MH3100 Real Analysis I – Revision Notes

Quantitative Research Society @NTU

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Abstract

Comprehensive revision notes for MH3100 Real Analysis I, rebuilt lecture by lecture from the AY2026/27 slides and covering the Midterm syllabus (Lectures 1–17). The topics are the real numbers and the Axiom of Completeness, sequences and their limits, subsequences, the Bolzano–Weierstrass and Monotone Convergence Theorems, series, and the topology of sets of real numbers (limit points, closures, closed sets and compact sets). Every theorem is proved in full. Each proof separates the *Idea* (scratch work) from an exam-ready *Proof* written in complete sentences, with explicit quantifiers, logical connectives and a stated inference at every step.

Chapter	Week	Lecture	Tutorial	Key Concepts
Ch. 1: *The Real Numbers*	1–2	L1–3	T1	sup/inf; Axiom of Completeness; ε-Criterion; Nested Interval Property; Archimedean Property
Ch. 2: *Sequences and Limits*	2–3	L4–6	T2	ε–N limits; Algebraic/Order Limit Theorems; Squeeze Theorem; Cauchy Criterion
Ch. 3: *Subsequences, BW, MCT*	3–4	L7–9	T3	subsequences; Divergence Criterion; Bolzano–Weierstrass; Monotone Convergence
Ch. 4: *Series*	4–5	L10–12	T4	partial sums; Cauchy Criterion for Series; alternating, comparison and absolute convergence tests
Ch. 5: *Limit Points and Closed Sets*	5–6	L13–15	T5	limit/isolated points; closure; closed sets; sequential criteria
Ch. 6: *Compact Sets*	6	L16–17	T5	extrema of closed bounded sets; sequential compactness; Nested Set Property
Ch. 7: Midterm Summary	–	L1–17	T1–T5	Synthesis chapter: comparison tables, method patterns, traps, integrated examples.

This revision guide follows the official MH3100 lecture sequence, one chapter per lecture group (as the slide decks are grouped), with a synthesis chapter before the Midterm. Week numbers are approximate; tutorial dates follow the tutorial sheets. Lectures 18–20 (function limits and continuity) are outside the Midterm and are not covered here.

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Official Version

How to Use This Document

Structure of the Notes

Lectures provide the definitions, axioms and theorems; these notes state them in the course's own words and prove every one of them. Tutorials are folded in as labelled worked examples ("Worked Example (Tutorial n, Qk)"); full tutorial solutions live in the separate Tutorial Solutions series. Each chapter is organised into:

1. **Core Theory** — definitions, axioms, theorems, and the intuition behind them.
2. **Proofs** — every theorem proved in full. Where the slides state a result without proof, a QRS proof using only earlier results is supplied and flagged.
3. **Worked Examples** — lecture and tutorial problems written as model exam answers.
4. **Exam Focus** — high-yield results, proof templates, and typical traps.

How Each Proof Is Laid Out

- **Idea.** Scratch work: guessing the supremum, choosing ε, finding N. It is *not* part of the written answer.
- **Proof.** The exam-ready write-up. It uses complete sentences, joins related statements with logical connectives, and shows every inference with "therefore", "hence", "thus", "so" or $\therefore$. Quantifiers are stated explicitly, and each definition or theorem used is cited by its slide name. It ends with $\square$.
- **Common slip.** Where marks are typically lost.

Suggested Revision Flow

- First pass: read the definitions and theorem statements, and the Key Formula boxes.
- Second pass: reprove every theorem without notes; attempt the Worked Examples *before* reading the solutions.
- Final pass: the Midterm Summary chapter, then the Common Mistake and Exam Focus boxes in the week before the test.

Colour and Box System (Legend)

Colour	Box Name	Use This For
Green	**Definition**	Formal definitions and assumptions (numbered).
Blue (band)	**Theorem**	Formal results; a proof/derivation follows immediately.
Blue	**Key Formula**	Equations to memorise or rederive quickly.
Brown	**Worked Example**	Fully worked questions. Try on your own first.
Red	**Common Mistake**	Typical errors and conceptual traps.
Purple	**Exam Focus**	High-yield checklists and exam strategy.
Cyan	**Extension**	Beyond-syllabus depth (non-examinable unless stated).

Part I
Main Notes

1 Lectures 1–3: The Real Numbers and the Axiom of Completeness

This chapter answers three questions: (1) what *are* the real numbers, and how do they differ from the rationals; (2) what does it mean for a set of real numbers to have a least upper bound or a greatest lower bound; and (3) what structural consequences follow from the single axiom that distinguishes $\mathbb{R}$ from $\mathbb{Q}$. It is the first chapter of the course, and it prepares Chapter 2 (Sequences and Limits), where the ε-Criterion and the Archimedean Property become the standard toolkit for every ε–N proof.

Lecture 1 motivates the need for $\mathbb{R}$ by exhibiting a concrete gap in $\mathbb{Q}$ (the $\sqrt{2}$ example). Lecture 2 makes precise what a "gap" means via upper bounds, suprema, and the Axiom of Completeness, and derives the ε-Criterion that turns the definition of supremum into a workable proof technique. Lecture 3 draws the two headline structural consequences of completeness – the Nested Interval Property and the Archimedean Property – that Chapter 2 onward will use repeatedly.

1.1 Lecture 1: Construction of the Real Numbers

Lecture 1 has no boxed definitions or theorems on the slides: it is a narrative build-up $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R}$, closing with one worked example. We reproduce the notation exactly, since every later lecture depends on it.

Key Formula: Number System Notation

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}, \qquad \mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}.$$

For $X \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$: X^+ denotes the positive elements of X (so $\mathbb{Z}^+ = \mathbb{N}$); X_0^+ denotes the nonnegative elements (so $\mathbb{Z}_0^+ = \mathbb{N}_0$); X^- denotes the negative elements; X_0^- denotes the nonpositive elements. The subscript 0 always marks the version that includes zero.

Sample memberships. $0 \in \mathbb{Z}_0^+$ but $0 \notin \mathbb{Z}^+ = \mathbb{N}$; $-\frac{1}{2} \in \mathbb{Q}^-$ but $-\frac{1}{2} \notin \mathbb{Z}$; $\sqrt{2} \in \mathbb{R}^+$ but $\sqrt{2} \notin \mathbb{Q}$.

The natural numbers $\mathbb{N}$ are the counting numbers; they are used to define *finite sets* $\{a_1, \ldots, a_n\}$ and *countable sets* $\{a_1, a_2, a_3, \ldots\}$ of distinct elements. The integers $\mathbb{Z}$ are constructed from $\mathbb{N}$ by adjoining 0 and negatives; the rationals $\mathbb{Q}$ are constructed from $\mathbb{Z}$ and $\mathbb{N}$ as

$$\mathbb{Q} = \left\{ \frac{i}{n} : (i, n) \in \mathbb{Z} \times \mathbb{N},\ |i| \text{ and } n \text{ have no common divisor} > 1 \right\}.$$

Both $\mathbb{Z}$ and $\mathbb{Q}$ are countable: each can be listed as a single sequence (e.g. $\mathbb{Z} = \{0, -1, 1, -2, 2, -3, 3, \ldots\}$). We can add, subtract, multiply, divide, and compare any two rational numbers.

Intuition. Each stage of $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R}$ repairs a defect of the previous system: $\mathbb{Z}$ adjoins the additive inverses that $\mathbb{N}$ lacks (so subtraction becomes total), $\mathbb{Q}$ adjoins the multiplicative inverses that $\mathbb{Z}$ lacks (so division by nonzero elements becomes total), and $\mathbb{R}$ – as Lectures 2–3 make precise – adjoins the suprema that $\mathbb{Q}$ lacks (so every bounded-above set gets a least upper bound). Each system after $\mathbb{N}$ is built so that the arithmetic and comparison operations already available restrict correctly to the smaller system; this is why, e.g., $2 + 3 = 5$ means the same thing whether $2, 3, 5$ are regarded as elements of $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$.

Common Mistake: Is 0 a Natural Number?

In this course, $\mathbb{N} = \{1, 2, 3, \ldots\}$ **does not** include 0; the symbol $\mathbb{N}_0$ is reserved for $\mathbb{N} \cup \{0\}$. This differs from some other textbooks (and from set-theoretic conventions) where $\mathbb{N}$ starts at 0. Writing "$0 \in \mathbb{N}$" or indexing a sequence $\{a_n\}_{n=1}^{\infty}$ starting from a_0 without switching to $\mathbb{N}_0$-notation is marked wrong under this course's convention.

Why is a new construction needed for $\mathbb{R}$? The slides pose this directly: "How do we formally (mathematically) define a real number or the real numbers $\mathbb{R}$?" The motivating worked example shows that $\mathbb{Q}$ itself is not enough.

Example (Lecture 1): Approximating $\sqrt{2}$ by Rationals

The decimal expansion $\sqrt{2} = 1.4142135623730950488\ldots$ can be approached by an increasing sequence of rationals whose squares stay below 2:

$$q_1 = 1 :\ 1^2 = 1 < 2, \quad q_2 = 1.4 :\ 1.4^2 = 1.96 < 2, \quad q_3 = 1.41 :\ 1.41^2 = 1.9881 < 2,$$

$$q_4 = 1.414 :\ 1.414^2 = 1.999396 < 2, \quad q_5 = 1.4142 :\ 1.4142^2 = 1.99996164 < 2.$$

We may associate $\sqrt{2}$ with the set of *rational numbers less than it*:

$$A_{\sqrt{2}} := \mathbb{Q}_0^- \cup \{q \in \mathbb{Q}^+ : q^2 < 2\}.$$

$A_{\sqrt{2}} \subseteq \mathbb{Q}$ is nonempty and bounded above in $\mathbb{Q}$ (by 2), but has no supremum in $\mathbb{Q}$. Write $A := A_{\sqrt{2}}$. A *supremum of A in $\mathbb{Q}$* is a rational s that is an upper bound for A and satisfies $s \leq u$ for every rational upper bound u of A.

Method 1: Two explicit rational witnesses and a parity argument

Proof. *Step 1: A is nonempty and bounded above by 2.* Since $1 \in \mathbb{Q}^+$ and $1^2 = 1 < 2$, we have $1 \in A$; therefore $A \neq \emptyset$. Let $a \in A$. If $a \leq 0$, then $a < 2$. If $a > 0$, then $a^2 < 2$ by the definition of A; if we had $a \geq 2$, then $a^2 \geq 4 > 2$, which is false; hence $a < 2$. Therefore every $a \in A$ satisfies $a \leq 2$, i.e. 2 is an upper bound for A.
Step 2: no rational number has square 2. Suppose, for contradiction, that $q \in \mathbb{Q}$ and $q^2 = 2$. Write $q = \frac{i}{n}$ with $(i, n) \in \mathbb{Z} \times \mathbb{N}$ such that $|i|$ and n have no common divisor > 1. Then

$$i^2 = 2n^2,$$

so i^2 is even. If i were odd, say $i = 2k + 1$ with $k \in \mathbb{Z}$, then $i^2 = 4k^2 + 4k + 1$ would be odd; hence i is even, $i = 2k$ for some $k \in \mathbb{Z}$. Substituting,

$$4k^2 = 2n^2 \implies n^2 = 2k^2,$$

so n^2 is even, and by the same parity argument n is even. Therefore 2 divides both $|i|$ and n, contradicting the choice of (i, n). Hence $q^2 \neq 2$ for every $q \in \mathbb{Q}$.
Step 3: if $q \in \mathbb{Q}^+$ and $q^2 < 2$, then some $q' \in \mathbb{Q}^+$ with $q' > q$ satisfies $q'^2 < 2$. Let

$$h := \frac{2 - q^2}{2(2q + 1)}.$$

Then $h \in \mathbb{Q}$, and $h > 0$ because $2 - q^2 > 0$ and $2q + 1 > 0$. Also $h < 1$, because $0 < 2 - q^2 < 2 < 2(2q + 1)$. Hence $0 < h < 1$, and therefore $h^2 < h$. Put $q' := q + h \in \mathbb{Q}^+$. Then $q' > q$ and

$$\begin{aligned} q'^2 &= q^2 + 2qh + h^2 \\ &< q^2 + 2qh + h = q^2 + (2q + 1)h \\ &= q^2 + \frac{2 - q^2}{2} < q^2 + (2 - q^2) = 2. \end{aligned}$$

Step 4: if $q \in \mathbb{Q}^+$ and $q^2 > 2$, then some $q' \in \mathbb{Q}^+$ with $q' < q$ satisfies $q'^2 > 2$. Let

$$q' := q - \frac{q^2 - 2}{2q} = \frac{q^2 + 2}{2q}.$$

Then $q' \in \mathbb{Q}^+$ since $q^2+2>0$ and $2q>0$, and $q'<q$ since $\frac{q^2-2}{2q}>0$. Moreover

$$q'^2-2=\frac{(q^2+2)^2-8q^2}{4q^2}=\frac{q^4-4q^2+4}{4q^2}=\frac{(q^2-2)^2}{4q^2}>0,$$

so $q'^2>2$.
Step 5: A has no supremum in $\mathbb{Q}$. Suppose, for contradiction, that $s\in\mathbb{Q}$ is a supremum of A in $\mathbb{Q}$. Since $1\in A$ (Step 1) and s is an upper bound for A, $s\geq 1>0$; hence $s\in\mathbb{Q}^+$. By Step 2, $s^2\neq 2$; therefore $s^2<2$ or $s^2>2$.
Case $s^2<2$. Then $s\in A$ by the definition of A, and Step 3 gives $q'\in A$ with $q'>s$. Hence s is not an upper bound for A, a contradiction.
Case $s^2>2$. Step 4 gives $q'\in\mathbb{Q}^+$ with $q'<s$ and $q'^2>2$. We show that q' is an upper bound for A. Let $a\in A$. If $a\leq 0$, then $a<q'$. If $a>0$, then $a^2<2<q'^2$, so

$$(q'-a)(q'+a)=q'^2-a^2>0;$$

since $q'+a>0$, this gives $q'-a>0$, i.e. $a<q'$. Hence q' is a rational upper bound for A with $q'<s$, which contradicts $s\leq q'$.
Both cases are impossible. Therefore A has no supremum in $\mathbb{Q}$. □

$A_{\sqrt{2}}$ is nonempty and bounded above in $\mathbb{Q}$, but has no supremum in $\mathbb{Q}$

Method 2: A single rational map and infinite descent

Proof. *Step 1: A is nonempty and bounded above by 2.* Since $1\in\mathbb{Q}^+$ and $1^2<2$, $1\in A$. Let $a\in A$. If $a\leq 0$, then $a<2$; if $a>0$, then $a^2<2<4$, and $a\geq 2$ would give $a^2\geq 4$; hence $a<2$. Therefore 2 is an upper bound for A.
Step 2: no rational number has square 2 (descent). Suppose, for contradiction, that $s\in\mathbb{Q}$ and $s^2=2$. Since $(-s)^2=s^2$, we may assume $s>0$. Let $K:=\{n\in\mathbb{N}: ns\in\mathbb{N}\}$. Writing $s=\frac{i}{n}$ with $i\in\mathbb{N}$ and $n\in\mathbb{N}$ gives $ns=i\in\mathbb{N}$, so $K\neq\emptyset$. By the Well-Ordering of $\mathbb{N}$ (every nonempty subset of $\mathbb{N}$ has a least element), K has a least element n; put $m:=ns\in\mathbb{N}$. If $s\leq 1$ then $s^2\leq 1$, and if $s\geq 2$ then $s^2\geq 4$; since $s^2=2$, we get $1<s<2$, and multiplying by $n>0$ gives $n<m<2n$. Put $n':=m-n$. Then $n'\in\mathbb{N}$ (as $m>n$) and $n'<n$ (as $m<2n$). Moreover $ms=(ns)s=ns^2=2n$, so

$$n's=ms-ns=2n-m\in\mathbb{N}\qquad(\text{since } m<2n).$$

Therefore $n'\in K$ and $n'<n$, contradicting the minimality of n. Hence $q^2\neq 2$ for every $q\in\mathbb{Q}$.
Step 3: one map moves every $q\in\mathbb{Q}^+$ towards $\sqrt{2}$ on its own side. For $q\in\mathbb{Q}^+$ let

$$p:=\frac{2q+2}{q+2}\in\mathbb{Q}^+.$$

Then

$$p-q=\frac{2q+2-q(q+2)}{q+2}=\frac{2-q^2}{q+2},$$

and

$$p^2-2=\frac{(2q+2)^2-2(q+2)^2}{(q+2)^2}=\frac{4q^2+8q+4-2q^2-8q-8}{(q+2)^2}=\frac{2(q^2-2)}{(q+2)^2}.$$

Since $q+2>0$ and $(q+2)^2>0$: if $q^2<2$, then $p>q$ and $p^2<2$; if $q^2>2$, then $p<q$ and $p^2>2$.
Step 4: A has no supremum in $\mathbb{Q}$. Suppose, for contradiction, that $s\in\mathbb{Q}$ is a supremum of A in $\mathbb{Q}$. Since $1\in A$, $s\geq 1$, so $s\in\mathbb{Q}^+$; by Step 2, $s^2<2$ or $s^2>2$. If $s^2<2$, then $s\in A$, and Step 3 gives $p\in\mathbb{Q}^+$ with $p>s$ and $p^2<2$, i.e. $p\in A$ with $p>s$; hence s is not an upper bound for A, a contradiction. If $s^2>2$, Step 3 gives $p\in\mathbb{Q}^+$ with $p<s$ and $p^2>2$. For every $a\in A$: if $a\leq 0$ then $a<p$; if $a>0$ then $p^2-a^2>0$ and $p+a>0$, so $p-a=\frac{p^2-a^2}{p+a}>0$. Hence p is a rational upper bound for A with $p<s$, contradicting $s\leq p$. Therefore A has no supremum in $\mathbb{Q}$. □

$A_{\sqrt{2}}$ is nonempty and bounded above in $\mathbb{Q}$, but has no supremum in $\mathbb{Q}$

Extension: The Dedekind-Cut-Style Construction of $\mathbb{R}$ (Not Carried Out in This Course)

The slides specify that a real number can be identified with a set $A \subset \mathbb{Q}$ satisfying: (1) A is nonempty and proper ($A \neq \emptyset$ and $A \neq \mathbb{Q}$); (2) A is downward closed ($\forall r, q \in \mathbb{Q}\ (r \in A) \wedge (q < r) \implies q \in A$); and (3) A has no maximum ($\forall r \in A\ \exists s \in A\ r < s$). Arithmetic on $\mathbb{R}$ $(+, -, \times, \div)$ and the comparison relation $<$ are then defined via the corresponding operations on these associated sets of rationals. The course states this construction but does not carry out the verification that it yields a field; the slides refer the details to Abbott, *Understanding Analysis*, §8.6.

Why each of the three conditions is needed (informal). Condition (1) rules out the two degenerate "numbers" $A = \emptyset$ (which would represent $-\infty$) and $A = \mathbb{Q}$ (which would represent $+\infty$). Condition (2), downward closure, is what makes A behave like "everything less than the real number it represents" – e.g. the set $A_{\sqrt{2}} = \mathbb{Q}_0^- \cup \{q \in \mathbb{Q}^+ : q^2 < 2\}$ from the worked example above is downward closed, since if $q \in A_{\sqrt{2}}$ and $r < q$ then $r^2 < q^2 < 2$ (for $r, q > 0$) or $r \leq 0$, so $r \in A_{\sqrt{2}}$ too. Condition (3), no maximum, is what rules out sets like $A = \{q \in \mathbb{Q} : q \leq 1\}$ that would otherwise represent the *rational* number 1 by two different associated sets ($\{q \leq 1\}$ and $\{q < 1\}$); requiring no maximum forces the representation to be unique.

1.2 Lecture 2: Bounds, Suprema, and Infima

Definition 1.1: Upper Bound and Lower Bound

A set $A \subseteq \mathbb{R}$ is said to be **bounded above** if there is some number $b \in \mathbb{R}$ satisfying

$$\forall a \in A \quad a \leq b.$$

The number b satisfying the above is called an **upper bound** for A. We similarly define a set to be **bounded below** and a **lower bound**. If A is both bounded above and below, then we say that it is **bounded**.

Intuition. An upper bound is any ceiling, not necessarily a tight one: if $A = (0, 1)$, then $1, 2, 100$ are all upper bounds for A. A set can have many upper bounds (indeed, if b is an upper bound, so is any $b' > b$), or none at all (e.g. $A = \mathbb{N}$ has no upper bound in $\mathbb{R}$, as Theorem 1.4 will confirm).

Definition 1.2: Supremum and Infimum

A number $s \in \mathbb{R}$ is said to be the **supremum** (a.k.a. **least upper bound**) of a set $A \subseteq \mathbb{R}$ if (1) s is an upper bound for A; and (2) s is not greater than any upper bound for A. We write $s = \sup A$ to mean that s is the supremum of A. We similarly define the **infimum** of a set and the notation $\inf A$.

Intuition. Among all the (possibly many) upper bounds of A, the supremum is the smallest – the tightest possible ceiling. For $A = (0, 1)$, every $b \geq 1$ is an upper bound, and 1 is the smallest of these, so $\sup A = 1$. Condition (2) of the definition is what makes $\sup A$ *unique* when it exists: if s and s' both satisfy the definition, then s being no greater than the upper bound s' gives $s \leq s'$, and symmetrically $s' \leq s$, so $s = s'$.

Definition 1.3: Maximum and Minimum

A number $m \in \mathbb{R}$ is said to be the **maximum** of a set $A \subseteq \mathbb{R}$ if $m \in A$ and m is an upper bound for A. We write $m = \max A$ to mean that m is the maximum of A. We similarly define the **minimum** of a set and the notation $\min A$.

Intuition. A maximum is a supremum that happens to be attained: for a finite set like $\{1, 2, 3\}$, the

maximum 3 is simply the largest listed element. The distinction only bites for infinite sets, where the "smallest ceiling" can sit just outside the set, as with $A = (0, 1)$ and $\sup A = 1 \notin A$.
Comparison between sup/inf and max/min. The maximum (respectively minimum) of A, if it exists, is always the supremum (respectively infimum) of A – but the converse is generally not true. The supremum (respectively infimum) of A, if it exists and also happens to be an element of A, is precisely the maximum (respectively minimum) of A.

Common Mistake: $\sup A$ Need Not Belong to A

The supremum of A is the *least upper bound*, not necessarily a member of A. For example $\sup(0, 1) = 1 \notin (0, 1)$. Writing "$\sup A = \max A$" is only justified once you have separately checked that $\sup A \in A$. Symmetrically for $\inf A$ and $\min A$. This is the single most common source of lost marks in Tutorial 1 Question 1.

Exam Focus: True/False Statements on Bounds

The sample midterm's Q1/Q2 format asks for true/false judgements on statements like these (no justification is required on the actual paper, but you should be able to justify each instantly):

- "Every set $A \subseteq \mathbb{R}$ that is bounded above has a maximum." **False** – e.g. $A = (0, 1)$ is bounded above but $\sup A = 1 \notin A$, so A has no maximum.
- "If $s = \sup A$ then $s \geq a$ for every $a \in A$." **True** – this is exactly the "upper bound" half of the definition of supremum.
- "If $A \subseteq B$ and both are nonempty and bounded above, then $\sup A \leq \sup B$." **True** – $\sup B$ is an upper bound for B, hence for $A \subseteq B$, so $\sup A \leq \sup B$ by the definition of supremum applied to A.

1.3 Lecture 2: The Axiom of Completeness

Theorem 1.1: Axiom of Completeness

Every nonempty set of real numbers that is bounded above has a supremum.

Proof. None is given, and none is possible: this is an *axiom* of the real numbers, not a consequence of anything more basic. It is precisely the property that $\mathbb{Q}$ lacks – the set $A_{\sqrt{2}}$ from Lecture 1's worked example is nonempty and bounded above in $\mathbb{Q}$ but has no rational supremum – and it is the property that $\mathbb{R}$ is *defined* to have.

