1. Order & Extrema of $\mathbb{R}$ (L1-2)

Bounds: $b \in \mathbb{R}$ is an **upper bound** for $A \subseteq \mathbb{R}$ if $\forall a \in A\ a \le b$; then A is **bounded above**. Similarly **lower bound/bounded below**. A is **bounded** if bounded above and below.
Sup/Inf: $s = \sup A$ if (1) s is an upper bound for A; (2) $s \le b$ for every upper bound b of A. Similarly $i = \inf A$ (greatest lower bound). $\sup \emptyset = -\infty$; A unbounded above $\Rightarrow \sup A = \infty$.
Max/Min: $m = \max A$ if $m \in A$ and m is an upper bound for A; similarly $\min A$. $\max A$ (if it exists) $= \sup A$, converse false. $\sup A \in A \iff \sup A = \max A$.
Ax. Def. of $\mathbb{R}$: Axiomatic Definition of $\mathbb{R}$: all ordered fields satisfying the Axiom of Completeness are isomorphic; $\mathbb{R}$ is defined to be an ordered field satisfying it. (Stated L2; not proved in the course.)

2. Sequences (L4-9)

$\overline{\mathbb{N}}$, sequence: $\mathbb{N} = \{1, 2, 3, \ldots\}$ (0 excluded); $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A **sequence** is a function $\mathbb{N} \to \mathbb{R}$, written $\{a_n\} = \{a_n\}_{n=1}^{\infty}$; a_n is its n-th term.
Limit: $a = \lim a_n$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n \ge N \implies |a_n - a| < \varepsilon)$. $\{a_n\}$ **converges** if it has a limit; else **diverges**.
Bounded seq.: $\{a_n\}$ **bounded** if $\exists M > 0\ \forall n \in \mathbb{N}\ |a_n| \le M$; bounded $\iff$ eventually bounded.
Tail / eventually: A **tail** of $\{a_n\}$ is $a_N, a_{N+1}, \ldots$ for some $N \in \mathbb{N}$. $\{a_n\}$ satisfies a property **eventually** if some tail satisfies it.
Cauchy: $\{a_n\}$ **Cauchy** if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n \ge N \implies |a_n - a_N| < \varepsilon)$; equivalently $\forall n, m \ge N\ |a_n - a_m| < \varepsilon$.
Subsequence: $\{a_{n_k}\}$ is a **subsequence** of $\{a_n\}$ if $n_1 < n_2 < n_3 < \cdots$ is strictly increasing in $\mathbb{N}$ (so $n_k \ge k$ for all k). **Proper** if $n_k \ne k$ for some k.
Monotone: $\{a_n\}$ **increasing** if $\forall n\ a_n \le a_{n+1}$; **decreasing** if $\forall n\ a_n \ge a_{n+1}$; either is **monotone** (**strictly** with $< / >$). **Eventually monotone**: eventually increasing or eventually decreasing.

3. Series & Sets (L10-17)

Series: $\sum a_n := a_1 + a_2 + \cdots$; n-th **partial sum** $s_n := a_1 + \cdots + a_n$. $\sum a_n$ **converges** if $\{s_n\}$ converges; then its **sum** $s = \sum a_n := \lim s_n$. Else **diverges**.
Abs./cond. conv.: $\sum a_n$ **absolutely convergent** if $\sum |a_n|$ converges. **Conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges. Every series is exactly one of: abs. conv., cond. conv., divergent.
Limit / isolated pt: $a \in \mathbb{R}$ is a **limit point** of $A \subseteq \mathbb{R}$ if $\forall \varepsilon > 0\ \exists x \in A \setminus \{a\}\ |x - a| < \varepsilon$ (a.k.a. cluster/accumulation pt). a is an **isolated point** of A if $a \in A$ and a is not a limit point of A. Every point of A is one or the other.
Closure: $\overline{A} := A \cup \{\text{limit points of } A\}$.
Closed set: $F \subseteq \mathbb{R}$ is **closed** if it contains all its limit points (equiv. F has no limit point outside F).
Compact set: $K \subseteq \mathbb{R}$ is **(sequentially) compact** if every sequence contained in K has a convergent subsequence with limit in K.

4. Axioms & Theorems I (L1-9)

Completeness: Axiom of Completeness: every nonempty $A \subseteq \mathbb{R}$ that is bounded above has a supremum.
ε-Crit. Sup/Inf: For $A \subseteq \mathbb{R}$, $s \in \mathbb{R}$: $s = \sup A \iff s$ is an upper bound for A and $\forall \varepsilon > 0\ \exists a \in A\ s - \varepsilon < a$. Similarly for $\inf A$.
NIP: Nested Interval Property: $I_n = [a_n, b_n]$ closed and bounded for each $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$ for all $n \Rightarrow \bigcap_{n \in \mathbb{N}} I_n \ne \emptyset$. Fails if not closed, or not bounded.
Archimedean: $\forall a \in \mathbb{R}\ \exists n \in \mathbb{N}\ n > a$; equivalently $\forall b > 0\ \exists n \in \mathbb{N}\ 1/n < b$.

