$; then $m_1 < m_2 < \cdots$. The sequence $\left\{a_{m_j}\right\}$ is a subsequence of $\{a_{n_k}\}$, so $a = \lim a_{m_j}$ (Subsequential Limits). Let $u_j := a_{m_j} - b_{m_j}$ and $u := a - b$; then

$$u = \lim u_j \qquad \text{(ALT)}.$$

By the Lemma,

$$\lim x_{m_j} = \lim \left|a_{m_j} - b_{m_j}\right| = \lim |u_j| = |u| = |a-b|,$$

and $|a-b| \in S$ because $a \in K_1$ and $b \in K_2$. Hence $\{x_{m_j}\}$ is a subsequence of $\{x_n\}$ converging to a limit in S. Since $\{x_n\}$ was arbitrary, S is compact.
Step 2: $0 \notin S$. If $|a-b| = 0$ with $a \in K_1$ and $b \in K_2$, then $a = b \in K_1 \cap K_2$, which contradicts $K_1 \cap K_2 = \emptyset$. Hence every element of S is positive.
Step 3: the minimum. Since S is compact, S is closed and bounded (Sequential Compactness of Closed and Bounded Sets), and S is nonempty. Hence $m := \min S$ exists (Extrema of Closed and Bounded Sets). Since $m \in S$, there are $a_1 \in K_1$ and $a_2 \in K_2$ with $m = |a_1 - a_2|$, and $m > 0$ by Step 2. Finally, m is a lower bound of S, and every lower bound ℓ of S satisfies $\ell \leq m$ because $m \in S$; hence $m = \inf S = \operatorname{dist}(K_1, K_2)$. □

$$\boxed{\operatorname{dist}(K_1, K_2) = |a_1 - a_2| = \min S > 0 \text{ for some } a_1 \in K_1,\ a_2 \in K_2.}$$

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Method 2: Minimising sequence

Proof. The set S is nonempty and bounded below by 0, so $d := \inf S = \operatorname{dist}(K_1, K_2)$ exists (AoC). Let $n \in \mathbb{N}$. Since $d + \frac{1}{n}$ is not a lower bound of S (ε-Criterion for Infima with $\varepsilon = \frac{1}{n}$), there are $a_n \in K_1$ and $b_n \in K_2$ with

$$d \le |a_n - b_n| < d + \frac{1}{n},$$

where $d \le |a_n - b_n|$ holds because d is a lower bound of S.

Since K_1 is compact, some subsequence satisfies $a := \lim a_{n_k} \in K_1$. Since K_2 is compact, the sequence $\{b_{n_k}\}$ has a subsequence with $b := \lim b_{n_{k_j}} \in K_2$. Put $m_j := n_{k_j}$, so $m_1 < m_2 < \cdots$ and $a = \lim a_{m_j}$ (Subsequential Limits). Also $b_{m_j} = b_{n_{k_j}}$, so $b = \lim b_{m_j}$, and therefore $a_{m_j} - b_{m_j} \to a - b$ (ALT). Let $\varepsilon > 0$; there is $J \in \mathbb{N}$ with $\big|(a_{m_j} - b_{m_j}) - (a - b)\big| < \varepsilon$ for every $j \ge J$ (def. of limit). For every $j \ge J$ the Reverse Triangle Inequality gives

$$\Big|\big|a_{m_j} - b_{m_j}\big| - |a - b|\Big| \le \big|(a_{m_j} - b_{m_j}) - (a - b)\big| < \varepsilon.$$

Hence $\big|a_{m_j} - b_{m_j}\big| \to |a - b|$ (def. of limit).

On the other hand, $m_j \ge j$ (Subsequence Index Bound), so

$$d \le \big|a_{m_j} - b_{m_j}\big| < d + \frac{1}{m_j} \le d + \frac{1}{j} \qquad \text{for every } j \in \mathbb{N}.$$

Since $d + \frac{1}{j} \to d$ (ALT), the Squeeze Theorem gives $\big|a_{m_j} - b_{m_j}\big| \to d$. By Uniqueness of Limits, $d = |a - b|$.

Finally $a \in K_1$, $b \in K_2$ and $K_1 \cap K_2 = \emptyset$, so $a \ne b$, and therefore $d = |a - b| > 0$. Let $a_1 := a$ and $a_2 := b$; then $\operatorname{dist}(K_1, K_2) = d = |a_1 - a_2| > 0$, and since $|a_1 - a_2| \in S$ is a lower bound of S, it is $\min S$. □

$$\boxed{\operatorname{dist}(K_1, K_2) = |a_1 - a_2| = \min S > 0 \text{ for some } a_1 \in K_1,\ a_2 \in K_2.}$$

Method Map

Q.	Core topic	Methods included
1	Limit points / isolated points / closure of five sets	M1: Sequential Criterion for Limit Points (injective-sequence lemma, density of $\mathbb{Q}$). M2: ε-neighbourhood definition of limit point.
2	(Sequentially) compact or not	M1: Sequential Compactness of Closed and Bounded Sets. M2: Definition of compactness with witnessing sequences. M3: Extrema of Closed and Bounded Sets ((b)–(e)).
3	$\sup A \in \overline{A}$	M1: ε-Criterion for Suprema and for Closures. M2: Sequential Criterion for Closures. M3: Contradiction from the least-upper-bound property.
4	Closed sets under union / arbitrary intersection	M1: Sequential Criterion for Closed Sets. M2: Definition of closed set via ε-neighbourhoods of limit points. M3: Complements of closed sets.
5	Closed subset of a compact set is compact	M1: Definition of compactness with the Sequential Criterion for Closed Sets. M2: Contradiction via the limit-point definition of closed. M3: Closed and bounded characterisation.
A1	Closed sets via distance-zero points	M1: ε-Criterion for Infima and for Closures. M2: Sequential Criterion for Closed Sets. M3: Limit-point definition of closed, with contrapositive.
A2	Disjoint closed sets, distance 0	M1: Explicit counterexample, closedness by the Sequential Criterion for Closed Sets. M2: Same counterexample, closedness by having no limit points.
A3	Disjoint compact sets, positive attained distance	M1: Compactness of the distance set and Extrema of Closed and Bounded Sets. M2: Minimising sequence.

Final Answers

1. A: limit points $\{-1, 1\}$, isolated $A \setminus \{1\}$, closure $A \cup \{-1\}$. B: $[0, 1]$, $\emptyset$, $[0, 1]$. C: $\mathbb{R}$, $\emptyset$, $\mathbb{R}$. D: $\{s\}$ ($s = \sum_{n=1}^{\infty} 1/n^2$), D, $D \cup \{s\}$. E: $\emptyset$, E, E.

2. (a) Compact. (b) Not compact. (c) Not compact. (d) Not compact. (e) Not compact.

3. $\sup A \in \overline{A}$ for every nonempty, bounded-above $A \subseteq \mathbb{R}$.

4. (a) F_1, F_2 closed $\implies$ $F_1 \cup F_2$ closed. (b) An arbitrary intersection of closed sets is closed.

5. A closed subset of a compact set is compact.

A1. A is closed $\iff$ every number at distance 0 from A lies in A.

A2. No: $A = \mathbb{N}$ and $B = \{n + 1/n : n \geq 2\}$ are disjoint, closed, at distance 0.

A3. $\operatorname{dist}(K_1, K_2) = |a_1 - a_2| > 0$ for some $a_1 \in K_1$, $a_2 \in K_2$.