Two conventions extending sup. The empty set $\emptyset$ does not have a supremum, because every real number is (vacuously) an upper bound for $\emptyset$; we may write $\sup \emptyset = -\infty$. A set $A \subseteq \mathbb{R}$ that is not bounded above does not have a supremum, because A has no upper bound at all; we may write $\sup A = \infty$. (By the symmetric argument, $\inf \emptyset = \infty$ and $\inf A = -\infty$ when A is unbounded below.)

Extension: Axiomatic Definition of the Real Numbers (Stated, Not Proved)

Theorem (Axiomatic Definition of the Real Numbers). All ordered fields satisfying the Axiom of Completeness are isomorphic to each other. The real numbers $\mathbb{R}$ can then be defined to be an ordered field that satisfies the Axiom of Completeness. Here a *field* is a set whose elements obey the laws of $+, -, \times, \div$; an *ordered field* is a field whose elements obey the laws of the comparison relation $<$, with the arithmetic operations respecting $<$; and an *isomorphism* of ordered fields is a bijection preserving all arithmetic operations and $<$. The slides state this theorem and unpack its terms, but give **no proof** of the isomorphism claim anywhere in Lectures 1–9; they instead refer the reader to the final chapter of Spivak's *Calculus*. This course does not examine the proof of this theorem.

Common Mistake: Forgetting the Two Hypotheses of Completeness

The Axiom of Completeness requires *both* that A is nonempty *and* that A is bounded above – dropping either breaks the conclusion. $A = \emptyset$ is bounded above (vacuously, by every real number) but has no supremum, since no real number fails to be an upper bound of $\emptyset$, so there is no *least* one. $A = \mathbb{N}$ is nonempty but not bounded above, and indeed has no supremum (Theorem 1.4 below shows precisely that no real number bounds $\mathbb{N}$). Before invoking the Axiom of Completeness on a set A, always state *both* checks explicitly: "A is nonempty (exhibit an element) and bounded above (exhibit a bound)."

1.4 Lecture 2: The ε-Criterion for Suprema and Infima

Theorem 1.2: ε-Criterion for Suprema and Infima

For any set $A \subseteq \mathbb{R}$ and any number $s \in \mathbb{R}$:

$$s = \sup A \quad \text{if and only if} \quad \begin{array}{l} s \text{ is an upper bound for } A \text{ and} \\ \forall \varepsilon > 0 \ \exists a \in A \quad s - \varepsilon < a. \end{array} \tag{1}$$

A similar criterion holds for $\inf A$: for any set $A \subseteq \mathbb{R}$ and any number $s \in \mathbb{R}$,

$$s = \inf A \quad \text{if and only if} \quad \begin{array}{l} s \text{ is a lower bound for } A \text{ and} \\ \forall \varepsilon > 0 \ \exists a \in A \quad a < s + \varepsilon. \end{array} \tag{2}$$

Idea. *(Scratch work – not part of the written answer.)* Recall that $\sup A$ is defined as an upper bound that is no greater than any other upper bound. The condition (1) says "no number strictly smaller than s is an upper bound of A" – which is exactly the second half of the definition of supremum, restated as a negative existence statement. The forward direction turns "s is least" into "every smaller candidate fails"; the backward direction turns "every smaller candidate fails" into a proof by contradiction that no upper bound can be strictly below s.

Proof. We first prove the supremum case; the infimum case is deduced from it afterwards.

(Only if.) Suppose $s = \sup A$. By the definition of supremum, s is an upper bound for A. Let $\varepsilon > 0$ be given. Since $s - \varepsilon < s$ and s is the *least* upper bound for A, the number $s - \varepsilon$ cannot itself be an upper bound for A (otherwise s would not be the least one). Therefore, by the definition of upper bound, there exists $a \in A$ such that a is not $\leq s - \varepsilon$; that is, $s - \varepsilon < a$. As $\varepsilon > 0$ was arbitrary, (1) holds.

(If.) Suppose s is an upper bound for A and (1) holds. We must show that s is not greater than any upper bound for A. Suppose, for contradiction, that b is an upper bound for A with $b < s$. Let $\varepsilon := s - b > 0$. By (1), there exists $a \in A$ such that $s - \varepsilon < a$, i.e., $b < a$. But b is an upper bound for A, so $a \leq b$, contradicting $b < a$. Hence no upper bound for A is strictly less than s; combined with s being an upper bound for A, the definition of supremum gives $s = \sup A$.

(Infimum case, via $A \to -A$.) This half is not carried out on the slides in Lectures 1–3 – they state only that "a similar criterion holds for $\inf A$" – so the argument below is written out by QRS using the standard reflection trick. Let $-A := \{-a : a \in A\}$.

Claim 1. For $b \in \mathbb{R}$: b is a lower bound for A if and only if $-b$ is an upper bound for $-A$. Indeed, b is a lower bound for A means $\forall a \in A \ b \leq a$, which (multiplying by -1, reversing the inequality) is equivalent to $\forall a \in A \ -a \leq -b$, i.e., $\forall (-a) \in -A \ (-a) \leq -b$, i.e., $-b$ is an upper bound for $-A$.

Claim 2. $s = \inf A$ if and only if $-s = \sup(-A)$. By the definition of infimum, $s = \inf A$ means s is a lower bound for A and s is not less than any lower bound for A. By Claim 1, "s is a lower bound for A" is equivalent to "$-s$ is an upper bound for $-A$", and (applying Claim 1 to every lower bound b of A, matched with the upper bound $-b$ of $-A$) "s is not less than any lower bound for A" is equivalent to "$-s$ is not greater than any upper bound for $-A$". Together these give exactly the definition of $-s = \sup(-A)$.

Now apply the already-proved supremum case of this theorem to the set $-A$ and the number $-s$: $-s = \sup(-A)$ if and only if $-s$ is an upper bound for $-A$ (by Claim 1, equivalent to s being a lower

bound for A) and, for every $\varepsilon > 0$, there exists an element $-a \in -A$ (that is, $a \in A$) such that

$$(-s) - \varepsilon < -a.$$

Multiplying by -1 reverses the inequality, so the displayed inequality is equivalent to $a < s+\varepsilon$. Therefore, combining with Claim 2, $s = \inf A$ if and only if s is a lower bound for A and for every $\varepsilon > 0$ there exists $a \in A$ with $a < s + \varepsilon$, which is exactly (2). □

Method 2: Direct Chain of Equivalences (the Slide's Route). The slides prove the supremum case as a single chain of logical equivalences rather than two separate directions; both are correct, and it is worth recognising this form too. Recall that $s = \sup A$ means, by definition, that s is an upper bound for A that is not greater than any other upper bound for A; write this second half as "every upper bound for A is at least s." Then, for a number already known to be an upper bound s,

$$\begin{aligned} \forall \varepsilon > 0\ \exists a \in A\ (s - \varepsilon < a) &\iff \forall \varepsilon > 0\ (s - \varepsilon \text{ is not an upper bound for } A) \\ &\iff \forall b < s\ (b \text{ is not an upper bound for } A) \\ &\iff \text{every upper bound for } A \text{ is at least } s. \end{aligned}$$

The first equivalence holds because "$s-\varepsilon < a$ for some $a \in A$" is exactly the negation of "$a \le s-\varepsilon$ for all $a \in A$", i.e. the negation of "$s-\varepsilon$ is an upper bound." The second equivalence is a relabelling $b := s-\varepsilon$ (as ε ranges over all positive reals, b ranges over all reals strictly less than s). The final expression is precisely the second half of the definition of supremum. Hence, given that s is an upper bound for A, condition (1) holds if and only if $s = \sup A$.

Common slip. The ε in (1) must work for *every* $\varepsilon > 0$, and the witness $a \in A$ is allowed to depend on ε (larger ε typically needs a less extreme a). Do not fix ε to a specific value (e.g. "take $\varepsilon = 1$") and call the criterion verified. Also, the criterion *presupposes* that s is already known to be an upper bound for A – checking only the ε-condition, without first checking s is an upper bound, proves nothing.

Mini-example: $\sup(0,1) = 1$ via the ε-Criterion. Let $A := (0,1)$. For every $a \in A$, $a < 1$, so 1 is an upper bound for A. Let $\varepsilon > 0$ be given. If $\varepsilon \ge 1$, then $1 - \varepsilon \le 0$, and any $a \in (0,1)$ (e.g. $a = 1/2$) satisfies $a > 0 \ge 1 - \varepsilon$. If $0 < \varepsilon < 1$, then $1 - \frac{\varepsilon}{2} \in (0,1) = A$ and $1 - \frac{\varepsilon}{2} > 1 - \varepsilon$. In either case, some $a \in A$ satisfies $1 - \varepsilon < a$. By the ε-Criterion for Suprema, $\sup A = 1$; and since $1 \notin A$, this supremum is not a maximum – confirming the Common Mistake noted after the definition of supremum in Lecture 2.

1.5 Lecture 3: The Nested Interval Property

Example (Lecture 3): The Bisection Method

Given $f : [a,b] \to \mathbb{R}$ with $f(a) < 0 < f(b)$, the bisection method locates some $c \in [a,b]$ with $f(c) = 0$:

1. Set $I_1 := [a_1, b_1] := [a, b]$.

2. For $k = 1, 2, 3, \ldots$, set $m_k = \frac{1}{2}(a_k + b_k)$ and set

$$I_{k+1} := [a_{k+1}, b_{k+1}] := \begin{cases} [m_k, b_k] & \text{if } f(m_k) < 0, \\ [a_k, m_k] & \text{if } f(m_k) \ge 0. \end{cases}$$

This produces nested closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

Method 1: Induction on k with a three-part invariant

Proof. Since $f(a) < 0 < f(b)$, we have $f(a) \ne f(b)$, so $a \ne b$; as $[a,b]$ is an interval, $a < b$. For $k \in \mathbb{N}$ let $P(k)$ be the statement

$$a_k < b_k, \qquad b_k - a_k = \frac{b-a}{2^{k-1}}, \qquad f(a_k) < 0 \le f(b_k).$$

Base case. $a_1 = a < b = b_1$, $b_1 - a_1 = b - a = \frac{b-a}{2^0}$, and $f(a_1) = f(a) < 0 < f(b) = f(b_1)$. Hence $P(1)$ holds.
Inductive step. Assume $P(k)$ for some $k \in \mathbb{N}$. Then

$$m_k - a_k = \frac{b_k - a_k}{2} > 0 \quad \text{and} \quad b_k - m_k = \frac{b_k - a_k}{2} > 0,$$

so $a_k < m_k < b_k$.
Case $f(m_k) < 0$. Then $I_{k+1} = [m_k, b_k]$, so $a_{k+1} = m_k < b_k = b_{k+1}$;

$$b_{k+1} - a_{k+1} = b_k - m_k = \frac{b_k - a_k}{2} = \frac{b-a}{2^k};$$

and $f(a_{k+1}) = f(m_k) < 0 \le f(b_k) = f(b_{k+1})$. Hence $P(k+1)$ holds. If $x \in I_{k+1}$, then $a_k < m_k \le x \le b_k$, so $x \in I_k$; therefore $I_{k+1} \subseteq I_k$.
Case $f(m_k) \ge 0$. Then $I_{k+1} = [a_k, m_k]$, so $a_{k+1} = a_k < m_k = b_{k+1}$;

$$b_{k+1} - a_{k+1} = m_k - a_k = \frac{b_k - a_k}{2} = \frac{b-a}{2^k};$$

and $f(a_{k+1}) = f(a_k) < 0 \le f(m_k) = f(b_{k+1})$. Hence $P(k+1)$ holds. If $x \in I_{k+1}$, then $a_k \le x \le m_k < b_k$, so $x \in I_k$; therefore $I_{k+1} \subseteq I_k$.
By induction, $P(k)$ holds and $I_{k+1} \subseteq I_k$ for every $k \in \mathbb{N}$. Since $a_k < b_k$ are real numbers, each $I_k = [a_k, b_k]$ is a closed and bounded interval, of length $\frac{b-a}{2^{k-1}}$. □

$\boxed{I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \text{ are nested closed and bounded intervals}}$

Theorem 1.3: Nested Interval Property

If $I_n = [a_n, b_n]$ is a closed and bounded interval for each $n \in \mathbb{N}$, and $I_n \supseteq I_{n+1}$ for each $n \in \mathbb{N}$, then the nested sequence $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ has a **nonempty intersection**; i.e., there is some $c \in \mathbb{R}$ with $c \in I_n$ for all $n \in \mathbb{N}$.

Idea. *(Scratch work – not part of the written answer.)* Collect all the left endpoints into a set $A = \{a_n\}$. Nesting forces every right endpoint b_n to sit above every left endpoint a_m, so each b_n is an upper bound for A. Take $c = \sup A$ (Axiom of Completeness); then c is above every a_n (since c is an upper bound of A) and below every b_n (since c is the *least* upper bound, and each b_n is already an upper bound). So $c \in [a_n, b_n]$ for every n.

Proof. Let $A := \{a_n : n \in \mathbb{N}\}$, which is nonempty since $\mathbb{N}$ is nonempty. Fix $n \in \mathbb{N}$; we show b_n is an upper bound for A. Let $m \in \mathbb{N}$ be arbitrary. Since the sequence of intervals is nested, either $I_n \supseteq I_m$ or $I_m \supseteq I_n$. If $I_n \supseteq I_m$, then $a_n \le a_m$ and $b_m \le b_n$, so $a_m \le b_m \le b_n$. If instead $I_m \supseteq I_n$, then $a_m \le a_n$, and since $I_n = [a_n, b_n]$ is a (nonempty) interval, $a_n \le b_n$; hence $a_m \le a_n \le b_n$. In either case $a_m \le b_n$. As $m \in \mathbb{N}$ was arbitrary, b_n is an upper bound for A.

Since $n \in \mathbb{N}$ was arbitrary, A is a nonempty set that is bounded above (by any b_n). Therefore the Axiom of Completeness asserts that A has a supremum; write $c := \sup A$.

We now show $c \in I_n$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since $a_n \in A$ and c is an upper bound for A (by the definition of supremum), $a_n \le c$. Since b_n is an upper bound for A (shown above) and $c = \sup A$ is not greater than any upper bound for A (by the definition of supremum), $c \le b_n$. Therefore $a_n \le c \le b_n$, i.e., $c \in I_n$. As $n \in \mathbb{N}$ was arbitrary, $c \in I_n$ for all $n \in \mathbb{N}$; hence $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$. □

Common Mistake: Dropping Closedness or Boundedness

Both hypotheses of Theorem 1.3 are essential.

- *Closedness matters.* $(0,1] \supset (0,\frac{1}{2}] \supset (0,\frac{1}{3}] \supset \cdots$ is nested with each interval bounded, yet $\bigcap_n (0,\frac{1}{n}] = \emptyset$: the proof above breaks because $a_n = 0$ is never itself included, so there is no element of the set of left endpoints that could be "caught" inside every interval.
- *Boundedness matters.* $[1,\infty) \supset [2,\infty) \supset [3,\infty) \supset \cdots$ is nested with each interval closed, yet $\bigcap_n [n,\infty) = \emptyset$: the proof above breaks because the set of left endpoints $A = \mathbb{N}$ is not bounded above, so $\sup A$ does not exist.

Do not quote the Nested Interval Property for a sequence of intervals unless you have checked *both* that each interval is closed and that at least one (equivalently, every) interval is bounded.

Remark (the Nested Interval Property fails for $\mathbb{Q}$). This is the same incompleteness phenomenon as Lecture 1's $\sqrt{2}$ example, now stated for nested intervals. Let a_n be the decimal expansion of $\sqrt{2}$ to n places and $b_n := a_n + 10^{-n}$:

$$a_1 = 1.4,\ b_1 = 1.5; \qquad a_2 = 1.41,\ b_2 = 1.42; \qquad a_3 = 1.414,\ b_3 = 1.415;$$

$$a_4 = 1.4142,\ b_4 = 1.4143; \qquad a_5 = 1.41421,\ b_5 = 1.41422; \qquad \ldots$$

The sets $I_n := \{q \in \mathbb{Q} : a_n \le q \le b_n\}$ form a nested sequence of closed bounded intervals *in* $\mathbb{Q}$ (each I_n is nonempty since $a_n, b_n \in \mathbb{Q}$, and $I_{n+1} \subseteq I_n$ since the decimal expansion only tightens), yet $\bigcap_n I_n = \emptyset$ in $\mathbb{Q}$: the only candidate real number trapped by every I_n is $\sqrt{2}$ itself, which is not rational, so no *rational* number lies in every I_n. Tracing through the proof of Theorem 1.3 with $A = \{a_n\} \subset \mathbb{Q}$ shows exactly where it breaks: the Axiom of Completeness (applied within $\mathbb{Q}$) would need to supply $\sup_{\mathbb{Q}} A$, but A has no supremum in $\mathbb{Q}$ at all. The Nested Interval Property is therefore a genuine consequence of completeness, not a fact about ordered fields in general.

1.6 Lecture 3: The Archimedean Property

Example (Lecture 3): Bisection Method, Continued

In the Bisection Method, the interval lengths are

$$b_1 - a_1,\ \frac{b_1-a_1}{2},\ \frac{b_1-a_1}{4},\ \frac{b_1-a_1}{8},\ \ldots,\ \frac{b_1-a_1}{2^{n-1}},\ \ldots$$

The lengths tend to 0, so $\bigcap_n I_n$ contains *at most one* real number:

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N} \quad n \ge N \implies \frac{b_1-a_1}{2^{n-1}} < \varepsilon.$$

Method 1: Archimedean Property with the bound $2^{n-1} \ge n$

Proof. Write $L := b_1 - a_1$. Since $f(a) < 0 < f(b)$ forces $a < b$, $L > 0$. At each step I_{k+1} is one of the halves $[m_k, b_k]$ or $[a_k, m_k]$ of I_k, each of length $\frac{b_k-a_k}{2}$; hence $b_{k+1} - a_{k+1} = \frac{b_k-a_k}{2}$ for every $k \in \mathbb{N}$, and by induction on n,

$$b_n - a_n = \frac{L}{2^{n-1}} \qquad \text{for every } n \in \mathbb{N}.$$

Step 1: $2^{n-1} \ge n$ for every $n \in \mathbb{N}$. For $n = 1$, $2^0 = 1 \ge 1$. If $2^{k-1} \ge k$ for some $k \in \mathbb{N}$, then $2^k = 2 \cdot 2^{k-1} \ge 2k = k + k \ge k + 1$. Hence the claim holds for all $n \in \mathbb{N}$ by induction.
Step 2: the lengths fall below every $\varepsilon > 0$. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $N > \frac{L}{\varepsilon}$. Let $n \in \mathbb{N}$ with $n \ge N$. By Step 1,

$$2^{n-1} \ge n \ge N > \frac{L}{\varepsilon}.$$

Multiplying by $\frac{\varepsilon}{2^{n-1}} > 0$ gives $\varepsilon > \frac{L}{2^{n-1}}$. Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that every $n \geq N$ satisfies $\frac{L}{2^{n-1}} < \varepsilon$.
Step 3: at most one point. Suppose, for contradiction, that $x, y \in \bigcap_n I_n$ with $x < y$. For every $n \in \mathbb{N}$, $a_n \leq x < y \leq b_n$, hence

$$y - x \leq b_n - a_n = \frac{L}{2^{n-1}}.$$

Apply Step 2 with $\varepsilon := y - x > 0$: there is $N \in \mathbb{N}$ with $\frac{L}{2^{N-1}} < y - x$, which contradicts $y - x \leq \frac{L}{2^{N-1}}$. Therefore $\bigcap_n I_n$ contains at most one real number. □

$$\boxed{\begin{array}{c}\forall \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ such that } \forall n \geq N \quad \dfrac{b_1 - a_1}{2^{n-1}} < \varepsilon; \\ \bigcap_n I_n \text{ contains at most one real number}\end{array}}$$

Method 2: Completeness and halving (infimum of the lengths)

Proof. Write $L := b_1 - a_1 > 0$ ($a < b$ because $f(a) < 0 < f(b)$). Each step keeps one half of the previous interval, so $b_{k+1} - a_{k+1} = \frac{b_k - a_k}{2}$, and by induction $b_n - a_n = \frac{L}{2^{n-1}}$ for every $n \in \mathbb{N}$. Let

$$S := \left\{ \frac{L}{2^{n-1}} : n \in \mathbb{N} \right\}.$$

Step 1: $\inf S$ *exists and is* ≥ 0. $L \in S$, so $S \neq \emptyset$; and $\frac{L}{2^{n-1}} > 0$ for every $n \in \mathbb{N}$, so 0 is a lower bound for S. Hence $-S$ is nonempty and bounded above by 0, so $\sup(-S)$ exists (AoC), and $t := \inf S = -\sup(-S)$ exists (Claim 2 in the proof of Theorem 1.2). Since 0 is a lower bound for S and t is the greatest lower bound, $0 \leq t$.
Step 2: $t = 0$. Suppose, for contradiction, that $t > 0$. Then $2t > t$, so $2t$ is not a lower bound for S (def. of infimum); hence there is $n \in \mathbb{N}$ with $\frac{L}{2^{n-1}} < 2t$. Dividing by 2,

$$\frac{L}{2^{(n+1)-1}} = \frac{L}{2^n} < t.$$

But $\frac{L}{2^{(n+1)-1}} \in S$ and t is a lower bound for S, so $t \leq \frac{L}{2^n}$, a contradiction. Therefore $t = 0$.
Step 3: the lengths fall below every $\varepsilon > 0$. Let $\varepsilon > 0$. By the ε-Criterion for infima applied to $\inf S = 0$, there is $N \in \mathbb{N}$ with $\frac{L}{2^{N-1}} < 0 + \varepsilon$. Let $n \in \mathbb{N}$ with $n \geq N$. Then $2^{n-1} = 2^{n-N} \cdot 2^{N-1} \geq 2^{N-1}$, hence

$$\frac{L}{2^{n-1}} \leq \frac{L}{2^{N-1}} < \varepsilon.$$

Step 4: at most one point. Suppose, for contradiction, that $x, y \in \bigcap_n I_n$ with $x < y$. For every $n \in \mathbb{N}$, $a_n \leq x < y \leq b_n$, so $y - x \leq \frac{L}{2^{n-1}}$. Hence $y - x$ is a lower bound for S, and therefore $y - x \leq \inf S = 0$ (def. of infimum), which contradicts $y - x > 0$. Therefore $\bigcap_n I_n$ contains at most one real number. □

$$\boxed{\begin{array}{c}\forall \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ such that } \forall n \geq N \quad \dfrac{b_1 - a_1}{2^{n-1}} < \varepsilon; \\ \bigcap_n I_n \text{ contains at most one real number}\end{array}}$$

Why the Archimedean Property is not trivially true. It can feel obvious that the natural numbers are unbounded within $\mathbb{R}$ – surely one can always count higher than any fixed real number. But this does not follow from $\mathbb{R}$ merely being an *ordered field*: there exist ordered fields (containing so-called infinite and infinitesimal elements) in which $\mathbb{N}$ *is* bounded above. What rules this out specifically for $\mathbb{R}$ is the Axiom of Completeness; the proof below manufactures the required natural number directly from $\sup\{n \in \mathbb{N} : n \leq a\}$, which exists only because $\mathbb{R}$ is complete. This is exactly Lecture 3's Objective 4: the Archimedean Property is a genuine consequence of completeness, not of order alone.

Theorem 1.4: Archimedean Property

Given any $a \in \mathbb{R}$, there is some $n \in \mathbb{N}$ satisfying $n > a$. Equivalently, given any $b > 0$, there is some $n \in \mathbb{N}$ satisfying

$$\frac{1}{n} < b. \tag{3}$$

Idea. *(Scratch work – not part of the written answer.)* Collect the natural numbers that fail to exceed a into $A = \{n \in \mathbb{N} : n \leq a\}$. If A is empty, any n works. Otherwise A is bounded above (by a), so it has a supremum s; since $s - 1 < s$ is not an upper bound, some $n \in A$ beats $s - 1$, which forces $n + 1$ to beat s – and hence to beat every element of A, so $n + 1 \notin A$, i.e. $n + 1 > a$.