4. Axioms & Theorems I (L1-9), cont'd

Uniq./Bdd. limits: A convergent sequence has a **unique** limit. Every **convergent** sequence is **bounded** (converse false).
Alg. Limit Thm: $a_n \to a$, $b_n \to b \Rightarrow ca_n \to ca$ ($c \in \mathbb{R}$); $a_n + b_n \to a + b$; $a_n b_n \to ab$; $1/a_n \to 1/a$ if $a \neq 0$.
Order Limit Thm: Convergent $\{a_n\}, \{b_n\}$: eventually $a_n \geq 0 \Rightarrow \lim a_n \geq 0$; eventually $a_n \leq b_n \Rightarrow \lim a_n \leq \lim b_n$; eventually $a_n \leq c \Rightarrow \lim a_n \leq c$. Strict $<$ need NOT pass to the limit.
Squeeze Thm: Eventually $a_n \leq b_n \leq c_n$, $\lim a_n = \lim c_n = l \Rightarrow \{b_n\}$ converges to l. (Also asserts convergence of $\{b_n\}$, unlike OLT.)
Cauchy Criterion: $\{a_n\}$ is convergent $\iff$ $\{a_n\}$ is Cauchy.
Subseq. Limits: $a_n \to a \Rightarrow$ every subsequence $a_{n_k} \to a$.
Divergence Crit.: A **bounded** $\{a_n\}$ is divergent $\iff$ it has two subsequences converging to different limits. (Needs boundedness – stated for bounded seqs. only.)
Bolzano–Weierstrass: Every **bounded** sequence has a convergent subsequence.
MCL / MCT: Lemma: eventually monotone $\{a_n\}$ with a convergent subsequence is convergent. **Theorem**: eventually monotone *and* bounded $\Rightarrow$ convergent.

5. Theorems II (L10-17)

Terms $\to 0$: $\sum a_n$ convergent $\Rightarrow a_n \to 0$. (Converse false: $\sum 1/n$.)
Alg. Limit Thm (series): $\sum a_n \to s$, $\sum b_n \to t \Rightarrow \sum ca_n \to cs$ ($c \in \mathbb{R}$); $\sum(a_n + b_n) \to s + t$.
Cauchy Crit. (series): $\sum a_n$ convergent $\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > m \geq N\ |a_{m+1} + \cdots + a_n| < \varepsilon$.
Alt. Series Test: $\{b_n\}$ decreasing and $b_n \to 0 \Rightarrow \sum(-1)^{n-1} b_n$ converges.
Comparison Test: $0 \leq a_n \leq b_n$ for all n, $\sum b_n$ convergent $\Rightarrow \sum a_n$ convergent.
Abs. Conv. Test: $\sum |a_n|$ convergent $\Rightarrow \sum a_n$ convergent.
Seq. Crit. Limit Pts: a is a limit point of $A \iff \exists \{a_n\}$ in $A \setminus \{a\}$ with $a_n \to a$.
ε-Crit. Closures: $a \in \overline{A} \iff \forall \varepsilon > 0\ \exists x \in A\ |x - a| < \varepsilon$.
Seq. Crit. Closures: $a \in \overline{A} \iff \exists \{a_n\}$ in A with $a_n \to a$; i.e. $\overline{A} = \{\text{limits of convergent sequences in } A\}$.
Limit Pts of Closures: a is a limit point of $\overline{A} \iff a$ is a limit point of A.
Closures of Closures: $\overline{\overline{A}} = \overline{A}$.
Closure Crit. Closed: F is closed $\iff \overline{F} \subseteq F$.
Seq. Crit. Closed: F is closed $\iff$ every convergent sequence in F has its limit in F.
Extrema Cl. & Bdd: A nonempty, closed, bounded $K \subseteq \mathbb{R}$ has a maximum *and* a minimum.
Seq. Compactness: K is closed and bounded $\iff$ K is (sequentially) compact.
Nested Set Property: $K_n \subseteq \mathbb{R}$ nonempty, closed, bounded for each $n \in \mathbb{N}$, $K_n \supseteq K_{n+1}$ for all $n \Rightarrow \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$.
Dependency chain: Completeness $\Rightarrow$ NIP, Archimedean. Completeness + ε-Crit. $\Rightarrow$ Bolzano–Weierstrass $\Rightarrow$ MCT. NIP $\Rightarrow$ Cauchy Criterion. BW $\Rightarrow$ Seq. Compactness $\Rightarrow$ Nested Set Property. Seq. Crit. Limit Pts $\Rightarrow$ Seq. Crit. Closures. ε-Crit. Closures $\Rightarrow$ Limit Pts of Closures $\Rightarrow$ Closure Crit. Closed $\Rightarrow$ Extrema (with Completeness). Closure Crit. Closed + Seq. Crit. Closures $\Rightarrow$ Seq. Crit. Closed $\Rightarrow$ Seq. Compactness.