Proof. Let $A := \{n \in \mathbb{N} : n \leq a\} \subseteq \mathbb{N}$, which is bounded above by a.

Case 1: $A = \emptyset$. Then no natural number satisfies $n \leq a$, so every $n \in \mathbb{N}$ satisfies $n > a$; in particular $n = 1$ works.

Case 2: $A \neq \emptyset$. Since A is nonempty and bounded above, the Axiom of Completeness asserts that A has a supremum; write $s := \sup A$. Since $s - 1 < s$ and s is the least upper bound of A, the number $s - 1$ is not an upper bound for A (by the definition of supremum). Therefore there exists $n \in A \subseteq \mathbb{N}$ such that $n > s - 1$, i.e., $n + 1 > s$. Since $s = \sup A$ is an upper bound for A (by the definition of supremum), every element of A is $\leq s < n + 1$; in particular $n + 1 \notin A$, i.e., $n + 1 \leq a$ is false, so $n + 1 > a$. In either case, some natural number strictly exceeds a. □

Equivalence of the two forms. (1) If $a \leq 0$, then $n = 1$ already satisfies $n > a$, so the first form holds trivially whenever $a \leq 0$. (2) Given $b > 0$, apply the first form to $a := 1/b$ to obtain $n \in \mathbb{N}$ with $n > 1/b$; since $b > 0$, this rearranges to $1/n < b$, which is (3). (3) Given $a > 0$, apply the second form to $b := 1/a$ to obtain $n \in \mathbb{N}$ with $1/n < 1/a$; since $a > 0$, this rearranges to $n > a$. Hence each form implies the other, so the theorem's two statements are logically equivalent.

Common Mistake: Quantifier Order in the Archimedean Property

The theorem reads $\forall a \in \mathbb{R}\ \exists n \in \mathbb{N}\ n > a$ – the natural number n is chosen *after*, and depending on, a. It is emphatically **not** $\exists n \in \mathbb{N}\ \forall a \in \mathbb{R}\ n > a$, which would claim a single natural number exceeding *every* real number simultaneously – an obviously false statement. Whenever you invoke the Archimedean Property (as in almost every ε-proof from Chapter 2 onward), the index n or N you produce is allowed, and expected, to depend on whatever quantity (a, b, or later ε) was fixed first.

Key Formula: Vanishing Interval Lengths (Lecture 3 Summary)

The Axiom of Completeness guarantees the Nested Interval Property (bracketing methods trap *at least one* real number) and the Archimedean Property (brackets whose lengths shrink at rate $\frac{1}{n}$ or faster trap *at most one* real number, since by the Archimedean Property the lengths can be forced below any $\varepsilon > 0$). Together, a nested sequence of closed bounded intervals whose lengths tend to 0 traps *exactly one* real number – this is what makes the Bisection Method converge to a single root.

1.7 Worked Examples (Tutorial 1)

Tutorial 1 (21 August) tests exactly the material of Lectures 1–3. The official tutorial hints are terse by design – Q1 asks for "suprema and infima, without proofs"; Q2, Q4, and Q5's hints give only the two-inequality *strategy*, not the write-up; Q3's hint omits the existence check for $\sup B$. The five worked examples below (plus one extra on the Archimedean Property) give the full exam-standard write-up for each, filling exactly the gaps the digest of the official hints identifies.

Worked Example (Tutorial 1, Q1): Suprema, Infima, Maxima, Minima of Four Sets

Compute, without proofs, the suprema and infima (if they exist) of the following sets, and determine whether they are, respectively, maxima and minima.

$$A = \left\{ \frac{m}{n} : m, n \in \mathbb{N},\ m \le n \right\}, \qquad B = \left\{ \frac{(-1)^m}{n} : m, n \in \mathbb{N} \right\},$$
$$C = \left\{ \frac{n}{3n+1} : n \in \mathbb{N} \right\}, \qquad D = \left\{ \frac{m}{m+n} : m, n \in \mathbb{N} \right\}.$$

(a) The set A. Method 1: ε-Criterion with the Archimedean Property

Proof. *Supremum.* Let $x \in A$; then $x = \frac{m}{n}$ for some $m, n \in \mathbb{N}$ with $m \le n$. Since $n > 0$, dividing $m \le n$ by n gives $\frac{m}{n} \le 1$. Hence 1 is an upper bound for A. Taking $m = n = 1$ gives $1 = \frac{1}{1} \in A$. Let $\varepsilon > 0$; then $a := 1 \in A$ satisfies $1 - \varepsilon < a$. Therefore $\sup A = 1$ (ε-Criterion), and since $1 \in A$, $\max A = 1$.
Infimum. Let $x = \frac{m}{n} \in A$. Since $m \ge 1 > 0$ and $n \ge 1 > 0$, $\frac{m}{n} > 0$. Hence 0 is a lower bound for A. Let $\varepsilon > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Since $1 \le n$, taking $m = 1$ gives $\frac{1}{n} \in A$, and $\frac{1}{n} < 0 + \varepsilon$. Therefore $\inf A = 0$ (ε-Criterion for infima). Every element of A is > 0, so $0 \notin A$. A minimum of A would be an element of A equal to $\inf A = 0$; therefore A has no minimum. □

$$\boxed{\sup A = \max A = 1} \qquad \boxed{\inf A = 0,\ \text{not a minimum}}$$

Method 2: Least-upper-bound definition, contradiction via the Archimedean Property

Proof. *Supremum.* For every $\frac{m}{n} \in A$, $m \le n$ and $n > 0$ give $\frac{m}{n} \le 1$; hence 1 is an upper bound for A. Let u be any upper bound for A. Since $1 = \frac{1}{1} \in A$, $1 \le u$. Hence 1 is an upper bound for A that is not greater than any upper bound for A, so $\sup A = 1$ (def. of supremum); since $1 \in A$, $\max A = 1$.
Infimum. For every $\frac{m}{n} \in A$, $m, n \ge 1$ give $\frac{m}{n} > 0$; hence 0 is a lower bound for A. Suppose, for contradiction, that some lower bound ℓ of A satisfies $\ell > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \ell$. But $\frac{1}{n} \in A$ (take $m = 1 \le n$) and ℓ is a lower bound for A, so $\ell \le \frac{1}{n}$, a contradiction. Hence every lower bound ℓ of A satisfies $\ell \le 0$, and therefore $\inf A = 0$ (def. of infimum). Since every element of A is > 0, $0 \notin A$, so A has no minimum. □

$$\boxed{\sup A = \max A = 1} \qquad \boxed{\inf A = 0,\ \text{not a minimum}}$$

(b) The set B. Method 1: Definition, using attained bounds

Proof. Let $x \in B$; then $x = \frac{(-1)^m}{n}$ for some $m, n \in \mathbb{N}$. Since $(-1)^m \in \{-1, 1\}$ and $n \ge 1$,

$$|x| = \frac{1}{n} \le 1,$$

hence $-1 \le x \le 1$. Therefore 1 is an upper bound and -1 is a lower bound for B. Taking $(m, n) = (2, 1)$ gives $1 \in B$. If u is any upper bound for B, then $1 \le u$ because $1 \in B$. Hence $\sup B = 1$ (def. of supremum), and since $1 \in B$, $\max B = 1$. Taking $(m, n) = (1, 1)$ gives $-1 \in B$. If ℓ is any lower bound for B, then $\ell \le -1$ because $-1 \in B$. Hence $\inf B = -1$ (def. of infimum), and since $-1 \in B$, $\min B = -1$. □

$$\boxed{\sup B = \max B = 1} \qquad \boxed{\inf B = \min B = -1}$$

Method 2: ε-Criterion with the attained element as witness

Proof. For every $m, n \in \mathbb{N}$, $\left|\frac{(-1)^m}{n}\right| = \frac{1}{n} \leq 1$, so $-1 \leq \frac{(-1)^m}{n} \leq 1$; hence 1 is an upper bound and -1 is a lower bound for B. Let $\varepsilon > 0$. The element $a := \frac{(-1)^2}{1} = 1 \in B$ satisfies $1 - \varepsilon < a$; therefore $\sup B = 1$ (ε-Criterion), and $1 \in B$ gives $\max B = 1$. The element $a := \frac{(-1)^1}{1} = -1 \in B$ satisfies $a < -1 + \varepsilon$; therefore $\inf B = -1$ (ε-Criterion for infima), and $-1 \in B$ gives $\min B = -1$. □

$$\boxed{\sup B = \max B = 1} \qquad \boxed{\inf B = \min B = -1}$$

(c) The set C. Method 1: ε-Criterion with the Archimedean Property

Proof. *Supremum.* Let $n \in \mathbb{N}$. Since $3n < 3n + 1$ and $3(3n + 1) > 0$, dividing by $3(3n + 1)$ gives $\frac{n}{3n+1} < \frac{1}{3}$. Hence $\frac{1}{3}$ is an upper bound for C, and every element of C is strictly less than $\frac{1}{3}$. For every $n \in \mathbb{N}$,

$$\frac{1}{3} - \frac{n}{3n+1} = \frac{(3n+1) - 3n}{3(3n+1)} = \frac{1}{3(3n+1)}.$$

Let $\varepsilon > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Since $3(3n+1) = 9n+3 > n > 0$,

$$\frac{1}{3} - \frac{n}{3n+1} = \frac{1}{3(3n+1)} < \frac{1}{n} < \varepsilon,$$

so the element $\frac{n}{3n+1} \in C$ satisfies $\frac{n}{3n+1} > \frac{1}{3} - \varepsilon$. Therefore $\sup C = \frac{1}{3}$ (ε-Criterion). Since every element of C is strictly less than $\frac{1}{3}$, $\frac{1}{3} \notin C$; therefore C has no maximum.
Infimum. Let $n \in \mathbb{N}$. Then $n \geq 1$, so $4n \geq 3n + 1$; dividing by $4(3n + 1) > 0$ gives $\frac{n}{3n+1} \geq \frac{1}{4}$. Hence $\frac{1}{4}$ is a lower bound for C. Taking $n = 1$ gives $\frac{1}{3\cdot 1+1} = \frac{1}{4} \in C$. Let $\varepsilon > 0$; then $a := \frac{1}{4} \in C$ satisfies $a < \frac{1}{4} + \varepsilon$. Therefore $\inf C = \frac{1}{4}$ (ε-Criterion for infima), and since $\frac{1}{4} \in C$, $\min C = \frac{1}{4}$. □

$$\boxed{\sup C = \tfrac{1}{3},\ \text{not a maximum}} \qquad \boxed{\inf C = \min C = \tfrac{1}{4}}$$

Method 2: Least-upper-bound definition, contradiction via the Archimedean Property

Proof. *Supremum.* For every $n \in \mathbb{N}$, $3n < 3n+1$ gives $\frac{n}{3n+1} < \frac{1}{3}$; hence $\frac{1}{3}$ is an upper bound for C and $\frac{1}{3} \notin C$. Suppose, for contradiction, that $u < \frac{1}{3}$ is an upper bound for C, and let $\delta := \frac{1}{3} - u > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \delta$. Since $3(3n + 1) > n > 0$,

$$\frac{1}{3} - \frac{n}{3n+1} = \frac{1}{3(3n+1)} < \frac{1}{n} < \delta = \frac{1}{3} - u,$$

hence $\frac{n}{3n+1} > u$ with $\frac{n}{3n+1} \in C$, contradicting that u is an upper bound for C. Therefore every upper bound u of C satisfies $u \geq \frac{1}{3}$, and $\sup C = \frac{1}{3}$ (def. of supremum). Since $\frac{1}{3} \notin C$, C has no maximum.
Infimum. For every $n \in \mathbb{N}$, $n \geq 1$ gives $4n \geq 3n + 1$, hence $\frac{n}{3n+1} \geq \frac{1}{4}$; so $\frac{1}{4}$ is a lower bound for C. Also $\frac{1}{4} = \frac{1}{3\cdot 1+1} \in C$, so every lower bound ℓ of C satisfies $\ell \leq \frac{1}{4}$. Therefore $\inf C = \frac{1}{4}$ (def. of infimum), and since $\frac{1}{4} \in C$, $\min C = \frac{1}{4}$. □

$$\boxed{\sup C = \tfrac{1}{3},\ \text{not a maximum}} \qquad \boxed{\inf C = \min C = \tfrac{1}{4}}$$

(d) The set D. Method 1: ε-Criterion with the Archimedean Property

Proof. *Supremum.* Let $m, n \in \mathbb{N}$. Since $n \geq 1$, $m < m+n$; dividing by $m+n > 0$ gives $\frac{m}{m+n} < 1$. Hence 1 is an upper bound for D, and every element of D is strictly less than 1. Let $\varepsilon > 0$. By the Archimedean Property there is $m \in \mathbb{N}$ with $\frac{1}{m} < \varepsilon$. Taking $n = 1$ gives the element

$$\frac{m}{m+1} = 1 - \frac{1}{m+1} \in D.$$

Since $m+1 > m > 0$, $\frac{1}{m+1} < \frac{1}{m} < \varepsilon$, hence $\frac{m}{m+1} > 1 - \varepsilon$. Therefore $\sup D = 1$ (ε-Criterion). Since every element of D is strictly less than 1, $1 \notin D$, so D has no maximum.
Infimum. Let $m, n \in \mathbb{N}$. Since $m \geq 1 > 0$ and $m + n > 0$, $\frac{m}{m+n} > 0$. Hence 0 is a lower bound for D, and $0 \notin D$. Let $\varepsilon > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon$. Taking $m = 1$ gives $\frac{1}{1+n} \in D$, and since $1 + n > n > 0$,

$$\frac{1}{1+n} < \frac{1}{n} < \varepsilon = 0 + \varepsilon.$$

Therefore $\inf D = 0$ (ε-Criterion for infima). Since $0 \notin D$, D has no minimum. □

$$\boxed{\sup D = 1, \text{ not a maximum}} \qquad \boxed{\inf D = 0, \text{ not a minimum}}$$

Method 2: Least-upper-bound definition, contradiction via the Archimedean Property

Proof. *Supremum.* For all $m, n \in \mathbb{N}$, $n \geq 1$ gives $m < m + n$, hence $\frac{m}{m+n} < 1$; so 1 is an upper bound for D and $1 \notin D$. Suppose, for contradiction, that $u < 1$ is an upper bound for D. Since $1 - u > 0$, the Archimedean Property gives $m \in \mathbb{N}$ with $\frac{1}{m} < 1 - u$. Then

$$\frac{m}{m+1} = 1 - \frac{1}{m+1} > 1 - \frac{1}{m} > u,$$

and $\frac{m}{m+1} \in D$ (take $n = 1$), contradicting that u is an upper bound for D. Hence every upper bound u of D satisfies $u \geq 1$, and $\sup D = 1$ (def. of supremum). Since $1 \notin D$, D has no maximum.
Infimum. For all $m, n \in \mathbb{N}$, $\frac{m}{m+n} > 0$; so 0 is a lower bound for D and $0 \notin D$. Suppose, for contradiction, that $\ell > 0$ is a lower bound for D. The Archimedean Property gives $n \in \mathbb{N}$ with $\frac{1}{n} < \ell$. Then $\frac{1}{1+n} \in D$ (take $m = 1$) and $\frac{1}{1+n} < \frac{1}{n} < \ell$, contradicting that ℓ is a lower bound for D. Hence every lower bound ℓ of D satisfies $\ell \leq 0$, and $\inf D = 0$ (def. of infimum). Since $0 \notin D$, D has no minimum. □

$$\boxed{\sup D = 1, \text{ not a maximum}} \qquad \boxed{\inf D = 0, \text{ not a minimum}}$$

□

Worked Example (Tutorial 1, Q2 = Sample Midterm Q3): $\max\{\sup A, \sup B\} = \sup(A \cup B)$

Problem. Prove that if the sets $A, B \subseteq \mathbb{R}$ are nonempty and bounded above, then $\max\{\sup A, \sup B\} = \sup(A \cup B)$. (Sample Midterm Q3 phrasing: write down the definition of $s = \sup A$, and use the definition to show this; do not use the ε-Criterion for suprema and infima.)

Method 1: Least-upper-bound definition, two inequalities

Proof. *Definition.* For $X \subseteq \mathbb{R}$, $\sigma = \sup X$ means: (i) $x \leq \sigma$ for every $x \in X$; and (ii) $\sigma \leq b$ for every upper bound b of X.
Since A is nonempty and bounded above, $s := \sup A$ exists (AoC); likewise $t := \sup B$ exists. Put $M := \max\{s, t\}$, so that $s \leq M$ and $t \leq M$.
Step 1: M is an upper bound for $A \cup B$. Let $x \in A \cup B$. If $x \in A$, then $x \leq s$ by (i) for $s = \sup A$,

and $s \le M$; hence $x \le M$. If $x \in B$, then $x \le t$ by (i) for $t = \sup B$, and $t \le M$; hence $x \le M$. Therefore every $x \in A \cup B$ satisfies $x \le M$. Also $A \cup B \supseteq A \neq \emptyset$. Hence $A \cup B$ is nonempty and bounded above, so $\sigma := \sup(A \cup B)$ exists (AoC).
Step 2: $\sigma \le M$. M is an upper bound for $A \cup B$ (Step 1), so (ii) for $\sigma = \sup(A \cup B)$ gives $\sigma \le M$.
Step 3: $M \le \sigma$. Let $a \in A$. Then $a \in A \cup B$, so $a \le \sigma$ by (i) for σ. Hence σ is an upper bound for A, and (ii) for $s = \sup A$ gives $s \le \sigma$. Let $b \in B$. Then $b \in A \cup B$, so $b \le \sigma$. Hence σ is an upper bound for B, and (ii) for $t = \sup B$ gives $t \le \sigma$. Since M equals s or t, $M \le \sigma$.
Step 4. Steps 2 and 3 give $\sigma \le M \le \sigma$; therefore $M = \sigma$, i.e. $\max\{\sup A, \sup B\} = \sup(A \cup B)$. □

$$\boxed{\max\{\sup A, \sup B\} = \sup(A \cup B)}$$

Method 2: ε-Criterion, case split on the larger supremum
Note: the Sample Midterm Q3 phrasing excludes the ε-Criterion; this method answers the Tutorial 1 Q2 phrasing only.

Proof. Since A and B are nonempty and bounded above, $s := \sup A$ and $t := \sup B$ exist (AoC). Put $M := \max\{s, t\}$.
Step 1: M is an upper bound for $A \cup B$. If $x \in A$, then $x \le s \le M$; if $x \in B$, then $x \le t \le M$. Hence every $x \in A \cup B$ satisfies $x \le M$.
Step 2: the ε-condition for M. Let $\varepsilon > 0$. *Case $s \ge t$.* Then $M = s$. By the ε-Criterion for $s = \sup A$, there is $a \in A$ with $s - \varepsilon < a$. Then $a \in A \cup B$ and $M - \varepsilon < a$. *Case $t > s$.* Then $M = t$. By the ε-Criterion for $t = \sup B$, there is $b \in B$ with $t - \varepsilon < b$. Then $b \in A \cup B$ and $M - \varepsilon < b$. In both cases some element of $A \cup B$ exceeds $M - \varepsilon$.
Step 3. By Steps 1 and 2 and the ε-Criterion applied to the set $A \cup B$ and the number M, $M = \sup(A \cup B)$. □

$$\boxed{\max\{\sup A, \sup B\} = \sup(A \cup B)}$$

Worked Example (Tutorial 1, Q3): $\sup B = \inf A$ for the Set of Lower Bounds B

Problem. Suppose $A \subseteq \mathbb{R}$ is nonempty and bounded below, and define $B := \{b \in \mathbb{R} : b \text{ is a lower bound for } A\}$. Prove that $\sup B = \inf A$.

Method 1: Existence of both numbers, then two inequalities

Proof. *Step 1: $\inf A$ exists.* Let $-A := \{-a : a \in A\}$. Since $A \neq \emptyset$, $-A \neq \emptyset$. Since A is bounded below, it has a lower bound ℓ_0, and by Claim 1 in the proof of Theorem 1.2, $-\ell_0$ is an upper bound for $-A$. Hence $-A$ is nonempty and bounded above, so $\sup(-A)$ exists (AoC), and by Claim 2 in the same proof, $\inf A = -\sup(-A)$ exists. Write $i := \inf A$.
Step 2: $\sup B$ exists. By the definition of infimum, i is a lower bound for A; hence $i \in B$, and $B \neq \emptyset$. Fix $a_0 \in A$ (possible since $A \neq \emptyset$). Every $b \in B$ is a lower bound for A, so $b \le a_0$. Hence a_0 is an upper bound for B. Therefore B is nonempty and bounded above, and $\sigma := \sup B$ exists (AoC).
Step 3: $i \le \sigma$. $i \in B$ (Step 2) and σ is an upper bound for B (def. of supremum); therefore $i \le \sigma$.
Step 4: $\sigma \le i$. Let $a \in A$. Every $b \in B$ is a lower bound for A, so $b \le a$. Hence a is an upper bound for B, and $\sigma \le a$ (def. of $\sup B$). As $a \in A$ was arbitrary, σ is a lower bound for A. Since $i = \inf A$ is not less than any lower bound for A (def. of infimum), $\sigma \le i$.
Step 5. Steps 3 and 4 give $i \le \sigma \le i$; therefore $\sigma = i$, i.e. $\sup B = \inf A$. □

$$\boxed{\sup B = \inf A}$$

Method 2: $\sup B$ satisfies the definition of $\inf A$

Proof. *Step 1: $\sigma := \sup B$ exists.* Since A is bounded below, there is a lower bound ℓ_0 of A, and $\ell_0 \in B$; hence $B \neq \emptyset$. Fix $a_0 \in A$. Every $b \in B$ is a lower bound for A, so $b \le a_0$; hence a_0 is an upper bound for B. Therefore $\sigma := \sup B$ exists (AoC).

Step 2: σ is a lower bound for A. Let $a \in A$. Every $b \in B$ satisfies $b \le a$, so a is an upper bound for B; hence $\sigma \le a$ (def. of supremum). Therefore every $a \in A$ satisfies $\sigma \le a$.
Step 3: σ is not less than any lower bound for A. Let ℓ be a lower bound for A. Then $\ell \in B$ by the definition of B, and σ is an upper bound for B, so $\ell \le \sigma$.
Step 4. By Steps 2 and 3, σ satisfies both conditions in the definition of the infimum of A. Therefore $\inf A$ exists and $\inf A = \sigma = \sup B$; no separate appeal to $-A$ is needed for the existence of $\inf A$. □

$$\boxed{\sup B = \inf A}$$

Method 3: ε-Criterion for infima, by contradiction

Proof. *Step 1: $\sigma := \sup B$ exists.* A lower bound ℓ_0 of A exists and lies in B, so $B \neq \emptyset$; any fixed $a_0 \in A$ satisfies $b \le a_0$ for every $b \in B$, so B is bounded above. Hence $\sigma := \sup B$ exists (AoC).
Step 2: σ is a lower bound for A. Let $a \in A$. Every $b \in B$ satisfies $b \le a$, so a is an upper bound for B, and therefore $\sigma \le a$ (def. of supremum).
Step 3: the ε-condition. Let $\varepsilon > 0$. Suppose, for contradiction, that no $a \in A$ satisfies $a < \sigma + \varepsilon$. Then every $a \in A$ satisfies $\sigma + \varepsilon \le a$, so $\sigma + \varepsilon$ is a lower bound for A, i.e. $\sigma + \varepsilon \in B$. Since σ is an upper bound for B, $\sigma + \varepsilon \le \sigma$, i.e. $\varepsilon \le 0$, a contradiction. Hence there is $a \in A$ with $a < \sigma + \varepsilon$.
Step 4. By Steps 2 and 3 and the ε-Criterion for infima, $\sigma = \inf A$, i.e. $\sup B = \inf A$. □

$$\boxed{\sup B = \inf A}$$

Worked Example (Tutorial 1, Q4): $\sup(A+B) = \sup A + \sup B$

Problem. Suppose that the sets $A, B \subseteq \mathbb{R}$ are nonempty and bounded above and that $s = \sup A$ and $t = \sup B$. Define the notation $A + B := \{a + b : a \in A \text{ and } b \in B\}$. (a) Prove that $s + t$ is an upper bound for $A + B$. (b) Prove that if $u \in \mathbb{R}$ is an upper bound for $A + B$ and $a \in A$, then $t \le u - a$. (c) Hence deduce that $\sup(A + B) = s + t$.

Method 1: Least-upper-bound definition through parts (a), (b), (c)

Proof. Since $A \neq \emptyset$ and $B \neq \emptyset$, pick $a_0 \in A$ and $b_0 \in B$; then $a_0 + b_0 \in A + B$, so $A + B \neq \emptyset$.
(a) $s + t$ is an upper bound for $A + B$. Let $c \in A + B$. Then $c = a + b$ for some $a \in A$ and $b \in B$. Since s is an upper bound for A, $a \le s$; since t is an upper bound for B, $b \le t$. Adding the two inequalities,

$$c = a + b \le s + t.$$

Hence every $c \in A + B$ satisfies $c \le s + t$, i.e. $s + t$ is an upper bound for $A + B$. Therefore $A + B$ is nonempty and bounded above, and $\sup(A + B)$ exists (AoC).
(b) If u is an upper bound for $A + B$ and $a \in A$, then $t \le u - a$. Let $b \in B$. Then $a + b \in A + B$, so $a + b \le u$, hence $b \le u - a$. As $b \in B$ was arbitrary, $u - a$ is an upper bound for B. Since $t = \sup B$ is not greater than any upper bound for B (def. of supremum), $t \le u - a$.
(c) $s + t = \sup(A + B)$. Let u be an upper bound for $A + B$, and let $a \in A$. By (b), $t \le u - a$, hence $a \le u - t$. As $a \in A$ was arbitrary, $u - t$ is an upper bound for A. Since $s = \sup A$ is not greater than any upper bound for A, $s \le u - t$, i.e. $s + t \le u$. Therefore $s + t$ is not greater than any upper bound for $A + B$; together with (a), the definition of supremum gives $s + t = \sup(A + B)$. □

$$\boxed{\sup(A + B) = s + t}$$

Method 2: ε-Criterion with $\varepsilon/2$ witnesses
Note: this method proves (a) and the conclusion of (c) without passing through (b).

Proof. *Step 1: $s + t$ is an upper bound for $A + B$.* Let $c \in A + B$, so $c = a + b$ with $a \in A$ and $b \in B$. Since $a \le s$ and $b \le t$, $c = a + b \le s + t$.
Step 2: the ε-condition. Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$. By the ε-Criterion for $s = \sup A$, there is $a \in A$ with $s - \frac{\varepsilon}{2} < a$. By the ε-Criterion for $t = \sup B$, there is $b \in B$ with $t - \frac{\varepsilon}{2} < b$. Adding the two

inequalities,

$$(s+t)-\varepsilon=\left(s-\frac{\varepsilon}{2}\right)+\left(t-\frac{\varepsilon}{2}\right)<a+b,$$

and $a+b\in A+B$.
Step 3. By Steps 1 and 2 and the ε-Criterion applied to $A+B$ and $s+t$, $s+t=\sup(A+B)$. □

$$\boxed{\sup(A+B)=s+t}$$

Worked Example (Tutorial 1, Q5): A Two-Sided $\frac{1}{n}$ Criterion Forces $s=\sup A$

Problem. Suppose $A\subseteq\mathbb{R}$ is nonempty and bounded above, and $s\in\mathbb{R}$ satisfies

$$\forall n\in\mathbb{N}\quad s+\frac{1}{n}\text{ is an upper bound for }A\quad\text{and}\quad s-\frac{1}{n}\text{ is not an upper bound for }A.$$

Prove that $s=\sup A$.

Method 1: ε-Criterion with the Archimedean Property

Proof. *Step 1: s is an upper bound for A.* Suppose, for contradiction, that some $a\in A$ satisfies $a>s$. Then $a-s>0$, so by the Archimedean Property there is $n\in\mathbb{N}$ with $\frac{1}{n}<a-s$, i.e. $s+\frac{1}{n}<a$. But $s+\frac{1}{n}$ is an upper bound for A by hypothesis, so $a\le s+\frac{1}{n}$, a contradiction. Therefore every $a\in A$ satisfies $a\le s$.
Step 2: the ε-condition. Let $\varepsilon>0$. By the Archimedean Property there is $n\in\mathbb{N}$ with $\frac{1}{n}<\varepsilon$, hence $s-\varepsilon<s-\frac{1}{n}$. By hypothesis $s-\frac{1}{n}$ is not an upper bound for A, so there is $a\in A$ with $a>s-\frac{1}{n}$. Therefore $a>s-\frac{1}{n}>s-\varepsilon$.
Step 3. By Steps 1 and 2 and the ε-Criterion for Suprema (Theorem 1.2), $s=\sup A$. □

$$\boxed{s=\sup A}$$

Method 2: Least-upper-bound definition, contradiction via the Archimedean Property

Proof. *Step 1: s is an upper bound for A.* Suppose, for contradiction, that $a\in A$ and $a>s$. The Archimedean Property gives $n\in\mathbb{N}$ with $\frac{1}{n}<a-s$, so $s+\frac{1}{n}<a$; but $s+\frac{1}{n}$ is an upper bound for A, so $a\le s+\frac{1}{n}$, a contradiction. Hence every $a\in A$ satisfies $a\le s$.
Step 2: s is not greater than any upper bound for A. Let u be an upper bound for A, and suppose, for contradiction, that $u<s$. Then $s-u>0$, so the Archimedean Property gives $n\in\mathbb{N}$ with $\frac{1}{n}<s-u$, i.e. $u<s-\frac{1}{n}$. By hypothesis $s-\frac{1}{n}$ is not an upper bound for A, so there is $a\in A$ with $a>s-\frac{1}{n}>u$, contradicting that u is an upper bound for A. Therefore $s\le u$.
Step 3. By Steps 1 and 2 and the definition of supremum, $s=\sup A$. □

$$\boxed{s=\sup A}$$

Method 3: Comparison with the supremum supplied by Completeness

Proof. Since A is nonempty and bounded above, $\sigma:=\sup A$ exists (AoC). Fix $n\in\mathbb{N}$.
Since $s+\frac{1}{n}$ is an upper bound for A and σ is not greater than any upper bound for A, $\sigma\le s+\frac{1}{n}$.
If $\sigma\le s-\frac{1}{n}$, then every $a\in A$ would satisfy $a\le\sigma\le s-\frac{1}{n}$, making $s-\frac{1}{n}$ an upper bound for A, contrary to hypothesis. Hence $s-\frac{1}{n}<\sigma$.
Combining the two inequalities,

$$-\frac{1}{n}\le s-\sigma<\frac{1}{n},$$

so $|s-\sigma|\le\frac{1}{n}$, and this holds for every $n\in\mathbb{N}$.

Suppose, for contradiction, that $s \neq \sigma$. Then $|s-\sigma| > 0$, and the Archimedean Property gives $n \in \mathbb{N}$ with $\frac{1}{n} < |s-\sigma|$, contradicting $|s-\sigma| \leq \frac{1}{n}$. Therefore $s = \sigma = \sup A$. □

$$\boxed{s = \sup A}$$

Worked Example: The Archimedean Property Gives $\inf\{1/n : n \in \mathbb{N}\} = 0$

Claim. $\inf\{1/n : n \in \mathbb{N}\} = 0$, and 0 is not attained (not a minimum). Let $A := \{1/n : n \in \mathbb{N}\}$.

Method 1: ε-Criterion for infima with the Archimedean Property

Proof. Let $n \in \mathbb{N}$. Since $n \geq 1 > 0$, $\frac{1}{n} > 0$. Hence 0 is a lower bound for A, and $0 \notin A$. Let $\varepsilon > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon = 0 + \varepsilon$, and $\frac{1}{n} \in A$. Therefore $\inf A = 0$ (ε-Criterion for Infima, Theorem 1.2). A minimum of A would be an element of A equal to $\inf A = 0$; since $0 \notin A$, A has no minimum. □

$$\boxed{\inf\{1/n : n \in \mathbb{N}\} = 0, \text{ not a minimum}}$$

Method 2: Greatest-lower-bound definition, contradiction via the Archimedean Property

Proof. For every $n \in \mathbb{N}$, $n > 0$ gives $\frac{1}{n} > 0$; hence 0 is a lower bound for A and $0 \notin A$. Let ℓ be a lower bound for A, and suppose, for contradiction, that $\ell > 0$. By the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < \ell$. But $\frac{1}{n} \in A$ and ℓ is a lower bound for A, so $\ell \leq \frac{1}{n}$, a contradiction. Hence every lower bound ℓ of A satisfies $\ell \leq 0$. Since 0 is itself a lower bound, $\inf A = 0$ (def. of infimum). Since $0 \notin A$, A has no minimum. □

$$\boxed{\inf\{1/n : n \in \mathbb{N}\} = 0, \text{ not a minimum}}$$

Method 3: Completeness and a doubling argument

Proof. *Step 1: $t := \inf A$ exists and $t \geq 0$.* $1 = \frac{1}{1} \in A$, so $A \neq \emptyset$; and $\frac{1}{n} > 0$ for every $n \in \mathbb{N}$, so 0 is a lower bound for A and $0 \notin A$. Hence $-A = \{-\frac{1}{n} : n \in \mathbb{N}\}$ is nonempty and bounded above by 0, so $\sup(-A)$ exists (AoC), and $t := \inf A = -\sup(-A)$ exists (Claim 2 in the proof of Theorem 1.2). Since 0 is a lower bound for A and t is the greatest lower bound, $0 \leq t$.
Step 2: $2t$ is a lower bound for A. Let $n \in \mathbb{N}$. Then $2n \in \mathbb{N}$, so $\frac{1}{2n} \in A$, and therefore $t \leq \frac{1}{2n}$. Multiplying by $2 > 0$ gives $2t \leq \frac{1}{n}$. As $n \in \mathbb{N}$ was arbitrary, $2t$ is a lower bound for A.
Step 3: $t = 0$. Since t is not less than any lower bound for A (def. of infimum), Step 2 gives $2t \leq t$, i.e. $t \leq 0$. With Step 1, $t = 0$. Since $0 \notin A$, A has no minimum. □

$$\boxed{\inf\{1/n : n \in \mathbb{N}\} = 0, \text{ not a minimum}}$$

1.8 Chapter Summary

Key Formula: Chapter 1 Recap

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \qquad \mathbb{N}_0 = \mathbb{N} \cup \{0\};$$

$$s = \sup A \iff s \text{ upper-bounds } A \text{ and } \forall \varepsilon > 0\ \exists a \in A\ s - \varepsilon < a;$$

$$s = \inf A \iff s \text{ lower-bounds } A \text{ and } \forall \varepsilon > 0\ \exists a \in A\ a < s + \varepsilon;$$

Axiom of Completeness: $A \neq \emptyset$ and A bounded above $\implies \sup A$ exists;

Archimedean Property: $\forall a \in \mathbb{R}\ \exists n \in \mathbb{N}\ n > a \iff \forall b > 0\ \exists n \in \mathbb{N}\ \frac{1}{n} < b$;

Nested Interval Property: $I_1 \supseteq I_2 \supseteq \cdots$ closed & bounded $\implies \bigcap_n I_n \neq \emptyset$.

Exam Focus: Chapter 1 Takeaways

- **Two-step template to prove $s = \sup A$ from the definition:** (1) show s is an upper bound for A (i.e. $\forall a \in A\ a \leq s$); (2) show s is not *greater* than any upper bound, either directly (assume $b < s$ is an upper bound and derive a contradiction) or via the ε-Criterion ($\forall \varepsilon > 0\ \exists a \in A\ s - \varepsilon < a$). Skipping step (1) is the most common way this template fails.
- $\sup A$ (respectively $\inf A$) need not belong to A; it equals $\max A$ (respectively $\min A$) only once membership is separately checked.
- Every ε-Criterion proof in this chapter, and almost every "find n" step from here on, ultimately reduces to the Archimedean Property – it is the standard device for turning a fixed bound like $1/n$ into an arbitrary $\varepsilon > 0$ (or vice versa).
- Quote the Nested Interval Property only when the intervals are both closed *and* bounded; quote the Axiom of Completeness only when the set is both nonempty *and* bounded above.
- $\mathbb{Q}$ fails all three headline results of this chapter for the *same* underlying reason: the set $\{q \in \mathbb{Q}^+ : q^2 < 2\}$ (Lecture 1) has no rational supremum, so it also gives a nested sequence of rational intervals with empty intersection (Lecture 3) – completeness, not order, is what $\mathbb{R}$ adds to $\mathbb{Q}$.
- "Theorem (Axiomatic Definition of the Real Numbers)" may be cited by name but is never examined for proof: the course explicitly defers it to Spivak.

2 Lectures 4–6: Sequences, Limit Theorems and Cauchy Sequences

This chapter builds the theory of sequences of real numbers: what it means for a sequence to converge, how limits interact with arithmetic and order, and a second, purely internal test for convergence – the Cauchy Criterion – that does not require guessing the limit in advance. Throughout, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ denote sequences of real numbers, i.e. functions $\mathbb{N} \to \mathbb{R}$, and every ε-based statement is understood exactly as it is written on the slides: quantifiers first, then a bare implication. Chapter 1 supplied the tools used repeatedly below (the Axiom of Completeness, the Nested Interval Property, the Archimedean Property); this chapter prepares Chapter 3, where subsequences and the Bolzano–Weierstrass Theorem give an alternative route to some of the results proved here.

2.1 Preliminaries: Absolute Value and the Triangle Inequality

Every ε-argument in this chapter is built on the absolute value and the triangle inequality, so we record them first.

Definition 2.1: Absolute Value

For $x \in \mathbb{R}$, the *absolute value* of x is

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Remark (Absolute-value inequalities). Directly from the definition, for any $x \in \mathbb{R}$ and $b \geq 0$,

$$|x| \leq b \iff -b \leq x \leq b, \qquad \text{and} \qquad |x| < b \iff -b < x < b.$$

This equivalence is used silently in almost every proof below to move between a single absolute-value bound and a pair of two-sided inequalities.

Theorem 2.1: Triangle Inequality

For all $x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|\,.$$

Idea. *(Scratch work – not part of the written answer.)* Bound x and y each between $\pm$ their own absolute value, add the two chains of inequalities, and read off the claim using the equivalence above.

Proof. Let $x, y \in \mathbb{R}$ be given. By the definition of absolute value, $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Adding these two chains of inequalities term by term gives

$$-(|x| + |y|) \leq x + y \leq |x| + |y|\,.$$

Since $|x| + |y| \geq 0$, the remark on absolute-value inequalities applied with $b = |x| + |y|$ shows that this two-sided inequality is equivalent to $|x + y| \leq |x| + |y|$. Therefore $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$. □

Remark (Three-point triangle inequality). Applying the theorem with x replaced by $x - y$ and y replaced by $y - z$ gives, for all $x, y, z \in \mathbb{R}$,

$$|x - z| = |(x - y) + (y - z)| \leq |x - y| + |y - z|\,.$$

This three-point form is the one that recurs constantly below, for instance in the Uniqueness of Limits and both directions of the Cauchy Criterion.

Theorem 2.2: Reverse Triangle Inequality

For all $x, y \in \mathbb{R}$,

$$\big||x| - |y|\big| \leq |x - y|\,.$$

Idea. *(Scratch work – not part of the written answer.)* Write $x = (x - y) + y$ and apply the Triangle Inequality to bound $|x| - |y|$; repeat with the roles of x, y swapped to bound $|y| - |x|$; combine both one-sided bounds into the single two-sided statement.

Proof. Let $x, y \in \mathbb{R}$ be given. Since $x = (x - y) + y$, the Triangle Inequality (Theorem 2.1) gives $|x| \leq |x - y| + |y|$, and hence $|x| - |y| \leq |x - y|$. By the same argument with x and y interchanged, $|y| - |x| \leq |y - x| = |x - y|$. Therefore

$$-|x - y| \leq |x| - |y| \leq |x - y|\,,$$

which, by the remark on absolute-value inequalities, is exactly the statement $\big||x| - |y|\big| \leq |x - y|$. □

2.2 Sequences, Tails and the Meaning of "Eventually"

Definition 2.2: Sequence

A *sequence of real numbers* is a function from $\mathbb{N}$ to $\mathbb{R}$. We usually write a sequence as $a_1, a_2, a_3, \ldots$, where a_n denotes the image of the number $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, the number a_n is called a *term* of the sequence. We use the notation $\{a_n\}_{n=1}^{\infty}$ (or simply $\{a_n\}$) to denote the above sequence.

Remark (Different starting points). It is sometimes useful to start labelling a sequence from an integer different from 1. For example, we may want to start a sequence from a_0 instead of a_1; in this case we denote the sequence by $\{a_n\}_{n=0}^{\infty}$.

A sequence can be described in several equivalent ways: by listing its terms, e.g. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$; by a closed-form expression, e.g. $\left\{\dfrac{n}{n^2 - 1}\right\}_{n=2}^{\infty}$; or by a recurrence, e.g. $\{x_n\}$ with $x_1 = 2$ and $x_{n+1} = \dfrac{x_n + 1}{2}$ for each $n \in \mathbb{N}$.

Definition 2.3: Tail and Eventually

A *tail* of a sequence $\{a_n\}$ is the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ for some number $N \in \mathbb{N}$. A sequence is said to satisfy a property *eventually* if the property is satisfied by some tail of the sequence.

For example, a sequence is *eventually constant* if there is some number $N \in \mathbb{N}$ such that $a_{n+1} = a_n$ for all $n \geq N$, and a sequence is *eventually positive* if there is some number $N \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq N$.

Remark ("Eventually" vs. "for all"). The word "eventually" is not decorative: it distinguishes a genuinely global property (true of *every* term) from a tail property (true from some point on). Convergence, boundedness, and being Cauchy are all tail properties in this course, and nearly every theorem statement below is phrased using "eventually" precisely so it can be read off directly from an ε-N argument.

2.3 The Limit of a Sequence

Definition 2.4: Limit of a Sequence

A sequence $\{a_n\}$ is said to have the *limit* $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N} \quad n \geq N \implies |a_n - a| < \varepsilon.$$

We use the notation $\lim_{n\to\infty} a_n$ (or simply $\lim a_n$) to denote the limit, and say that $\{a_n\}$ *converges* to a. We write $a = \lim_{n\to\infty} a_n$ or $a_n \to a$ as $n \to \infty$ (or simply $a = \lim a_n$ or $a_n \to a$) to mean that $\{a_n\}$ has limit a. A sequence $\{a_n\}$ is *convergent* if it has a limit; otherwise it is *divergent* and *diverges*.

Remark (Intuitive meaning). Intuitively, a sequence $\{a_n\}$ converges to a if $\{a_n\}$ is eventually arbitrarily close to a: no matter how small a tolerance $\varepsilon > 0$ is demanded, some tail of the sequence lies entirely inside the interval $(a - \varepsilon, a + \varepsilon)$.

Common slip. The index N in the definition is allowed to depend on ε – indeed it typically must, since smaller ε forces larger N – but N may never depend on n. A "proof" that picks N after fixing a particular n has the quantifiers in the wrong order and proves nothing.

Worked Example (Tutorial 2, Q2(a)): $\frac{2}{5} = \lim \frac{2n+1}{5n+4}$

Method 1: ε-N definition with a bounding chain

Proof. For every $n \in \mathbb{N}$, writing the difference over the common denominator $5(5n+4)$ gives

$$\frac{2n+1}{5n+4} - \frac{2}{5} = \frac{5(2n+1) - 2(5n+4)}{5(5n+4)} = \frac{10n + 5 - 10n - 8}{5(5n+4)} = \frac{-3}{5(5n+4)}.$$

Since $5(5n+4) > 0$ for every $n \in \mathbb{N}$, it follows that for every $n \in \mathbb{N}$,

$$\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| = \frac{3}{5(5n+4)} = \frac{3}{25n+20}.$$

Let $\varepsilon > 0$ be given. Then $\frac{3}{25\varepsilon}$ is a real number, so by the Archimedean Property there exists $N \in \mathbb{N}$ such that $N > \frac{3}{25\varepsilon}$ (Archimedean). Multiplying this inequality by $25\varepsilon > 0$ gives $25N\varepsilon > 3$, and dividing by $25N > 0$ gives $\frac{3}{25N} < \varepsilon$. Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$\begin{aligned}
\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \frac{3}{25n+20} \\
&< \frac{3}{25n} && (25n + 20 > 25n > 0) \\
&\leq \frac{3}{25N} && (n \geq N > 0) \\
&< \varepsilon && (25N\varepsilon > 3).
\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$, $n \geq N \implies \left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $\frac{2}{5} = \lim \frac{2n+1}{5n+4}$ (def. of limit). □

$$\boxed{N > \frac{3}{25\varepsilon}}$$

Method 2: Squeeze Theorem with an explicit lower sequence
This method should be used only if the course allows this interpretation (the tutorial asks for the definition of limit; the Squeeze Theorem is Lecture 5).

Proof. For every $n \in \mathbb{N}$ let $b_n := \dfrac{2n+1}{5n+4}$, $x_n := \dfrac{2}{5} - \dfrac{3}{25n}$ and $y_n := \dfrac{2}{5}$.
Step 1: the sandwich $x_n \le b_n \le y_n$. For every $n \in \mathbb{N}$,

$$b_n - \frac{2}{5} = \frac{5(2n+1) - 2(5n+4)}{5(5n+4)} = \frac{-3}{25n+20}.$$

Since $25n+20 > 0$, we have $b_n - \frac{2}{5} < 0$, hence $b_n < \frac{2}{5} = y_n$. Since $25n+20 > 25n > 0$, we have $\dfrac{3}{25n+20} < \dfrac{3}{25n}$, hence $b_n - \dfrac{2}{5} = -\dfrac{3}{25n+20} > -\dfrac{3}{25n}$, i.e. $b_n > \dfrac{2}{5} - \dfrac{3}{25n} = x_n$. Therefore for all $n \in \mathbb{N}$, $x_n < b_n < y_n$; in particular the sandwich holds for all $n \ge 1$.
Step 2: $\frac{2}{5} = \lim y_n$. Let $\varepsilon > 0$ be given. For all $n \ge 1$, $\left|y_n - \frac{2}{5}\right| = 0 < \varepsilon$. Hence $\frac{2}{5} = \lim y_n$ (def. of limit), with index 1.
Step 3: $\frac{2}{5} = \lim x_n$. Let $\varepsilon > 0$ be given. By the Archimedean Property there exists $N \in \mathbb{N}$ such that $N > \dfrac{3}{25\varepsilon}$, and hence $25N\varepsilon > 3$, i.e. $\dfrac{3}{25N} < \varepsilon$ (Archimedean). Let $n \in \mathbb{N}$ with $n \ge N$. Then

$$\begin{aligned} \left|x_n - \frac{2}{5}\right| &= \frac{3}{25n} && (n > 0) \\ &\le \frac{3}{25N} && (n \ge N) \\ &< \varepsilon. \end{aligned}$$

Therefore for all $n \ge N$, $\left|x_n - \frac{2}{5}\right| < \varepsilon$, and since $\varepsilon > 0$ was arbitrary, $\frac{2}{5} = \lim x_n$ (def. of limit).
Step 4: conclusion. By Steps 1–3, $x_n \le b_n \le y_n$ for all $n \in \mathbb{N}$ and $\frac{2}{5} = \lim x_n = \lim y_n$. Therefore, by the Squeeze Theorem, $\frac{2}{5} = \lim b_n$. Following the proof of the Squeeze Theorem with the three indices N (Step 3), 1 (Step 2) and 1 (Step 1), the index $\max\{N, 1, 1\} = N$ works: for all $n \ge N$,

$$\frac{2}{5} - \varepsilon < x_n \le b_n \le y_n < \frac{2}{5} + \varepsilon,$$

hence $\left|\dfrac{2n+1}{5n+4} - \dfrac{2}{5}\right| < \varepsilon$. □

$$\boxed{N > \frac{3}{25\varepsilon}}$$

Worked Example (Tutorial 2, Q2(b)): $0 = \lim \dfrac{2n^2}{n^3+3}$

Method 1: ε-N definition with a bounding chain

Proof. For every $n \in \mathbb{N}$, $2n^2 > 0$ and $n^3 + 3 > 0$, hence $\dfrac{2n^2}{n^3+3} > 0$ and therefore

$$\left|\frac{2n^2}{n^3+3} - 0\right| = \frac{2n^2}{n^3+3}.$$

Let $\varepsilon > 0$ be given. Then $\frac{2}{\varepsilon}$ is a real number, so by the Archimedean Property there exists $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$ (Archimedean). Multiplying by $\varepsilon > 0$ gives $N\varepsilon > 2$, and dividing by $N > 0$ gives $\frac{2}{N} < \varepsilon$. Let $n \in \mathbb{N}$ with $n \geq N$. Since $n^3 + 3 > n^3 > 0$ and $2n^2 > 0$, we have $\frac{2n^2}{n^3+3} < \frac{2n^2}{n^3}$. Hence

$$\begin{aligned}
\left|\frac{2n^2}{n^3+3} - 0\right| &= \frac{2n^2}{n^3+3} \\
&< \frac{2n^2}{n^3} = \frac{2}{n} && (n^3+3 > n^3 > 0) \\
&\leq \frac{2}{N} && (n \geq N > 0) \\
&< \varepsilon && (N\varepsilon > 2).
\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$, $n \geq N \implies \left|\frac{2n^2}{n^3+3} - 0\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $0 = \lim \frac{2n^2}{n^3+3}$ (def. of limit). □

$$\boxed{N > \frac{2}{\varepsilon}}$$

Method 2: Squeeze Theorem between 0 and $\frac{2}{n}$
This method should be used only if the course allows this interpretation (the tutorial asks for the definition of limit; the Squeeze Theorem is Lecture 5).