6. Proof-Technique Menu

Prove $s = \sup A$: (i) s upper bound: $\forall a \in A\ a \leq s$; (ii) no smaller upper bound: $\forall \varepsilon > 0\ \exists a \in A\ a > s - \varepsilon$ (or: every upper bound $b \geq s$, if ε-Crit. is banned).
Set has no max: Assume $m = \max A$; exhibit $a \in A$ with $a > m$ (e.g. average m with a known larger element, or an explicit formula) $\Rightarrow$ contradiction.
ε-N limit skeleton: Let $\varepsilon > 0$ be given. Bound $|a_n - a|$ by an expression $f(n)$; choose $N \in \mathbb{N}$ (Archimedean) so $f(n) < \varepsilon$ for all $n \geq N$; conclude via the definition of limit.
Cauchy-from-defn.: *Bound first*: let $\varepsilon > 0$; bound $|a_n - a_m|$ via algebra/triangle inequality in terms of $1/n, 1/m$ (or similar); pick N so each piece is $< \varepsilon/2$ for $n, m \geq N$.
Archimedean choice of n: To beat a target $b > 0$: Archimedean Property gives $n \in \mathbb{N}$ with $1/n < b$ directly – cite it, do not re-derive from Completeness.
Induction: State $P(n)$; verify base case $P(1)$ (or $P(n_0)$); assume $P(n)$, derive $P(n+1)$; conclude $\forall n \in \mathbb{N}\ P(n)$.
Contradiction / contrapositive: Contradiction: assume the negation, derive an absurdity. Contrapositive: to show $P \Rightarrow Q$, show $\neg Q \Rightarrow \neg P$ (useful for Divergence Crit., Seq. Compactness's boundedness half).
Subsequence extraction: Build n_k recursively: pick n_1 satisfying the target condition; given n_k, search the tail $n > n_k$ for the next qualifying index n_{k+1} – guarantees $n_1 < n_2 < \cdots$, i.e. $n_k \geq k$.
Not closed / not compact: Not closed: exhibit a limit point of A outside A (sequence in A converging outside A). **Not compact**: exhibit a sequence in K with no subsequence converging in K (unbounded, or bounded with all subsequential limits outside K).

7. Writing Expectations

Exam-graded style: (1) Complete sentences, e.g. "b is an upper bound for A if for all $a \in A$, $a \leq b$," not "$a \in A, a \leq b$." (2) Join related statements with connectives/ $\Longrightarrow$ / $\Longleftrightarrow$. (3) Show inference with therefore/hence/thus/so or $\therefore$.

Triggers & Traps

"Show $\sup A = s$": (i) s upper bound; (ii) $\forall \varepsilon > 0\ \exists a \in A\ a > s - \varepsilon$ (or: every upper bound $b \geq s$, if ε-Crit. is banned).
"...use the definition. Do not use X": Find every named lemma/theorem secretly relying on X; reprove that step from the raw definition instead of citing it.
"Is A closed?": Hunt for a limit point of A outside A (a sequence in A with limit outside A) $\Rightarrow$ **No**; else show every limit point lies in $A \Rightarrow$ **Yes**.
"Is K compact?": Sequential Compactness: K compact $\iff$ closed *and* bounded – check both separately.
T/F: sup vs max: $\sup A$ need not be in A (e.g. $\sup(0,1) = 1 \notin (0,1)$); $\max A$ always is, when it exists.
T/F: bdd but divergent: Bounded does *not* imply convergent, e.g. $\{(-1)^n\}$ – use Divergence Criterion (two subsequences, limits ± 1).
T/F: $a_n \to 0$ vs $\sum a_n$: $a_n \to 0$ does *not* imply $\sum a_n$ converges, e.g. harmonic series $\sum 1/n$.
T/F: subsequence indices: A subsequence needs strictly increasing indices $n_1 < n_2 < \cdots$; non-monotone or repeated index lists are not subsequences.
T/F: limits and $<$: Limits preserve only $\leq$: eventually $a_n < b_n$ gives only $\lim a_n \leq \lim b_n$, never strict (e.g. $1/n > 0$ but $\lim(1/n) = 0$).
"two subsequences, diff. limits": Divergence Criterion needs $\{a_n\}$ **bounded** first – state boundedness before invoking it.
Series algebra trap: $\sum a_n, \sum b_n$ both divergent can still give $\sum(a_n + b_n)$ or $\sum a_n b_n$ convergent; the Algebraic Limit Theorem for Series needs *both* series convergent before concluding about their sum/scalar multiple.