Proof. For every $n \in \mathbb{N}$ let $b_n := \frac{2n^2}{n^3+3}$, $x_n := 0$ and $y_n := \frac{2}{n}$.
Step 1: the sandwich $x_n \leq b_n \leq y_n$. For every $n \in \mathbb{N}$, $2n^2 > 0$ and $n^3 + 3 > 0$, hence $b_n > 0 = x_n$. Since $n^3+3 > n^3 > 0$, we have $b_n = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n} = y_n$. Therefore for all $n \in \mathbb{N}$, $x_n < b_n < y_n$.
Step 2: $0 = \lim x_n$. Let $\varepsilon > 0$ be given. For all $n \geq 1$, $|x_n - 0| = 0 < \varepsilon$. Hence $0 = \lim x_n$ (def. of limit), with index 1.
Step 3: $0 = \lim y_n$. Let $\varepsilon > 0$ be given. By the Archimedean Property there exists $N \in \mathbb{N}$ such that $N > \frac{2}{\varepsilon}$, and hence $N\varepsilon > 2$, i.e. $\frac{2}{N} < \varepsilon$ (Archimedean). Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$\begin{aligned}
|y_n - 0| &= \frac{2}{n} && (n > 0) \\
&\leq \frac{2}{N} && (n \geq N) \\
&< \varepsilon.
\end{aligned}$$

Therefore for all $n \geq N$, $|y_n - 0| < \varepsilon$, and since $\varepsilon > 0$ was arbitrary, $0 = \lim y_n$ (def. of limit).
Step 4: conclusion. By Steps 1–3, $x_n \leq b_n \leq y_n$ for all $n \in \mathbb{N}$ and $0 = \lim x_n = \lim y_n$. Therefore, by the Squeeze Theorem, $0 = \lim b_n$. Following the proof of the Squeeze Theorem with the three indices 1 (Step 2), N (Step 3) and 1 (Step 1), the index $\max\{1, N, 1\} = N$ works: for all $n \geq N$,

$$-\varepsilon < 0 = x_n \leq b_n \leq y_n < \varepsilon,$$

hence $\left|\frac{2n^2}{n^3+3} - 0\right| < \varepsilon$. □

$$\boxed{N > \frac{2}{\varepsilon}}$$

2.4 Uniqueness of Limits

Theorem 2.3: Uniqueness of Limits

A convergent sequence $\{a_n\}$ has a unique limit.

Idea. *(Scratch work – not part of the written answer.)* Suppose a and a' are both limits. For every $\varepsilon > 0$ the tail is eventually within ε of both a and a', so by the three-point triangle inequality $|a - a'| < 2\varepsilon$ for every $\varepsilon > 0$. Plugging in the specific value $\varepsilon = |a - a'|/2$ forces $|a - a'| = 0$.

Proof. Let a and a' both be limits of $\{a_n\}$. Suppose, for contradiction, that $a \neq a'$, so that $|a - a'| > 0$. Let $\varepsilon := \dfrac{|a - a'|}{2} > 0$. By the definition of $a = \lim a_n$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \varepsilon$. By the definition of $a' = \lim a_n$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - a'| < \varepsilon$. Let $N := \max\{N_1, N_2\}$, and fix any $n \geq N$; such an n exists since $\mathbb{N}$ is unbounded. Then, by the three-point triangle inequality,

$$\begin{aligned} |a - a'| &\leq |a - a_n| + |a_n - a'| \\ &< \varepsilon + \varepsilon \\ &= 2\varepsilon = |a - a'|. \end{aligned}$$

This gives $|a - a'| < |a - a'|$, which is false. Therefore the assumption $a \neq a'$ is untenable, and hence $a = a'$. □

Common slip. It is tempting to conclude "$|a - a'| < 2\varepsilon$ for a particular ε, so $a = a'$." The argument only works because the bound $|a - a'| < 2\varepsilon$ holds for *every* $\varepsilon > 0$; substituting the specific value $\varepsilon = |a - a'|/2$ is what turns this into a contradiction (equivalently, $|a - a'| \geq 0$ and $|a - a'| < 2\varepsilon$ for all $\varepsilon > 0$ forces $|a - a'| = 0$).

2.5 Boundedness of Convergent Sequences

Definition 2.5: Bounded Sequence

A sequence $\{a_n\}$ is said to be *bounded* if $\exists M > 0\ \forall n \in \mathbb{N}\ |a_n| \leq M$. Otherwise the sequence is said to be *unbounded*.

Remark (Comparison with bounded sets). Given a sequence $\{a_n\}$ we can form the set $A := \{a_n : n \in \mathbb{N}\}$ and compare boundedness of the set with boundedness of the sequence. The set A is bounded above and below if $\exists b \in \mathbb{R}\ \forall a \in A\ a \leq b$ and $\exists c \in \mathbb{R}\ \forall a \in A\ c \leq a$; the sequence $\{a_n\}$ is bounded if $\exists M > 0\ \forall n \in \mathbb{N}\ -M \leq a_n \leq M$. The two notions agree once the upper and lower bounds are combined into a single symmetric bound $M := \max\{|b|, |c|\}$.

Remark (Eventual boundedness). A sequence $\{a_n\}$ is bounded if and only if it is eventually bounded, i.e. some tail has all its terms satisfying $|a_n| \leq M$ for some $M > 0$. Indeed, if $|a_n| \leq M_1$ for all $n \geq N$, then taking $M := \max\{|a_1|, \ldots, |a_{N-1}|, M_1\}$ (with the finite list omitted if $N = 1$) bounds every term of the sequence.

Theorem 2.4: Boundedness of Convergent Sequences

Every convergent sequence $\{a_n\}$ is bounded.

Idea. *(Scratch work – not part of the written answer.)* Take the specific value $\varepsilon = 1$ in the definition of the limit: the tail is eventually within 1 of the limit a, hence eventually bounded by $|a| + 1$, hence bounded by the Remark on eventual boundedness.

Proof. Let a denote the limit of $\{a_n\}$. By the definition of $a = \lim a_n$, applied with $\varepsilon = 1 > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < 1$. By the Reverse Triangle Inequality, $\big||a_n| - |a|\big| \leq |a_n - a| < 1$ for all $n \geq N$, and hence $|a_n| < |a| + 1$ for all $n \geq N$. This shows that $\{a_n\}$ is eventually bounded by $|a| + 1$. Therefore, by the Remark on eventual boundedness, $\{a_n\}$ is bounded. □

Remark (The converse is not true). Not every bounded sequence is convergent. For instance, the sequence $\{(-1)^n\}$ satisfies $|(-1)^n| = 1 \leq 1$ for every n, so it is bounded, yet it does not converge: it oscillates permanently between -1 and 1 and so has no single tail that stays close to one number. (A fully rigorous divergence proof is available once subsequences are introduced in Lecture 7; for now this serves only as a counterexample to the converse.)

2.6 The Algebraic Limit Theorem

Theorem 2.5: Algebraic Limit Theorem

If a sequence $\{a_n\}$ converges to $a \in \mathbb{R}$ and a sequence $\{b_n\}$ converges to $b \in \mathbb{R}$, then

(1) for all numbers $c \in \mathbb{R}$, the sequence $\{ca_n\}$ converges to ca;

(2) the sequence $\{a_n + b_n\}$ converges to $a + b$;

(3) the sequence $\{a_n b_n\}$ converges to ab; and

(4) the sequence $\left\{\dfrac{1}{a_n}\right\}$ converges to $\dfrac{1}{a}$, provided $a \neq 0$.

Idea (Part 1). *(Scratch work – not part of the written answer.)* We want $|ca_n - ca| = |c|\,|a_n - a| < \varepsilon$. Choose the tail of $\{a_n\}$ to be within $\dfrac{\varepsilon}{|c| + 1}$ of a (the "+1" avoids dividing by $c = 0$); multiplying by $|c|$ gives a bound strictly below ε.

Proof (Part 1). Let $\varepsilon > 0$ be given. By the definition of $a = \lim a_n$, applied with the positive number $\dfrac{\varepsilon}{|c| + 1}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \dfrac{\varepsilon}{|c| + 1}$. Then for all $n \geq N$,

$$|ca_n - ca| = |c|\ |a_n - a| < |c| \cdot \frac{\varepsilon}{|c| + 1} < \varepsilon,$$

since $\dfrac{|c|}{|c| + 1} < 1$. Therefore, for all $n \geq N$, $|ca_n - ca| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $ca = \lim(ca_n)$. □

Idea (Part 2). *(Scratch work – not part of the written answer.)* Split the target tolerance in half: make $\{a_n\}$ and $\{b_n\}$ each eventually within $\varepsilon/2$ of their own limits, then add.

Proof (Part 2). Let $\varepsilon > 0$ be given. By the definition of $a = \lim a_n$, applied with $\dfrac{\varepsilon}{2}$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \dfrac{\varepsilon}{2}$. By the definition of $b = \lim b_n$, applied with $\dfrac{\varepsilon}{2}$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \dfrac{\varepsilon}{2}$. Let $N := \max\{N_1, N_2\}$. Then for all $n \geq N$, by the Triangle Inequality,

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, for all $n \geq N$, $|(a_n + b_n) - (a + b)| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $a + b = \lim(a_n + b_n)$. □

Idea (Part 3). *(Scratch work – not part of the written answer.)* Add and subtract a cross term to split $a_nb_n - ab$ into three controllable pieces:

$$a_nb_n - ab = (a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a).$$

Bound $|a_n - a|, |b_n - b|$ by $\min\left\{\sqrt{\varepsilon/2},\ \dfrac{\varepsilon}{2(|a| + |b| + 1)}\right\}$ so the quadratic term contributes at most $\varepsilon/2$ and the two linear terms together contribute less than $\varepsilon/2$.

Proof (Part 3). Let $\varepsilon > 0$ be given, and set $\delta := \min\left\{\sqrt{\dfrac{\varepsilon}{2}},\ \dfrac{\varepsilon}{2(|a| + |b| + 1)}\right\} > 0$. By the definition of $a = \lim a_n$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \delta$. By the definition of $b = \lim b_n$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \delta$. Let $N := \max\{N_1, N_2\}$. For all $n \geq N$, the identity $a_nb_n - ab = (a_n - a)(b_n - b) + a(b_n - b) + b(a_n - a)$ together with the Triangle Inequality gives

$$\begin{aligned} |a_nb_n - ab| &\leq |a_n - a|\,|b_n - b| + |a|\,|b_n - b| + |b|\,|a_n - a| \\ &< \delta^2 + |a|\ \delta + |b|\ \delta \\ &\leq \frac{\varepsilon}{2} + \frac{(|a| + |b|)\varepsilon}{2(|a| + |b| + 1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

using $\delta^2 \leq \varepsilon/2$, $\delta \leq \dfrac{\varepsilon}{2(|a| + |b| + 1)}$, and $\dfrac{|a| + |b|}{|a| + |b| + 1} < 1$. Therefore, for all $n \geq N$, $|a_nb_n - ab| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $ab = \lim(a_nb_n)$. □

Idea (Part 4 – QRS proof, not on the slides). *(Scratch work – not part of the written answer.)* The slides state part 4 but prove only parts 1–3 (see the digest notes on Lecture 5); we supply a proof here using only results available by Lecture 5. Since $a \neq 0$, the Reverse Triangle Inequality shows $\{a_n\}$ is eventually bounded away from 0: taking $\varepsilon = |a|/2$ in the definition of $a = \lim a_n$ forces $|a_n| > |a|/2$ eventually, which both makes $1/a_n$ well defined and lets us bound the denominator of $\dfrac{1}{a_n} - \dfrac{1}{a} = \dfrac{a - a_n}{a_na}$ from below.

Proof (Part 4 – QRS proof, not on the slides). Since $a \neq 0$, let $\varepsilon_1 := \dfrac{|a|}{2} > 0$. By the definition of $a = \lim a_n$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \varepsilon_1 = \dfrac{|a|}{2}$. By the Reverse Triangle Inequality, for all $n \geq N_1$,

$$\big||a_n| - |a|\big| \leq |a_n - a| < \frac{|a|}{2} \implies |a_n| > |a| - \frac{|a|}{2} = \frac{|a|}{2}.$$

In particular $a_n \neq 0$ for all $n \geq N_1$, so $\dfrac{1}{a_n}$ is defined for these n. Now let $\varepsilon > 0$ be given, and let $\varepsilon_2 := \dfrac{\varepsilon\,|a|^2}{2} > 0$. By the definition of $a = \lim a_n$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - a| < \varepsilon_2$. Let $N := \max\{N_1, N_2\}$. Then for all $n \geq N$,

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| = \frac{|a - a_n|}{|a_n|\,|a|} < \frac{|a_n - a|}{\left(\dfrac{|a|}{2}\right)|a|} = \frac{2\,|a_n - a|}{|a|^2} < \frac{2}{|a|^2} \cdot \frac{\varepsilon\,|a|^2}{2} = \varepsilon,$$

using $|a_n| > |a|/2$ (valid since $n \geq N_1$) for the first inequality and $|a_n - a| < \varepsilon_2$ (valid since $n \geq N_2$) for the second. Therefore, for all $n \geq N$, $\left|\dfrac{1}{a_n} - \dfrac{1}{a}\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $\dfrac{1}{a} = \lim\left(\dfrac{1}{a_n}\right)$. □

Common slip. Writing $\left|\frac{1}{a_n}-\frac{1}{a}\right| = \frac{|a_n-a|}{|a_n|\,|a|}$ and dividing by $|a_n|$ is only legitimate once a_n is known to be nonzero – and to get a *useful* bound, $|a_n|$ must first be shown to be eventually bounded *away from* 0 (not merely nonzero). Skipping the step that produces $|a_n| > |a|/2$ and dividing anyway is a common way to lose marks.

Worked Example (Tutorial 2, Q4(c)): $2 = \lim a_n \implies \frac{1}{2} = \lim \frac{1}{a_n}$, without the Algebraic Limit Theorem

Here $\{a_n\}$ is a sequence of *positive* real numbers with $2 = \lim a_n$. We must prove $\frac{1}{2} = \lim \frac{1}{a_n}$ directly from the definition of limit, without citing the Algebraic Limit Theorem (even though the argument mirrors the proof of its Part 4 above).

Method 1: ε-N definition with a lower bound on a_n

Proof. Step 1: a_n is eventually bounded away from 0. By the definition of $2 = \lim a_n$ (def. of limit) with $\varepsilon_1 = 1 > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - 2| < 1$. For such n,

$$|a_n - 2| < 1 \iff -1 < a_n - 2 < 1 \iff 1 < a_n < 3.$$

Therefore for all $n \geq N_1$, $a_n > 1$.
Step 2: the ε-N argument. Let $\varepsilon > 0$ be given. Then $\varepsilon_2 := 2\varepsilon > 0$, so by the definition of $2 = \lim a_n$ there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - 2| < 2\varepsilon$. Let $N := \max\{N_1, N_2\}$ and let $n \in \mathbb{N}$ with $n \geq N$. Then $n \geq N_1$ and $n \geq N_2$. Since $a_n > 0$, the term $\frac{1}{a_n}$ is defined and

$$\frac{1}{a_n} - \frac{1}{2} = \frac{2-a_n}{2a_n}, \qquad \text{hence} \qquad \left|\frac{1}{a_n} - \frac{1}{2}\right| = \frac{|2-a_n|}{|2a_n|} = \frac{|a_n-2|}{2a_n} \quad (2a_n > 0).$$

Since $n \geq N_1$, Step 1 gives $a_n > 1$, so $2a_n > 2$ and therefore $\frac{1}{2a_n} < \frac{1}{2}$. Hence

$$\begin{aligned} \left|\frac{1}{a_n} - \frac{1}{2}\right| &= \frac{|a_n-2|}{2a_n} \\ &\leq \frac{|a_n-2|}{2} && (a_n > 1, \text{ since } n \geq N_1) \\ &< \frac{2\varepsilon}{2} = \varepsilon && (|a_n - 2| < 2\varepsilon, \text{ since } n \geq N_2). \end{aligned}$$

Therefore, for all $n \in \mathbb{N}$, $n \geq N \implies \left|\frac{1}{a_n} - \frac{1}{2}\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $\frac{1}{2} = \lim \frac{1}{a_n}$ (def. of limit). □

$$\boxed{\varepsilon_1 = 1 \text{ (forces } a_n > 1)} \qquad \boxed{\varepsilon_2 = 2\varepsilon} \qquad \boxed{N = \max\{N_1, N_2\}}$$

Method 2: Two one-sided bounds by order reversal of reciprocals

Proof. For all $x, y \in \mathbb{R}$ with $0 < x < y$, we have $\frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy} > 0$, hence $\frac{1}{y} < \frac{1}{x}$ (taking reciprocals of positive numbers reverses the order).

Step 1: the two windows. By the definition of $2 = \lim a_n$ with $\varepsilon_1 = 1$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - 2| < 1$, i.e. $1 < a_n < 3$. Let $\varepsilon > 0$ be given. By the definition of $2 = \lim a_n$ with $\varepsilon_2 = 2\varepsilon$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - 2| < 2\varepsilon$, i.e. $2 - 2\varepsilon < a_n < 2 + 2\varepsilon$. Let $N := \max\{N_1, N_2\}$ and let $n \in \mathbb{N}$ with $n \geq N$. Then $a_n > 1$ and $a_n > 2 - 2\varepsilon$, hence $a_n > m := \max\{1, 2 - 2\varepsilon\} \geq 1 > 0$; also $a_n < 2 + 2\varepsilon$. Since $0 < m < a_n < 2 + 2\varepsilon$, order reversal gives

$$\frac{1}{2 + 2\varepsilon} < \frac{1}{a_n} < \frac{1}{m}.$$

Step 2: lower side. We compute

$$\frac{1}{2} - \frac{1}{2 + 2\varepsilon} = \frac{(2 + 2\varepsilon) - 2}{2(2 + 2\varepsilon)} = \frac{2\varepsilon}{2(2 + 2\varepsilon)} = \frac{\varepsilon}{2 + 2\varepsilon} < \varepsilon,$$

since $2 + 2\varepsilon > 1$. Therefore $\dfrac{1}{a_n} > \dfrac{1}{2 + 2\varepsilon} > \dfrac{1}{2} - \varepsilon$.

Step 3: upper side, by cases on ε. If $\varepsilon \geq \frac{1}{2}$, then $2 - 2\varepsilon \leq 1$, so $m = 1$ and $\dfrac{1}{a_n} < 1 = \dfrac{1}{2} + \dfrac{1}{2} \leq \dfrac{1}{2} + \varepsilon$. If $0 < \varepsilon < \frac{1}{2}$, then $2 - 2\varepsilon > 1$, so $m = 2 - 2\varepsilon$ and

$$\frac{1}{m} - \frac{1}{2} = \frac{2 - (2 - 2\varepsilon)}{2(2 - 2\varepsilon)} = \frac{2\varepsilon}{2(2 - 2\varepsilon)} = \frac{\varepsilon}{2 - 2\varepsilon} < \varepsilon,$$

since $2 - 2\varepsilon > 1$; hence $\dfrac{1}{a_n} < \dfrac{1}{m} < \dfrac{1}{2} + \varepsilon$. In both cases $\dfrac{1}{a_n} < \dfrac{1}{2} + \varepsilon$.

Step 4: conclusion. By Steps 2 and 3, $\dfrac{1}{2} - \varepsilon < \dfrac{1}{a_n} < \dfrac{1}{2} + \varepsilon$, which is equivalent to $\left|\dfrac{1}{a_n} - \dfrac{1}{2}\right| < \varepsilon$. Therefore, for all $n \in \mathbb{N}$, $n \geq N \implies \left|\dfrac{1}{a_n} - \dfrac{1}{2}\right| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\dfrac{1}{2} = \lim \dfrac{1}{a_n}$ (def. of limit). □

$$\boxed{\varepsilon_1 = 1 \text{ (forces } a_n > 1)}$$

$$\boxed{\varepsilon_2 = 2\varepsilon}$$

$$\boxed{N = \max\{N_1, N_2\}}$$

2.7 The Order Limit Theorem

Theorem 2.6: Order Limit Theorem

For any convergent sequences $\{a_n\}$ and $\{b_n\}$,

(1) if $\{a_n\}$ eventually satisfies $a_n \geq 0$, then $\lim a_n \geq 0$;

(2) if $\{a_n\}$ and $\{b_n\}$ eventually satisfy $a_n \leq b_n$, then $\lim a_n \leq \lim b_n$; and

(3) for all numbers $c \in \mathbb{R}$, if $\{a_n\}$ eventually satisfies $a_n \leq c$, then $\lim a_n \leq c$.

Remark (Strict inequalities). If the convergent sequence $\{a_n\}$ eventually satisfies the *strict* inequality $a_n > 0$, it is generally **not** true that its limit $a > 0$: the sequence $\left\{\dfrac{1}{n}\right\}$ satisfies $\dfrac{1}{n} > 0$ for every n, yet its limit is 0. Limits only ever preserve non-strict inequalities.

Common slip. Concluding "$a_n < b_n$ eventually, so $\lim a_n < \lim b_n$" (with a strict inequality in the conclusion) is one of the most common errors on this theorem; the correct conclusion is always non-strict, $\lim a_n \leq \lim b_n$, exactly as in part (2) above.

Idea (Part 1). *(Scratch work – not part of the written answer.)* From the definition of $a = \lim a_n$, the tail is eventually within ε of a *and* eventually ≥ 0; combining the two tails gives $a > a_n - \varepsilon \geq -\varepsilon$ for every $\varepsilon > 0$, which forces $a \geq 0$.

Proof (Part 1). Let a denote the limit of $\{a_n\}$. Let $\varepsilon > 0$ be given. By the definition of $a = \lim a_n$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $a_n - a < \varepsilon$. By hypothesis, $\{a_n\}$ is eventually ≥ 0, so there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $a_n \geq 0$. Let $N := \max\{N_1, N_2\}$, and fix any $n \geq N$. Then

$$a = a_n - (a_n - a) > 0 - \varepsilon = -\varepsilon,$$

using $a_n \geq 0$ and $a_n - a < \varepsilon$. Therefore $a > -\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $a \geq 0$ (if instead $a < 0$, taking $\varepsilon = -a/2 > 0$ above would give $a > -\varepsilon = a/2 > a$, a contradiction). □

Idea (Part 2 – QRS proof, not on the slides). *(Scratch work – not part of the written answer.)* The slides prove only part 1 and move directly to the Squeeze Theorem (see the digest notes on Lecture 5). Apply part 1, already proved, to the sequence $\{b_n - a_n\}$, which is eventually ≥ 0 by hypothesis and converges to $b - a$ by the Algebraic Limit Theorem.

Proof (Part 2 – QRS proof, not on the slides). Let $a = \lim a_n$ and $b = \lim b_n$. By the Algebraic Limit Theorem (parts 1 and 2, with $c = -1$), the sequence $\{b_n - a_n\}$ converges to $b - a$. By hypothesis, $\{a_n\}$ and $\{b_n\}$ eventually satisfy $a_n \leq b_n$, so $\{b_n - a_n\}$ eventually satisfies $b_n - a_n \geq 0$. Therefore, by part 1 of the Order Limit Theorem (already proved), $\lim(b_n - a_n) \geq 0$, i.e. $b - a \geq 0$. Hence $a \leq b$, i.e. $\lim a_n \leq \lim b_n$. □

Idea (Part 3 – QRS proof, not on the slides). *(Scratch work – not part of the written answer.)* Compare $\{a_n\}$ with the constant sequence $b_n := c$, which trivially converges to c, and apply part 2, already proved.

Proof (Part 3 – QRS proof, not on the slides). Let $a = \lim a_n$ and let $c \in \mathbb{R}$. Define $b_n := c$ for every $n \in \mathbb{N}$. Then $c = \lim b_n$: given any $\varepsilon > 0$, for every $n \in \mathbb{N}$ (so trivially for all $n \geq 1$), $|b_n - c| = |c - c| = 0 < \varepsilon$, which is exactly the definition of $c = \lim b_n$. By hypothesis, $\{a_n\}$ eventually satisfies $a_n \leq c = b_n$. Therefore, by part 2 of the Order Limit Theorem (already proved), applied to $\{a_n\}$ and $\{b_n\}$, $\lim a_n \leq \lim b_n = c$. □

2.8 The Squeeze Theorem

Theorem 2.7: Squeeze Theorem

If the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ eventually satisfy $a_n \leq b_n \leq c_n$ and $\{a_n\}$ and $\{c_n\}$ have the same limit l, then the sequence $\{b_n\}$ converges to l.

Remark (Comparison with the Order Limit Theorem). The Squeeze Theorem is **not** a direct consequence of the Order Limit Theorem: the Order Limit Theorem only ever compares limits of sequences that are *already known to converge*, whereas here the convergence of $\{b_n\}$ is itself part of the conclusion. The proof below works directly from the two ε-bounds on $\{a_n\}$ and $\{c_n\}$ and the sandwiching hypothesis, without citing the Order Limit Theorem.

Idea. *(Scratch work – not part of the written answer.)* Both $\{a_n\}$ and $\{c_n\}$ are eventually inside $(l - \varepsilon, l + \varepsilon)$; since b_n is sandwiched between them, b_n is forced into the same interval.

Proof. Let $\varepsilon > 0$ be given. By the definition of $l = \lim a_n$, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - l| < \varepsilon$, i.e. $l - \varepsilon < a_n < l + \varepsilon$. By the definition of $l = \lim c_n$, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|c_n - l| < \varepsilon$, i.e. $l - \varepsilon < c_n < l + \varepsilon$. By hypothesis, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ eventually satisfy $a_n \leq b_n \leq c_n$, so there exists $N_3 \in \mathbb{N}$ such that for all $n \geq N_3$, $a_n \leq b_n \leq c_n$. Let $N := \max\{N_1, N_2, N_3\}$. Then for all $n \geq N$,

$$l - \varepsilon < a_n \leq b_n \leq c_n < l + \varepsilon,$$

and hence $|b_n - l| < \varepsilon$. Therefore, for all $n \geq N$, $|b_n - l| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $l = \lim b_n$. □

Worked Example (Tutorial 2, Q3(a)): bounded times null is null

Here $\{a_n\}$ is bounded and $\{b_n\}$ is convergent with $0 = \lim b_n$. We show $0 = \lim(a_n b_n)$.
Method 1: Squeeze Theorem with the Algebraic Limit Theorem

Proof. Step 1: the sandwich. Since $\{a_n\}$ is bounded, by the definition of a bounded sequence there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$. Let $n \in \mathbb{N}$. Since $|b_n| \geq 0$, multiplying $|a_n| \leq M$ by $|b_n|$ gives

$$|a_n b_n| = |a_n|\,|b_n| \leq M\,|b_n|\,.$$

Since $M\,|b_n| \geq 0$, the equivalence $|x| \leq b \iff -b \leq x \leq b$ (for $b \geq 0$), applied with $x = a_n b_n$ and $b = M\,|b_n|$, gives

$$-M\,|b_n| \leq a_n b_n \leq M\,|b_n|\,.$$

Therefore for all $n \in \mathbb{N}$, $-M\,|b_n| \leq a_n b_n \leq M\,|b_n|$.
Step 2: $0 = \lim |b_n|$. Let $\varepsilon > 0$ be given. By the definition of $0 = \lim b_n$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|b_n - 0| < \varepsilon$. For all $n \geq N$,

$$\big||b_n| - 0\big| = |b_n| = |b_n - 0| < \varepsilon.$$

Therefore for all $n \geq N$, $\big||b_n| - 0\big| < \varepsilon$, and since $\varepsilon > 0$ was arbitrary, $0 = \lim |b_n|$ (def. of limit).
Step 3: the outer sequences tend to 0. By the Algebraic Limit Theorem, part 1, applied to $\{|b_n|\}$ with $c = M$ and with $c = -M$,

$$\lim(M\,|b_n|) = M \cdot 0 = 0 \qquad \text{and} \qquad \lim(-M\,|b_n|) = (-M) \cdot 0 = 0.$$

Step 4: conclusion. By Step 1, the three sequences $\{-M\,|b_n|\}$, $\{a_n b_n\}$, $\{M\,|b_n|\}$ satisfy $-M\,|b_n| \leq a_n b_n \leq M\,|b_n|$ for all $n \in \mathbb{N}$ (in particular eventually), and by Step 3 the outer two have the same limit 0. Therefore, by the Squeeze Theorem, $0 = \lim(a_n b_n)$. □

Method 2: ε-N definition with the bound M

Proof. Since $\{a_n\}$ is bounded, there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$. Let $\varepsilon > 0$ be given. Since $M > 0$, the number $\dfrac{\varepsilon}{M}$ is positive, so by the definition of $0 = \lim b_n$ (def. of limit) applied with $\dfrac{\varepsilon}{M}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|b_n - 0| < \dfrac{\varepsilon}{M}$. Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$\begin{aligned} |a_n b_n - 0| &= |a_n|\,|b_n| \\ &\leq M\,|b_n| && (|a_n| \leq M,\ |b_n| \geq 0) \\ &< M \cdot \frac{\varepsilon}{M} && (|b_n| < \tfrac{\varepsilon}{M},\ M > 0) \\ &= \varepsilon. \end{aligned}$$

Therefore, for all $n \in \mathbb{N}$, $n \geq N \implies |a_n b_n - 0| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $0 = \lim(a_n b_n)$. □

2.9 Cauchy Sequences

Definition 2.6: Cauchy Sequence

A sequence $\{a_n\}$ is said to be *Cauchy* if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N} \quad n \geq N \implies |a_n - a_N| < \varepsilon.$$

Equivalently (and more commonly), a sequence $\{a_n\}$ is Cauchy if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n, m \in \mathbb{N} \quad n, m \geq N \implies |a_n - a_m| < \varepsilon.$$

Remark (Comparison with convergent sequences). Convergent: $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N} \; n \geq N \implies |a_n - \lim a_n| < \varepsilon$. Cauchy: $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \in \mathbb{N} \; n \geq N \implies |a_n - a_N| < \varepsilon$. The two definitions have exactly the same shape, but the Cauchy definition compares every tail term to another term *of the sequence itself* (a_N), never to an external candidate limit – this is what makes it possible to test convergence without knowing the limit in advance.

Remark (Cauchy sequences in $\mathbb{Q}$). In $\mathbb{Q}$, Cauchy sequences need not converge (in $\mathbb{Q}$). For instance, any sequence of rational numbers converging (in $\mathbb{R}$) to an irrational number, such as successive decimal truncations of $\sqrt{2}$, is Cauchy but has no limit inside $\mathbb{Q}$.

Common slip. The two forms of the Cauchy definition quantify over "for all $n, m \geq N$" (a statement about *every* pair of tail indices), not "for some $n, m \geq N$." Producing a single pair of close terms says nothing; the bound must hold uniformly across the whole tail.

Worked Example (Tutorial 2, Q5(b) / Sample Midterm Q4): $\{a_n^2\}$ is Cauchy

Here $\{a_n\}$ is a Cauchy sequence. We show, using only the definition of a Cauchy sequence (no Cauchy Criterion, no Algebraic Limit Theorem), that $\{a_n^2\}$ is Cauchy. This is a model answer for the exam format: "write down the definition of … and use it to show …"
Method 1: Direct proof from the definition (bound, then ε')

Proof. *Definition.* $\{a_n\}$ is Cauchy iff $\forall \varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\forall n, m \geq N$, $|a_n - a_m| < \varepsilon$.
Step 1: $\{a_n\}$ is bounded. By the definition of $\{a_n\}$ being Cauchy with $\varepsilon = 1$, there exists $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$, $|a_n - a_m| < 1$. Let $n \in \mathbb{N}$ with $n \geq N_1$. Taking $m = N_1$ (allowed, since $N_1 \geq N_1$) gives $|a_n - a_{N_1}| < 1$, and hence, writing $a_n = (a_n - a_{N_1}) + a_{N_1}$,

$$\begin{aligned} |a_n| &\leq |a_n - a_{N_1}| + |a_{N_1}| && \text{(Triangle Inequality)} \\ &< 1 + |a_{N_1}|. \end{aligned}$$

Let $M := \max\{|a_1|, \ldots, |a_{N_1-1}|, 1 + |a_{N_1}|\}$ (the list $|a_1|, \ldots, |a_{N_1-1}|$ is empty if $N_1 = 1$). Since $M \geq 1 + |a_{N_1}| \geq 1$, we have $M > 0$. Let $n \in \mathbb{N}$. If $n < N_1$, then $|a_n|$ is one of the numbers in the list, so $|a_n| \leq M$. If $n \geq N_1$, then $|a_n| < 1 + |a_{N_1}| \leq M$. Therefore for every $n \in \mathbb{N}$, $|a_n| \leq M$.
Step 2: $\{a_n^2\}$ is Cauchy. Let $\varepsilon > 0$ be given and let $\varepsilon' := \dfrac{\varepsilon}{2M}$; since $M > 0$, $\varepsilon' > 0$. By the definition of $\{a_n\}$ being Cauchy with ε', there exists $N_2 \in \mathbb{N}$ such that for all $n, m \geq N_2$, $|a_n - a_m| < \varepsilon'$. Let $n, m \in \mathbb{N}$ with $n, m \geq N_2$. Since $a_n^2 - a_m^2 = (a_n - a_m)(a_n + a_m)$,

$$\begin{aligned} |a_n^2 - a_m^2| &= |a_n - a_m| \, |a_n + a_m| \\ &\leq |a_n - a_m| \, (|a_n| + |a_m|) && \text{(Triangle Inequality, } |a_n - a_m| \geq 0) \\ &\leq |a_n - a_m| \, (M + M) = 2M \, |a_n - a_m| && \text{(Step 1)} \\ &< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon && (|a_n - a_m| < \varepsilon', \; 2M > 0). \end{aligned}$$

Therefore, for all $n, m \in \mathbb{N}$, $n, m \geq N_2 \implies |a_n^2 - a_m^2| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\{a_n^2\}$ is Cauchy (def. of Cauchy sequence). □

$$\boxed{M := \max\{|a_1|, \ldots, |a_{N_1-1}|, 1 + |a_{N_1}|\}}$$

$$\boxed{\varepsilon' = \frac{\varepsilon}{2M}}$$

Method 2: Contradiction via the negated definitions

Proof. Step 1: $\{a_n\}$ is bounded (by contradiction). By the definition of $\{a_n\}$ being Cauchy with $\varepsilon = 1$, there exists $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$, $|a_n - a_m| < 1$. Let $M := \max\{|a_1|, \ldots, |a_{N_1-1}|, 1 + |a_{N_1}|\}$ (the list $|a_1|, \ldots, |a_{N_1-1}|$ is empty if $N_1 = 1$), so $M \geq 1 > 0$. Suppose, for contradiction, that some $n \in \mathbb{N}$ satisfies $|a_n| > M$. If $n < N_1$, then $|a_n|$ is in the list defining M, so $|a_n| \leq M$, which contradicts $|a_n| > M$. If $n \geq N_1$, then taking $m = N_1$ gives $|a_n - a_{N_1}| < 1$, hence by the Triangle Inequality

$$|a_n| \leq |a_n - a_{N_1}| + |a_{N_1}| < 1 + |a_{N_1}| \leq M,$$

which again contradicts $|a_n| > M$. Both cases are impossible. Therefore for every $n \in \mathbb{N}$, $|a_n| \leq M$. Step 2: $\{a_n^2\}$ is Cauchy (by contradiction). Suppose, for contradiction, that $\{a_n^2\}$ is not Cauchy. Negating the definition, there is $\varepsilon > 0$ such that $\forall N \in \mathbb{N}$ there are $n, m \geq N$ with $|a_n^2 - a_m^2| \geq \varepsilon$. Fix this ε and let $\varepsilon' := \dfrac{\varepsilon}{2M} > 0$. By the definition of $\{a_n\}$ being Cauchy with ε', there exists $N_2 \in \mathbb{N}$ such that for all $n, m \geq N_2$, $|a_n - a_m| < \varepsilon'$. Applying the negated statement with $N = N_2$ gives $n, m \geq N_2$ with $|a_n^2 - a_m^2| \geq \varepsilon$. For these n, m, Step 1 and the Triangle Inequality give

$$\begin{aligned} |a_n^2 - a_m^2| &= |a_n - a_m|\,|a_n + a_m| \\ &\leq |a_n - a_m|\,(|a_n| + |a_m|) \\ &\leq 2M\,|a_n - a_m| && \text{(Step 1)} \\ &< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon && (n, m \geq N_2). \end{aligned}$$

Hence $\varepsilon \leq |a_n^2 - a_m^2| < \varepsilon$, i.e. $\varepsilon < \varepsilon$, which is false. Therefore the assumption is untenable, and $\{a_n^2\}$ is Cauchy. □

$$\boxed{M := \max\{|a_1|, \ldots, |a_{N_1-1}|, 1 + |a_{N_1}|\}}$$

$$\boxed{\varepsilon' = \frac{\varepsilon}{2M}}$$

2.10 The Cauchy Criterion

Theorem 2.8: Cauchy Criterion

A sequence $\{a_n\}$ is convergent if and only if it is Cauchy.

Idea ("only if"). *(Scratch work – not part of the written answer.)* If $a_n \to a$, then two tail terms $a_n, a_{N'}$ are each within $\varepsilon/2$ of a, so within ε of each other by the triangle inequality.

Proof ("only if": convergent $\implies$ Cauchy). Let a denote the limit of $\{a_n\}$. Let $\varepsilon > 0$ be given. By the definition of $a = \lim a_n$, applied with $\frac{\varepsilon}{2}$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N \implies |a_n - a| < \frac{\varepsilon}{2}$. Let $N' := N$, and let $n \in \mathbb{N}$ with $n \geq N'$. Since also $N' = N \geq N$, the same bound applies to the term $a_{N'}$: $|a_{N'} - a| < \frac{\varepsilon}{2}$. By the Triangle Inequality,

$$|a_n - a_{N'}| \leq |a_n - a| + |a - a_{N'}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, for all $n \geq N'$, $|a_n - a_{N'}| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $\{a_n\}$ being Cauchy. □

Idea ("if"). *(Scratch work – not part of the written answer.)* This direction cannot use the Bolzano–Weierstrass Theorem (it is only proved in Lecture 8); the course's route uses the Nested Interval Property (Lecture 3) instead. Fix $\varepsilon_m := \frac{1}{m} \to 0$. For each m, Cauchyness gives an index N_m past which all terms lie within ε_m of a_{N_m}. Record, for each m, the running maximum c_m of the lower endpoints $a_{N_k} - \varepsilon_k$ $(k \leq m)$ and the running minimum d_m of the upper endpoints $a_{N_k} + \varepsilon_k$ $(k \leq m)$. Because any sufficiently late term a_N (with $N \geq \max\{N_1, \ldots, N_m\}$) lies inside *every* one of these m windows simultaneously, $c_m < a_N < d_m$, forcing $c_m \leq d_m$ and producing nested intervals $I_m := [c_m, d_m]$. The Nested Interval Property gives a point a common to all of them, and a is then shown to be the limit.

Proof ("if": Cauchy $\implies$ convergent). Suppose $\{a_n\}$ is Cauchy. For each $m \in \mathbb{N}$, let $\varepsilon_m := \frac{1}{m}$. For every $\varepsilon > 0$ the Archimedean Property gives $N \in \mathbb{N}$ with $1/N < \varepsilon$, and since $\varepsilon_m = 1/m$ decreases in m, we have $\varepsilon_m < \varepsilon$ for all $m \geq N$; hence $0 = \lim \varepsilon_m$.

Step 1: construct $\{N_m\}$. For each $m \in \mathbb{N}$, the definition of $\{a_n\}$ being Cauchy, applied with ε_m, gives some $N_m \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N_m \implies |a_n - a_{N_m}| < \varepsilon_m$.

Step 2: construct nested intervals. For each $m \in \mathbb{N}$, define

$$c_m := \max\{a_{N_1} - \varepsilon_1,\ a_{N_2} - \varepsilon_2,\ \ldots,\ a_{N_m} - \varepsilon_m\}, \qquad d_m := \min\{a_{N_1} + \varepsilon_1,\ a_{N_2} + \varepsilon_2,\ \ldots,\ a_{N_m} + \varepsilon_m\}.$$

Since c_m is a maximum over a list that only grows with m, and d_m is a minimum over the same growing list, $c_1 \leq c_2 \leq c_3 \leq \cdots$ and $d_1 \geq d_2 \geq d_3 \geq \cdots$.

We claim $c_m < d_m$ for every $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$ and let $N := \max\{N_1, \ldots, N_m\}$. For each $k \in \{1, \ldots, m\}$, since $N \geq N_k$, Step 1 gives $|a_N - a_{N_k}| < \varepsilon_k$, i.e. $a_{N_k} - \varepsilon_k < a_N < a_{N_k} + \varepsilon_k$. As this holds simultaneously for every $k = 1, \ldots, m$, the term a_N exceeds each lower endpoint $a_{N_k} - \varepsilon_k$ and is exceeded by each upper endpoint $a_{N_k} + \varepsilon_k$; therefore a_N exceeds their maximum c_m and is exceeded by their minimum d_m, i.e. $c_m < a_N < d_m$. In particular $c_m < d_m$.

Combining $c_m < d_m$ for every m with the monotonicity $c_1 \leq \cdots \leq c_m$ and $d_m \leq \cdots \leq d_1$: for any $p \leq q$, $c_p \leq c_q < d_q \leq d_p$, so $c_p < d_p$ and the intervals $I_m := [c_m, d_m]$ satisfy $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, a nested sequence of closed and bounded intervals.

Step 3: extract a limit candidate. By the Nested Interval Property, there exists $a \in \mathbb{R}$ such that $a \in I_m$ for every $m \in \mathbb{N}$, i.e.

$$c_1 \leq c_2 \leq \cdots \leq c_m \leq a \leq d_m \leq \cdots \leq d_2 \leq d_1 \qquad \text{for every } m \in \mathbb{N}.$$

In particular, since c_m is the maximum of a list containing $a_{N_m} - \varepsilon_m$, we have $a_{N_m} - \varepsilon_m \leq c_m \leq a$; similarly $a \leq d_m \leq a_{N_m} + \varepsilon_m$. Hence

$$a_{N_m} - \varepsilon_m \leq a \leq a_{N_m} + \varepsilon_m \qquad \text{for every } m \in \mathbb{N},$$

which is equivalent to $|a - a_{N_m}| \leq \varepsilon_m$ for every $m \in \mathbb{N}$.

Step 4: show $a = \lim a_n$. Let $\varepsilon > 0$ be given. Since $0 = \lim \varepsilon_m$, there exists $m \in \mathbb{N}$ such that $\varepsilon_m < \frac{\varepsilon}{2}$. Let $N := N_m$, and let $n \in \mathbb{N}$ with $n \geq N$. By Step 1, $|a_n - a_{N_m}| < \varepsilon_m$; by Step 3, $|a - a_{N_m}| \leq \varepsilon_m$. By the Triangle Inequality,

$$|a_n - a| \leq |a_n - a_{N_m}| + |a_{N_m} - a| < \varepsilon_m + \varepsilon_m = 2\varepsilon_m < \varepsilon.$$

Therefore, for all $n \geq N$, $|a_n - a| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this is the definition of $a = \lim a_n$, so $\{a_n\}$ is convergent. □

Extension: An Alternative Route via Bolzano–Weierstrass

The proof above of "Cauchy $\Longrightarrow$ convergent" is the course's official route: it uses only the Nested Interval Property, which is available from Lecture 3. Once the Bolzano–Weierstrass Theorem is proved in Lecture 8 (Chapter 3), a shorter alternative becomes available: every Cauchy sequence is bounded (Worked Example above, Step 1, adapted), so by Bolzano–Weierstrass it has a convergent subsequence with some limit a; one then shows the *whole* Cauchy sequence converges to a by combining the Cauchy bound (terms are eventually close to each other) with the subsequence's convergence (some term is eventually close to a). This route is not available yet at this point of the course and must not be used until after Lecture 8.

2.11 Chapter Summary

Key Formula: Chapter 2 recap

$$
\begin{aligned}
a = \lim a_n: &\quad \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N \quad |a_n - a| < \varepsilon,\\
\{a_n\}\ \text{Cauchy}: &\quad \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \geq N \quad |a_n - a_m| < \varepsilon,\\
\text{ALT}: &\quad \lim(ca_n) = ca, \quad \lim(a_n + b_n) = a + b,\\
&\quad \lim(a_n b_n) = ab, \quad \lim\left(\frac{1}{a_n}\right) = \frac{1}{a}\ \ (a \neq 0),\\
\text{OLT}: &\quad a_n \leq b_n \text{ ev.} \Longrightarrow \lim a_n \leq \lim b_n,\\
\text{Squeeze}: &\quad a_n \leq b_n \leq c_n \text{ ev.},\ \lim a_n = \lim c_n = l \Longrightarrow \lim b_n = l,\\
\text{Cauchy Crit.}: &\quad \{a_n\}\ \text{convergent} \Longleftrightarrow \{a_n\}\ \text{Cauchy}.
\end{aligned}
$$

Exam Focus: the ε-N proof template

- Structure every "verify using the definition" proof as: *Idea* (scratch algebra to find a bound and choose N, not part of the written answer) then *Proof* ("Let $\varepsilon > 0$ be given. By the Archimedean Property [or: by the definition of ...], choose $N \in \mathbb{N}$ such that .... Then for all $n \geq N$, ...").
- N may depend on ε (and, for Cauchy proofs, on constants like M derived from ε) – it may *never* depend on n.
- Limits preserve only non-strict inequalities ($\leq$, never $<$); the Squeeze Theorem is not a shortcut for the Order Limit Theorem, since it additionally proves convergence.
- Before dividing by a_n (Algebraic Limit Theorem part 4, or any reciprocal-sequence argument), first show $\{a_n\}$ is eventually bounded *away from* 0, using a specific ε (commonly $\varepsilon = |a|/2$ or $\varepsilon = 1$) in the definition of the limit.
- "For all $n, m \geq N$" (Cauchy) and "for all $n \geq N$" (convergent) are universal, not existential, statements: a single close pair of terms proves nothing.

3 Lectures 7–9: Subsequences, Bolzano–Weierstrass and Monotone Convergence

This chapter answers three questions: (1) how can the convergence or divergence of a sequence be determined by looking only at pieces of it (its *subsequences*)? (2) why does every bounded sequence, however wild, always contain a convergent piece (the *Bolzano–Weierstrass Theorem*)? (3) why does monotonicity, combined with boundedness, guarantee convergence outright (the *Monotone Convergence Theorem*)?

These three results – the Divergence Criterion, Bolzano–Weierstrass, and Monotone Convergence – are the main toolkit the course adds to the ε-N definitions of Lectures 4–6.

Tutorial 3 shows they are, in fact, logically equivalent to one another.

3.1 Subsequences

Two motivating slide examples set up the need for subsequences.

The alternating harmonic sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ can be split into its odd-length partial sums and its even-length partial sums.

The first sequence is decreasing, the second increasing, and both converge to the same value, hinting that the whole series converges.

Conversely, the sequence $\sin(\pi/2), \sin(2\pi/2), \sin(3\pi/2), \ldots$ splits into an even-indexed piece converging to 0 and a piece indexed by $\{4k-3\}$ converging to 1.

Since these two pieces have different limits, the original sequence must diverge.

Both examples extract convergent *pieces* of a sequence while retaining the original order of terms – this is the idea a subsequence makes precise.

Definition 3.1: Subsequence

Let $\{a_n\}$ be a sequence. A *subsequence* of $\{a_n\}$ is a sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$, written $\{a_{n_k}\}$, where $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers. The subsequence $\{a_{n_k}\}$ is said to be *proper* if $n_k \neq k$ for some $k \in \mathbb{N}$.

The definition of proper is deliberately weak: it fails only for the single index sequence $n_k = k$ for every k, i.e. the sequence $\{a_n\}$ counted against itself.

Every other subsequence – dropping a term, keeping only even indices, keeping only the indices past 10^6 – is proper.

Common slip. Two conditions are easy to blur but must both be checked: the indices n_k must be *strictly* increasing (not merely non-decreasing), and they must be a genuine sequence of terms drawn from $\{a_n\}$ in order.

For $\{a_n\} = \{(n-1)/n\}$, the list $\frac{9}{10}, \frac{4}{5}, \frac{99}{100}, \frac{49}{50}, \ldots$ is *not* a subsequence, because its index list $10, 5, 100, 50, \ldots$ is not increasing at all.

The list $\frac{2}{3}, \frac{2}{3}, \frac{4}{5}, \frac{4}{5}, \ldots$ is also *not* a subsequence, because its index list $3, 3, 5, 5, \ldots$ is increasing but not strictly so: a repeated index is not allowed.

The following fact is used repeatedly (in the proofs of Subsequential Limits and the Monotone Convergence Lemma below) to relate a subsequence's own counting index k to its position n_k inside the original sequence.

Theorem 3.1: Subsequence Index Bound

If $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, then $n_k \geq k$ for every $k \in \mathbb{N}$.

Idea. *(Scratch work – not part of the written answer.)*

Induct on k.

The base case is immediate, since every natural number is at least 1.

The inductive step uses that consecutive indices are distinct integers with $n_{k+1} > n_k$, so in fact $n_{k+1} \geq n_k + 1$.

Proof. We argue by induction on k.
Base case ($k = 1$). Since $n_1 \in \mathbb{N} = \{1, 2, 3, \ldots\}$, we have $n_1 \geq 1$.
Therefore the claim holds when $k = 1$.
Inductive step. Let $k \in \mathbb{N}$ be given, and suppose $n_k \geq k$.
Since $\{n_k\}$ is a strictly increasing sequence of integers, $n_{k+1} > n_k$, and as n_{k+1}, n_k are both integers this forces $n_{k+1} \geq n_k + 1$.
Combining this with the inductive hypothesis $n_k \geq k$ gives

$$n_{k+1} \geq n_k + 1 \geq k + 1.$$

Hence the claim holds for $k + 1$ as well.
By the Principle of Induction, $n_k \geq k$ for every $k \in \mathbb{N}$. □
A first consequence, used in Lecture 9, is that convergence is unaffected by shifting a sequence's starting index by a fixed finite amount.

Remark (Tail Shift). If $\{a_n\}$ is a sequence, $c \in \mathbb{N}_0$ is fixed, and $b_k := a_{k+c}$ for $k \in \mathbb{N}$, then $\{b_k\}$ converges if and only if $\{a_n\}$ converges, and in that case $\lim b_k = \lim a_n$.
Indeed, $\{b_k\}$ is a subsequence of $\{a_n\}$, since its indices $n_k = k + c$ are strictly increasing; so if $a = \lim a_n$, then the theorem on Subsequential Limits (proved below) gives $a = \lim b_k$ as well.
Conversely, suppose $L = \lim b_k$. Let $\varepsilon > 0$ be given.
Since $L = \lim b_k$, there exists $K \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq K$ implies $|b_k - L| = |a_{k+c} - L| < \varepsilon$.
Let $N := K + c$. Let $n \in \mathbb{N}$ with $n \geq N$; then $k := n - c \geq K$ and $k \in \mathbb{N}$, since $n \geq N = K + c \geq 1 + c \geq 1$ gives $n - c \geq K \geq 1$.
Applying the previous sentence to this k gives $|a_n - L| = |a_{k+c} - L| < \varepsilon$.
Therefore, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$; hence $L = \lim a_n$.

3.2 Subsequential Limits and the Divergence Criterion

Theorem 3.2: Subsequential Limits

If $\{a_n\}$ converges to $a \in \mathbb{R}$ and $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, then $\{a_{n_k}\}$ converges to a.

Idea. *(Scratch work – not part of the written answer.)*
Given $\varepsilon > 0$, the definition of $a = \lim a_n$ supplies an N that works for *all* terms with index at least N.
The Subsequence Index Bound ($n_k \geq k$) lets us reuse this same N as the required K for the subsequence: once $k \geq N$, automatically $n_k \geq N$ too.

Proof. Let $\varepsilon > 0$ be given. Since $a = \lim a_n$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $|a_n - a| < \varepsilon$.
Let $K := N$. Let $k \in \mathbb{N}$ be given and suppose $k \geq K$.
By the Subsequence Index Bound (Theorem 3.1), $n_k \geq k$.
Therefore $n_k \geq k \geq K = N$, so $n_k \geq N$.
Applying the definition of $a = \lim a_n$ to the index n_k gives $|a_{n_k} - a| < \varepsilon$.
Hence, for every $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq K$ implies $|a_{n_k} - a| < \varepsilon$.
This is precisely the definition of $a = \lim a_{n_k}$. □
We can now state the course's main tool for detecting divergence from pieces of a sequence.

Theorem 3.3: Divergence Criterion

A *bounded* sequence $\{a_n\}$ is divergent if and only if it has two subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ converging to different limits.

This is an "if and only if" claim, and the course proves its two directions at two different points.
The "if" direction is proved now, in Lecture 7, directly from Subsequential Limits.
The "only if" direction needs the Bolzano–Weierstrass Theorem, which is not yet available, so its proof is deferred to the next subsection (Lecture 8, immediately after Bolzano–Weierstrass is proved).
We record the "if" direction first, exactly in this order.

Idea ("if," by contrapositive). *(Scratch work – not part of the written answer.)*
The contrapositive of "two different subsequential limits $\Longrightarrow$ divergent" is "convergent $\Longrightarrow$ no two different subsequential limits."
This is an immediate consequence of Subsequential Limits: every subsequence of a convergent sequence converges to the same value.

Proof of "if." We prove the contrapositive: if $\{a_n\}$ is convergent, then $\{a_n\}$ does not have two subsequences converging to different limits.
Suppose $\{a_n\}$ converges, with limit a. Let $\{a_{n_k}\}$ and $\{a_{m_k}\}$ be any two subsequences of $\{a_n\}$.
By the theorem on Subsequential Limits, $\lim a_{n_k} = a$ and $\lim a_{m_k} = a$.
Therefore any two subsequences of $\{a_n\}$ converge to the same limit a, so $\{a_n\}$ cannot have two subsequences converging to different limits.
This establishes the contrapositive, and hence the "if" direction of the Divergence Criterion.
Notice that boundedness was not used anywhere in this argument. □

Common Mistake: The Divergence Criterion needs boundedness

The sequence $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots$ (odd positions $2k-1$ give $a_{2k-1} = 2k-1 \to \infty$, even positions give $a_{2k} = 1/(2k) \to 0$) is divergent, since it is unbounded, yet *every* one of its convergent subsequences converges to 0: there is no second subsequential limit to contrast against. This sequence is not a counterexample to the Divergence Criterion because it is not bounded, but it does show that an unbounded divergent sequence need not exhibit two different subsequential limits. Unboundedness is therefore a separate, sufficient reason for divergence (via the Boundedness of Convergent Sequences, Lecture 4), and should be argued directly rather than through the Divergence Criterion.

3.3 The Bolzano–Weierstrass Theorem

An optimisation algorithm confined to a bounded feasible set produces a bounded sequence $\{a_n\}$.
Splitting the bounding interval $[-M, M]$ into n equal pieces, at least one piece must contain infinitely many terms of the sequence, for every n.
Infinitely many terms keep gathering inside arbitrarily small intervals – this suggests that a bounded sequence, whatever its individual behaviour, has terms clustering around some specific real number, closely enough to extract a convergent subsequence.

Theorem 3.4: Bolzano–Weierstrass Theorem

If a sequence $\{a_n\}$ is bounded, then it has a convergent subsequence.

Idea. *(Scratch work – not part of the written answer.)*
Build a candidate limit a as the supremum of the set of numbers that lie below infinitely many terms of the sequence.
Show every interval $(a - 1/k, a + 1/k)$ still contains infinitely many terms, using the ε-Criterion for Suprema on one side and the definition of upper bound on the other.
Then greedily pick out a strictly increasing sequence of indices, each landing inside a shrinking such interval.

Proof. Since $\{a_n\}$ is bounded, there exists $M > 0$ such that $|a_n| \le M$ for every $n \in \mathbb{N}$. Define

$$A := \{x \in \mathbb{R} : x \le a_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

The set A is nonempty, since $-M \leq a_n$ for every $n \in \mathbb{N}$ (so in particular for infinitely many n), hence $-M \in A$.

Moreover M is an upper bound for A: if $x \in A$, then $x \leq a_n$ for some $n \in \mathbb{N}$, and $a_n \leq M$, so $x \leq M$.

Since A is nonempty and bounded above, the Axiom of Completeness implies that A has a supremum; let $a := \sup A$.

Claim: for every $k \in \mathbb{N}$, the interval $(a - 1/k,\, a + 1/k)$ contains a_n for infinitely many $n \in \mathbb{N}$.

Let $k \in \mathbb{N}$ be given. By the ε-Criterion for Suprema and Infima applied to $a = \sup A$ with $\varepsilon = 1/k$, there exists $x_k \in A$ such that $a - 1/k < x_k$.

Since $x_k \in A$, $x_k \leq a_n$ for infinitely many $n \in \mathbb{N}$; for each such n, $a - 1/k < x_k \leq a_n$.

On the other hand, $a + 1/k > a$, and since a is an upper bound for A, the number $a + 1/k$ is not itself an element of A.

By the definition of A, this means $a + 1/k \leq a_n$ for only finitely many $n \in \mathbb{N}$.

Consequently, among the infinitely many indices n satisfying $a_n > a - 1/k$, all but finitely many also satisfy $a_n < a + 1/k$, since only finitely many can fail this.

Removing a finite set from an infinite set leaves infinitely many elements, so there are infinitely many $n \in \mathbb{N}$ with $a - 1/k < a_n < a + 1/k$, i.e. $|a_n - a| < 1/k$.

This proves the claim.

We now construct a subsequence recursively.

By the claim with $k = 1$, there is an index $n_1 \in \mathbb{N}$ with $|a_{n_1} - a| < 1$.

Having chosen $n_1 < n_2 < \cdots < n_k$, the claim with $k + 1$ gives infinitely many indices n satisfying $|a_n - a| < 1/(k + 1)$.

Since only finitely many natural numbers do not exceed n_k, at least one such index n exceeds n_k, and we let n_{k+1} be such an index.

Then $n_{k+1} > n_k$ and $|a_{n_{k+1}} - a| < 1/(k + 1)$.

Repeating this construction produces a strictly increasing sequence of indices $n_1 < n_2 < n_3 < \cdots$ – that is, a genuine subsequence $\{a_{n_k}\}$ – satisfying $|a_{n_k} - a| < 1/k$ for every $k \in \mathbb{N}$.

Finally, we verify $a = \lim a_{n_k}$. Let $\varepsilon > 0$ be given.

By the Archimedean Property, there exists $K \in \mathbb{N}$ with $1/K < \varepsilon$. Then for every $k \in \mathbb{N}$ with $k \geq K$,

$$|a_{n_k} - a| < \frac{1}{k} \leq \frac{1}{K} < \varepsilon.$$

Hence, for every $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $|a_{n_k} - a| < \varepsilon$; this is the definition of $a = \lim a_{n_k}$.

Therefore $\{a_n\}$ has a convergent subsequence. □

Common Mistake: Bolzano–Weierstrass without boundedness

Boundedness cannot be dropped. The unbounded sequence $\{n\} = 1, 2, 3, \ldots$ has no convergent subsequence whatsoever: if $\{a_{n_k}\} = \{n_k\}$ were any subsequence, the Subsequence Index Bound gives $n_k \geq k \to \infty$, so $\{n_k\}$ is itself unbounded and, by the Boundedness of Convergent Sequences (Lecture 4), cannot converge.

Extension: The Bisection Proof of Bolzano–Weierstrass

The textbook route (not the one examined in this course, but logically equivalent) proves Bolzano–Weierstrass via the Nested Interval Property instead of a supremum. Start with $I_1 := [-M, M]$, which contains all terms of $\{a_n\}$. Bisect I_1 into two closed halves; at least one half contains a_n for infinitely many n (otherwise both halves, and hence I_1, would contain only finitely many terms), and it is chosen as I_2. Repeating this bisection produces nested closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$, each containing infinitely many terms of $\{a_n\}$, with $\text{length}(I_k) = 2M/2^{k-1} \to 0$. The Nested Interval Property gives a point $a \in \bigcap_{k=1}^{\infty} I_k$. Choosing n_k strictly increasing with $a_{n_k} \in I_k$ (possible since I_k contains infinitely many terms, so one can always be found past the previous index), the Squeeze Theorem applied to $|a_{n_k} - a| \leq \text{length}(I_k) \to 0$ gives $a_{n_k} \to a$.

We can now complete the Divergence Criterion (Theorem 3.3), whose "if" direction was proved above using only Subsequential Limits.

Idea ("only if", via Bolzano–Weierstrass). *(Scratch work – not part of the written answer.)* Since $\{a_n\}$ is bounded, Bolzano–Weierstrass names a first subsequential limit a. Because $\{a_n\}$ does not converge to a, infinitely many terms stay at distance at least some ε from a; these terms form a bounded subsequence, so a second application of Bolzano–Weierstrass extracts a subsequence converging to some b with $|b-a| \geq \varepsilon$, giving two different subsequential limits.

Proof of the Divergence Criterion, "only if" direction (completed). Suppose $\{a_n\}$ is bounded and divergent.

Since $\{a_n\}$ is bounded, the Bolzano–Weierstrass Theorem implies that $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$; let $a := \lim a_{n_k}$.

Since $\{a_n\}$ is divergent, $\{a_n\}$ does not converge to a.

By the negation of the definition of $a = \lim a_n$, there exists $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $n \geq N$ and $|a_n - a| \geq \varepsilon$.

In particular, for every $N \in \mathbb{N}$ there is an index $n \geq N$ with $|a_n - a| \geq \varepsilon$, so the set of such indices is not bounded above; since every finite subset of $\mathbb{N}$ is bounded above, there are infinitely many such n.

Let $\{b_j\}$ be the subsequence of $\{a_n\}$ consisting exactly of the terms with $|a_n - a| \geq \varepsilon$, listed in increasing order of index; this is a genuine subsequence since infinitely many indices qualify.

Since $\{a_n\}$ is bounded and $\{b_j\}$ consists of terms of $\{a_n\}$, $\{b_j\}$ is bounded as well.

Applying the Bolzano–Weierstrass Theorem a second time, now to $\{b_j\}$, gives a convergent subsequence $\{a_{m_k}\}$ of $\{b_j\}$.

A subsequence of a subsequence of $\{a_n\}$ is itself a subsequence of $\{a_n\}$, since composing two strictly increasing index maps gives a strictly increasing index map; so $\{a_{m_k}\}$ is a subsequence of $\{a_n\}$.

Let $b := \lim a_{m_k}$.

We claim $b \neq a$. Indeed, every term of $\{a_{m_k}\}$ satisfies $|a_{m_k} - a| \geq \varepsilon$.

Suppose, for contradiction, that $b = a$. Since $\{a_{m_k}\}$ converges to b, there exists $k \in \mathbb{N}$ with $|a_{m_k} - b| < \varepsilon$.

But $b = a$ would then give $|a_{m_k} - a| < \varepsilon$, contradicting $|a_{m_k} - a| \geq \varepsilon$ for every k.

Therefore $b \neq a$.

Consequently, $\{a_{n_k}\}$ and $\{a_{m_k}\}$ are two subsequences of $\{a_n\}$ converging to the different limits a and b respectively.

This proves the "only if" direction, and completes the proof of the Divergence Criterion. □

Common slip. The two applications of Bolzano–Weierstrass above serve different purposes: the first extracts *some* convergent subsequence to name a candidate limit a; the second is applied not to $\{a_n\}$ itself but to the (still bounded) subsequence of terms that stay at distance at least ε from a, to produce a *second*, different, subsequential limit. Forgetting to re-check boundedness of $\{b_j\}$ before the second application is a common gap in exam write-ups.

3.4 Monotone Sequences and the Monotone Convergence Theorem

The Babylonian method for $\sqrt{a}$ ($x_{n+1} := \frac{1}{2}(x_n + a/x_n)$) applied to $a = 31$ from $x_1 = 1$ produces

$$1,\ 16,\ 8.96875,\ 6.2126\ldots,\ 5.6012\ldots,\ 5.5679\ldots,\ldots,$$

a sequence that appears to be eventually decreasing.

This motivates asking whether monotonicity alone – together with boundedness – is already enough to guarantee convergence, without having to guess the limit in advance.

Definition 3.2: Monotone Sequence

A sequence $\{a_n\}$ is said to be *increasing* (resp. *strictly increasing*) if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$ (resp. $a_n < a_{n+1}$), and *decreasing* (resp. *strictly decreasing*) if for all $n \in \mathbb{N}$, $a_n \geq a_{n+1}$ (resp. $a_n > a_{n+1}$). In either case $\{a_n\}$ is said to be *monotone* (resp. *strictly monotone*). The sequence $\{a_n\}$ is *eventually monotone* if there exists $N_0 \in \mathbb{N}$ such that $\{a_n\}_{n \geq N_0}$ is monotone – i.e. if some tail of $\{a_n\}$ is monotone.

Theorem 3.5: Monotone Convergence Lemma

If a sequence $\{a_n\}$ is eventually monotone and has a convergent subsequence, then $\{a_n\}$ is convergent.

Idea. *(Scratch work – not part of the written answer.)*

By the Tail-Shift Remark, we may discard finitely many leading terms and assume $\{a_n\}$ is monotone outright.

Given $\varepsilon > 0$, the convergent subsequence supplies indices $n_{k'}$ and $n_{k'+1}$ that bracket an arbitrary n.

Monotonicity then sandwiches a_n between $a_{n_{k'}}$ and $a_{n_{k'+1}}$, both already known to be ε-close to the limit.

Proof. By passing to a tail, we may assume without loss of generality that $\{a_n\}$ is monotone.

Indeed, if $N_0 \in \mathbb{N}$ is such that $\{a_n\}_{n \geq N_0}$ is monotone, then proving the Lemma for this tail (which inherits a convergent subsequence from any tail of the given convergent subsequence, since removing finitely many terms from a subsequence leaves a subsequence) and invoking the Tail-Shift Remark transfers the conclusion back to $\{a_n\}$ itself.

So suppose $\{a_n\}$ is monotone.

We first record a sandwiching fact: *if $\{a_n\}$ is monotone and $p \leq q \leq r$ are natural numbers, then a_q lies between a_p and a_r.*

If $\{a_n\}$ is increasing, then $p \leq q$ gives $a_p \leq a_q$, and $q \leq r$ gives $a_q \leq a_r$; so $a_p \leq a_q \leq a_r$.

If $\{a_n\}$ is decreasing, the reverse inequalities hold: $a_p \geq a_q \geq a_r$.

In either case a_q lies between a_p and a_r, so

$$|a_q - a| \leq \max\{|a_p - a|, |a_r - a|\} \qquad \text{for every } a \in \mathbb{R}.$$

Let $\{a_{n_k}\}$ be a convergent subsequence of $\{a_n\}$, and let $a := \lim a_{n_k}$. Let $\varepsilon > 0$ be given.

Since $a = \lim a_{n_k}$, there exists $K \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq K$ implies $|a_{n_k} - a| < \varepsilon$.

Let $N := n_K$.

Let $n \in \mathbb{N}$ with $n \geq N$. Consider the set $S := \{k \in \mathbb{N} : n_k \leq n\}$.

This set is nonempty, since $n_K \leq n$ (as $n \geq N = n_K$) shows $K \in S$; and it is finite, since the Subsequence Index Bound gives $n_k \geq k$, so any $k > n$ satisfies $n_k \geq k > n$ and hence $k \notin S$.

Let $k' := \max S$; then $n_{k'} \leq n$, and by maximality of k', $n_{k'+1} > n$ (else $k' + 1 \in S$, contradicting maximality), so $n \leq n_{k'+1}$.

Since $K \in S$ and $k' = \max S$, we have $k' \geq K$, and hence also $k' + 1 \geq K$.

Applying the sandwiching fact with $p = n_{k'} \leq n \leq n_{k'+1} = r$ (both valid natural-number indices with $p \leq q = n \leq r$) gives

$$|a_n - a| \leq \max\{|a_{n_{k'}} - a|, |a_{n_{k'+1}} - a|\}.$$

Since $k' \geq K$ and $k' + 1 \geq K$, the choice of K gives $|a_{n_{k'}} - a| < \varepsilon$ and $|a_{n_{k'+1}} - a| < \varepsilon$.

Therefore $|a_n - a| < \varepsilon$.

Hence, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$; this is the definition of $a = \lim a_n$.

Therefore $\{a_n\}$ is convergent. □

Fidelity note. The slides write this argument using the shorthand n_n for "the subsequence index n_k evaluated at $k = n$" and conclude with $|a_n - a| \leq \max\{|a_{n_n} - a|, |a_{n_K} - a|\}$.

The write-up above avoids that shorthand for clarity, using the explicit bracketing indices $n_{k'} \leq n \leq n_{k'+1}$ instead; the underlying argument is the one on the slides.

Theorem 3.6: Monotone Convergence Theorem

If a sequence $\{a_n\}$ is eventually monotone and bounded, then $\{a_n\}$ is convergent.

Idea (Method 1: via Bolzano–Weierstrass). *(Scratch work – not part of the written answer.)*

Chain two already-proved results: boundedness gives *some* convergent subsequence, by Bolzano–Weierstrass.

Eventual monotonicity plus that convergent subsequence then gives convergence of the whole sequence, by the Monotone Convergence Lemma.

Proof (Method 1). Since $\{a_n\}$ is bounded, the Bolzano–Weierstrass Theorem implies that $\{a_n\}$ has a convergent subsequence.

Since $\{a_n\}$ is, in addition, eventually monotone and now known to have a convergent subsequence, the Monotone Convergence Lemma implies that $\{a_n\}$ is convergent. □

Idea (Method 2: directly from the Axiom of Completeness). *(Scratch work – not part of the written answer.)* (Tutorial 3, Additional Question 4; QRS proof – the official hint only sketches it.)

Reduce to the increasing case by passing to a tail.

Let $a := \sup\{a_n : n \in \mathbb{N}\}$, which exists by the Axiom of Completeness.

Show $a_n \to a$ using the ε-Criterion for Suprema: for any $\varepsilon > 0$, some a_N exceeds $a - \varepsilon$, and increasingness pushes every later term above a_N, while a itself is an upper bound.

Proof (Method 2). By passing to a tail and invoking the Tail-Shift Remark exactly as in the proof of the Monotone Convergence Lemma, we may assume without loss of generality that $\{a_n\}$ is monotone, not merely eventually monotone.

We treat the increasing case; the decreasing case follows by applying the increasing case to $\{-a_n\}$ (which is increasing and bounded whenever $\{a_n\}$ is decreasing and bounded) and using the Algebraic Limit Theorem to negate the resulting limit.

So suppose $\{a_n\}$ is increasing and bounded. Let $A := \{a_n : n \in \mathbb{N}\}$.

Then A is nonempty (it contains a_1) and bounded above (since $\{a_n\}$ is bounded), so the Axiom of Completeness implies that A has a supremum; let $a := \sup A$.

We show $a = \lim a_n$. Let $\varepsilon > 0$ be given.

Since $a - \varepsilon < a = \sup A$, the ε-Criterion for Suprema and Infima implies that $a - \varepsilon$ is not an upper bound for A; hence there exists $N \in \mathbb{N}$ with $a_N > a - \varepsilon$.

Let $n \in \mathbb{N}$ with $n \geq N$. Since $\{a_n\}$ is increasing, $a_n \geq a_N > a - \varepsilon$.

On the other hand, since $a = \sup A$ is an upper bound for A and $a_n \in A$, $a_n \leq a < a + \varepsilon$.

Combining the two displayed bounds,

$$a - \varepsilon < a_n < a + \varepsilon, \qquad \text{i.e.} \qquad |a_n - a| < \varepsilon.$$

Hence, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$; this is the definition of $a = \lim a_n$.

Therefore $\{a_n\}$ is convergent, with limit $\sup\{a_n : n \in \mathbb{N}\}$. □

Common slip. In Method 2, the bound $a_n \leq a$ uses only that a is *an* upper bound (true for every n, no index needed), while the bound $a_n > a - \varepsilon$ uses that $a - \varepsilon$ is *not* an upper bound, which only guarantees *some* index N works, and increasingness is needed to extend this to every $n \geq N$. Conflating "an upper bound" with "the least upper bound" at the wrong step is the most common way this proof goes wrong on an exam.

Key Formula: Partial Converse

Combining the Monotone Convergence Theorem with the Boundedness of Convergent Sequences (Lecture 4): an eventually monotone sequence is bounded if and only if it is convergent. (Without monotonicity, only one direction survives: every convergent sequence is bounded, but a bounded sequence need not converge.)

3.5 Worked Examples (Tutorial 3)

Worked Example (Tutorial 3, Question 3): The Shuffled Sequence

Problem. Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences with the same limit, and let $\{c_n\}$ be the shuffled sequence $a_1, b_1, a_2, b_2, a_3, b_3, \ldots$ (odd-indexed terms of $\{c_n\}$ coincide with $\{a_n\}$, even-indexed terms coincide with $\{b_n\}$). Use the Divergence Criterion to show $\{c_n\}$ is convergent, without using the definition of limit of a sequence directly.

Method 1: Divergence Criterion by contradiction

Proof. Let $L := \lim a_n = \lim b_n$. By the definition of the shuffle, $c_{2m-1} = a_m$ and $c_{2m} = b_m$ for every $m \in \mathbb{N}$.
Step 1: $\{c_n\}$ is bounded. Since $\{a_n\}$ and $\{b_n\}$ converge, they are bounded (Boundedness of Convergent Sequences). Hence there are $M_a, M_b > 0$ such that $|a_m| \le M_a$ and $|b_m| \le M_b$ for every $m \in \mathbb{N}$.
Let $M := \max\{M_a, M_b\}$, and let $n \in \mathbb{N}$. If n is odd, then $n = 2m-1$ with $m = \frac{n+1}{2} \in \mathbb{N}$, and therefore $|c_n| = |a_m| \le M_a \le M$. If n is even, then $n = 2m$ with $m = \frac{n}{2} \in \mathbb{N}$, and therefore $|c_n| = |b_m| \le M_b \le M$. Hence $|c_n| \le M$ for every $n \in \mathbb{N}$; that is, $\{c_n\}$ is bounded.
Step 2: every convergent subsequence of $\{c_n\}$ has limit L. Let $\{c_{p_k}\}$ be a subsequence of $\{c_n\}$, and suppose $L' = \lim_k c_{p_k}$. Let $O := \{k \in \mathbb{N} : p_k \text{ is odd}\}$ and $E := \{k \in \mathbb{N} : p_k \text{ is even}\}$, so that $O \cup E = \mathbb{N}$. If both O and E were finite, then $\mathbb{N}$ would be a union of two finite sets and hence finite, which is false. Therefore O is infinite or E is infinite.
Case O infinite. List the elements of O in increasing order as $k_1 < k_2 < k_3 < \cdots$. For each $j \in \mathbb{N}$ the index p_{k_j} is odd, so

$$p_{k_j} = 2i_j - 1, \qquad \text{where } i_j := \frac{p_{k_j}+1}{2} \in \mathbb{N}.$$

Let $j \in \mathbb{N}$. Then $k_j < k_{j+1}$, so $p_{k_j} < p_{k_{j+1}}$ because the indices p_k are strictly increasing; therefore $i_j < i_{j+1}$. Hence $\left\{c_{p_{k_j}}\right\}_j$ is a subsequence of $\{c_{p_k}\}_k$, and $\left\{a_{i_j}\right\}_j$ is a subsequence of $\{a_i\}$. Moreover $c_{p_{k_j}} = c_{2i_j-1} = a_{i_j}$ for every $j \in \mathbb{N}$, so these two subsequences are the same sequence. Applying Subsequential Limits to $\{c_{p_k}\}$ and then to $\{a_i\}$,

$$\begin{aligned} \lim_j c_{p_{k_j}} &= L' && \text{(Subsequential Limits)} \\ \lim_j c_{p_{k_j}} = \lim_j a_{i_j} &= L && \text{(Subsequential Limits)} \\ \therefore\ L' &= L && \text{(Uniqueness of Limits)} \end{aligned}$$

Case E infinite. List the elements of E as $k_1 < k_2 < k_3 < \cdots$ and write $p_{k_j} = 2i_j$ with $i_j := \frac{p_{k_j}}{2} \in \mathbb{N}$. As in the odd case, $k_j < k_{j+1}$ gives $p_{k_j} < p_{k_{j+1}}$ and hence $i_j < i_{j+1}$, and $c_{p_{k_j}} = c_{2i_j} = b_{i_j}$ for every $j \in \mathbb{N}$. Therefore $L' = \lim_j c_{p_{k_j}}$ and $L = \lim_j b_{i_j}$ (Subsequential Limits), so $L' = L$ (Uniqueness of Limits).
Step 3: $\{c_n\}$ converges. Suppose, for contradiction, that $\{c_n\}$ diverges. By Step 1, $\{c_n\}$ is bounded; therefore the Divergence Criterion gives two subsequences $\{c_{p_k}\}$ and $\{c_{q_k}\}$ of $\{c_n\}$ with limits $L_1 \ne L_2$. By Step 2, $L_1 = L$ and $L_2 = L$; hence $L_1 = L_2$, which contradicts $L_1 \ne L_2$. Therefore $\{c_n\}$ converges.
Step 4: the limit is L. Since $\{c_n\}$ converges, it is a convergent subsequence of itself (with indices $p_k = k$). Therefore Step 2 gives $\lim c_n = L$. □

$$\boxed{\{c_n\} \text{ converges (to } L).}$$

Method 2: Direct ε-N argument from the definition of limit
This method should be used only if the course allows this interpretation: it uses the definition of limit, which the question excludes.

Proof. Let $L := \lim a_n = \lim b_n$, so that $c_{2m-1} = a_m$ and $c_{2m} = b_m$ for every $m \in \mathbb{N}$. Let $\varepsilon > 0$ be given.
Since $L = \lim a_m$, there is $K_a \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $m \ge K_a$ implies $|a_m - L| < \varepsilon$. Since $L = \lim b_m$, there is $K_b \in \mathbb{N}$ such that for all $m \in \mathbb{N}$, $m \ge K_b$ implies $|b_m - L| < \varepsilon$. Let $N := \max\{2K_a - 1, 2K_b\}$, and let $n \in \mathbb{N}$ with $n \ge N$.
If n is odd, write $n = 2m-1$ with $m \in \mathbb{N}$. Then $2m - 1 = n \ge N \ge 2K_a - 1$, so $m \ge K_a$, and therefore

$$|c_n - L| = |a_m - L| < \varepsilon.$$

If n is even, write $n = 2m$ with $m \in \mathbb{N}$. Then $2m = n \ge N \ge 2K_b$, so $m \ge K_b$, and therefore

$$|c_n - L| = |b_m - L| < \varepsilon.$$

Hence, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|c_n - L| < \varepsilon$; that is, $L = \lim c_n$ (def. of limit). Therefore $\{c_n\}$ converges. □

$$\boxed{\{c_n\} \text{ converges (to } L).}$$

Worked Example (Tutorial 3, Question 4): Even, Odd and Triple Subsequences

Problem. Suppose $\{a_n\}$ is a sequence for which $\{a_{2k}\}$, $\{a_{2k+1}\}$ and $\{a_{3k}\}$ are all convergent. Show $\{a_n\}$ is convergent.

Method 1: Common limit via $\{a_{6k}\}$ and $\{a_{6k-3}\}$, then the shuffled sequence

Proof. Let $L := \lim_k a_{3k}$, $P := \lim_k a_{2k}$ and $Q := \lim_k a_{2k+1}$, where k runs through $\mathbb{N}$.
Step 1: $P = L$. For every $k \in \mathbb{N}$,

$$a_{6k} = a_{2(3k)} = a_{3(2k)}.$$

For every $k \in \mathbb{N}$, $3(k+1) = 3k+3 > 3k$; therefore the index $j = 3k$ is strictly increasing in k, and $\{a_{6k}\}_k$ is a subsequence of $\{a_{2j}\}_j$. For every $k \in \mathbb{N}$, $2(k+1) = 2k+2 > 2k$; therefore the index $j = 2k$ is strictly increasing in k, and $\{a_{6k}\}_k$ is a subsequence of $\{a_{3j}\}_j$. Hence

$$\begin{aligned} \lim_k a_{6k} &= P && \text{(Subsequential Limits)} \\ \lim_k a_{6k} &= L && \text{(Subsequential Limits)} \\ \therefore\ P &= L && \text{(Uniqueness of Limits)} \end{aligned}$$

Step 2: $Q = L$. For every $k \in \mathbb{N}$,

$$a_{6k-3} = a_{2(3k-2)+1} = a_{3(2k-1)}.$$

For every $k \in \mathbb{N}$, the index $j = 3k-2$ satisfies $j \geq 1$, so $j \in \mathbb{N}$, and it increases by 3 when k increases by 1; therefore $\{a_{6k-3}\}_k$ is a subsequence of $\{a_{2j+1}\}_j$. For every $k \in \mathbb{N}$, the index $j = 2k-1$ satisfies $j \geq 1$, so $j \in \mathbb{N}$, and it increases by 2 when k increases by 1; therefore $\{a_{6k-3}\}_k$ is a subsequence of $\{a_{3j}\}_j$. Hence

$$\begin{aligned} \lim_k a_{6k-3} &= Q && \text{(Subsequential Limits)} \\ \lim_k a_{6k-3} &= L && \text{(Subsequential Limits)} \\ \therefore\ Q &= L && \text{(Uniqueness of Limits)} \end{aligned}$$

Step 3: the odd-indexed terms. For $k \in \mathbb{N}$ let $u_k := a_{2k-1}$ and $v_k := a_{2k}$. Then $\{v_k\} = \{a_{2k}\}$, so $\lim v_k = P = L$ by Step 1. For every $k \in \mathbb{N}$, $u_{k+1} = a_{2(k+1)-1} = a_{2k+1}$; therefore $\lim_k u_{k+1} = Q = L$ by Step 2. Hence, by the Tail-Shift Remark with $c = 1$ applied to $\{u_k\}$, the sequence $\{u_k\}$ converges and $\lim u_k = L$.
Step 4: shuffle. Every $n \in \mathbb{N}$ is of exactly one of the forms $n = 2k-1$ or $n = 2k$ with $k \in \mathbb{N}$, and then $a_n = u_k$ or $a_n = v_k$ respectively. Therefore $\{a_n\}$ is the shuffled sequence $u_1, v_1, u_2, v_2, u_3, v_3, \ldots$ of the two sequences $\{u_k\}$ and $\{v_k\}$, which have the same limit L. Hence, by the previous Worked Example (Tutorial 3, Question 3) applied to $\{u_k\}$ and $\{v_k\}$, the sequence $\{a_n\}$ converges to L. □

$$\boxed{\{a_n\} \text{ converges (to } L = \lim a_{3k}).}$$

Method 2: Direct ε-N argument on even and odd indices

Proof. Let $L := \lim_k a_{3k}$. By Steps 1 and 2 of Method 1, which use only Subsequential Limits and Uniqueness of Limits, $\lim_k a_{2k} = L$ and $\lim_k a_{2k+1} = L$. Let $\varepsilon > 0$ be given.
Since $L = \lim_k a_{2k}$, there is $K_1 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq K_1$ implies $|a_{2k} - L| < \varepsilon$. Since $L = \lim_k a_{2k+1}$, there is $K_2 \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq K_2$ implies $|a_{2k+1} - L| < \varepsilon$. Let $N := \max\{2K_1, 2K_2 + 1\}$, and let $n \in \mathbb{N}$ with $n \geq N$.
If n is even, then $n = 2k$ with $k \in \mathbb{N}$, and $2k = n \geq N \geq 2K_1$; therefore $k \geq K_1$, and

$$|a_n - L| = |a_{2k} - L| < \varepsilon.$$

If n is odd, then $n \geq N \geq 2K_2 + 1 \geq 3$, so $n = 2k + 1$ with $k = \frac{n-1}{2} \in \mathbb{N}$, and $2k + 1 = n \geq 2K_2 + 1$; therefore $k \geq K_2$, and

$$|a_n - L| = |a_{2k+1} - L| < \varepsilon.$$

Hence, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$; that is, $L = \lim a_n$ (def. of limit). Therefore $\{a_n\}$ converges, to $L = \lim a_{3k}$. □

$$\boxed{\{a_n\} \text{ converges (to } L = \lim a_{3k}).}$$

Worked Example (Tutorial 3, Question 5): Extracting a Subsequence Converging to Zero

Problem. Suppose $\{a_n\}$ is a sequence such that for every $\varepsilon > 0$, infinitely many terms satisfy $|a_n| < \varepsilon$. Show $\{a_n\}$ has a subsequence converging to 0.
Method 1: Recursive construction by Well-Ordering with $\varepsilon_k = 1/k$

Proof. For $k \in \mathbb{N}$ let

$$S_k := \left\{n \in \mathbb{N} : |a_n| < \frac{1}{k}\right\}.$$

Let $k \in \mathbb{N}$. Since $\frac{1}{k} > 0$, the hypothesis with $\varepsilon = \frac{1}{k}$ shows that S_k is infinite.
Construction. Since S_1 is infinite, it is nonempty; therefore it has a least element (Well-Ordering), and we let $n_1 := \min S_1$. Suppose $k \in \mathbb{N}$ and $n_1 < n_2 < \cdots < n_k$ have been chosen with $n_j \in S_j$ for $j = 1, \ldots, k$. If $S_{k+1} \cap \{n \in \mathbb{N} : n > n_k\}$ were empty, then $S_{k+1} \subseteq \{1, 2, \ldots, n_k\}$, so S_{k+1} would be finite, which is false. Therefore $S_{k+1} \cap \{n \in \mathbb{N} : n > n_k\}$ is nonempty, and we let

$$n_{k+1} := \min\big(S_{k+1} \cap \{n \in \mathbb{N} : n > n_k\}\big) \qquad \text{(Well-Ordering)}.$$

Then $n_{k+1} > n_k$ and $n_{k+1} \in S_{k+1}$. By induction on k, this defines natural numbers $n_1 < n_2 < n_3 < \cdots$ with $n_k \in S_k$ for every $k \in \mathbb{N}$. Hence $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, and

$$|a_{n_k}| < \frac{1}{k} \qquad \text{for every } k \in \mathbb{N}.$$

Convergence. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $K \in \mathbb{N}$ with $K > \frac{1}{\varepsilon}$, i.e. $\frac{1}{K} < \varepsilon$. Let $k \in \mathbb{N}$ with $k \geq K$. Then $\frac{1}{k} \leq \frac{1}{K}$, and therefore

$$|a_{n_k} - 0| = |a_{n_k}| < \frac{1}{k} \leq \frac{1}{K} < \varepsilon.$$

Hence, for every $\varepsilon > 0$ there is $K \in \mathbb{N}$ such that for all $k \geq K$, $|a_{n_k} - 0| < \varepsilon$; that is, $0 = \lim a_{n_k}$ (def. of limit). □

$$\boxed{\{a_{n_k}\} \text{ converges to } 0.}$$

Method 2: Unbounded index sets and the Squeeze Theorem

Proof. For $k \in \mathbb{N}$ let $S_k := \{n \in \mathbb{N} : |a_n| < 1/k\}$. By the hypothesis with $\varepsilon = 1/k > 0$, each S_k is infinite.
An infinite subset of $\mathbb{N}$ is not bounded above. Let $S \subseteq \mathbb{N}$ be infinite, and suppose, for contradiction, that $B \in \mathbb{R}$ is an upper bound of S. By the Archimedean Property there is $m \in \mathbb{N}$ with $m > B$. Then every $n \in S$ satisfies $n \le B < m$, so $S \subseteq \{1, 2, \dots, m\}$ is finite, a contradiction. Therefore S has no upper bound.
Construction. Choose any $n_1 \in S_1$. Having chosen $n_k \in S_k$, the number n_k is not an upper bound of the infinite set S_{k+1}; therefore there is $n_{k+1} \in S_{k+1}$ with $n_{k+1} > n_k$. By induction, $n_1 < n_2 < n_3 < \cdots$ and $n_k \in S_k$ for every $k \in \mathbb{N}$. Hence $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, and

$$0 \le |a_{n_k}| < \frac{1}{k} \qquad \text{for every } k \in \mathbb{N}.$$

Squeeze. The sequence $\{1/k\}$ converges to 0: given $\varepsilon > 0$, the Archimedean Property gives $K \in \mathbb{N}$ with $K > 1/\varepsilon$, and then for all $k \ge K$, $|1/k - 0| = 1/k \le 1/K < \varepsilon$. The constant sequence $0, 0, 0, \dots$ also converges to 0. Since $0 \le |a_{n_k}| \le 1/k$ for every $k \in \mathbb{N}$, the Squeeze Theorem gives $\lim_k |a_{n_k}| = 0$. By the Algebraic Limit Theorem with $c = -1$, $\lim_k (-|a_{n_k}|) = -1 \cdot 0 = 0$. For every $k \in \mathbb{N}$, $-|a_{n_k}| \le a_{n_k} \le |a_{n_k}|$. Therefore the Squeeze Theorem gives $0 = \lim a_{n_k}$. □

$\boxed{\{a_{n_k}\} \text{ converges to } 0.}$

Worked Example (Sample Midterm, Question 1(b) = Tutorial 3, Question 2(b))

Problem. True or false: there is a sequence with a subsequence that is bounded but contains no subsequence that converges.
Method 1: Bolzano–Weierstrass on the bounded subsequence

Proof. The statement is false; we show that every bounded subsequence of every sequence contains a convergent subsequence. Let $\{a_n\}$ be any sequence, and let $\{a_{n_k}\}_k$ be a subsequence of $\{a_n\}$ that is bounded, where $n_1 < n_2 < n_3 < \cdots$.
Since $\{a_{n_k}\}_k$ is bounded, the Bolzano–Weierstrass Theorem implies that $\{a_{n_k}\}_k$ has a convergent subsequence. That is, there are natural numbers $k_1 < k_2 < k_3 < \cdots$ such that $\left\{a_{n_{k_j}}\right\}_j$ converges.
Let $j \in \mathbb{N}$. Then $k_j < k_{j+1}$, and since the indices n_k are strictly increasing in k, this gives $n_{k_j} < n_{k_{j+1}}$. Therefore $n_{k_1} < n_{k_2} < n_{k_3} < \cdots$, so $\left\{a_{n_{k_j}}\right\}_j$ is a convergent subsequence of $\{a_{n_k}\}_k$ and also a subsequence of $\{a_n\}$.
Hence every bounded subsequence of a sequence contains a convergent subsequence, and therefore no sequence of the described kind exists. □

$\boxed{\text{False.}}$

Method 2: Monotone subsequence via peak terms and the MCT

Proof. The statement is false. Let $\{a_n\}$ be any sequence and let $\{a_{n_k}\}_k$ be a bounded subsequence of it. Write $x_k := a_{n_k}$ for $k \in \mathbb{N}$, and let $M > 0$ be such that $|x_k| \le M$ for every $k \in \mathbb{N}$. Call $m \in \mathbb{N}$ a *peak* if $x_m \ge x_j$ for every $j \in \mathbb{N}$ with $j > m$, and let $P \subseteq \mathbb{N}$ be the set of peaks.
Case 1: P is infinite. Let $k_1 := \min P$ (Well-Ordering). Having chosen $k_j \in P$, the set $P \cap \{m \in \mathbb{N} : m > k_j\}$ is nonempty, since otherwise $P \subseteq \{1, \dots, k_j\}$ would be finite; let k_{j+1} be its least element (Well-Ordering). Then $k_1 < k_2 < k_3 < \cdots$. For each $j \in \mathbb{N}$, k_j is a peak and $k_{j+1} > k_j$; therefore $x_{k_j} \ge x_{k_{j+1}}$. Hence $\left\{x_{k_j}\right\}_j$ is decreasing.
Case 2: P is finite. Let $k_1 := 1$ if $P = \emptyset$, and $k_1 := 1 + \max P$ otherwise; then k_1 exceeds every element of P. Suppose k_j has been chosen and exceeds every element of P. Then $k_j \notin P$, so k_j is not a peak: there is $m \in \mathbb{N}$ with $m > k_j$ and $x_m > x_{k_j}$. Let k_{j+1} be such an m. Then $k_{j+1} > k_j$,

so k_{j+1} also exceeds every element of P, and $x_{k_{j+1}} > x_{k_j}$. By induction, $k_1 < k_2 < k_3 < \cdots$ and $x_{k_j} < x_{k_{j+1}}$ for every $j \in \mathbb{N}$. Hence $\{x_{k_j}\}_j$ is increasing.
Conclusion. In both cases $\{x_{k_j}\}_j$ is a monotone subsequence of $\{x_k\}$, and $|x_{k_j}| \leq M$ for every $j \in \mathbb{N}$, so it is bounded. Therefore $\{x_{k_j}\}_j$ converges (MCT, whose Method 2 proof above uses only the Axiom of Completeness). Since $j \mapsto k_j$ is strictly increasing, $\{x_{k_j}\}_j = \{a_{n_{k_j}}\}_j$ is a convergent subsequence of $\{a_{n_k}\}_k$. Hence every bounded subsequence of a sequence contains a convergent subsequence, and therefore no sequence of the described kind exists. □

False.

Worked Example (Sample Midterm, Question 1(c) = Tutorial 3, Question 1(a))

Problem. True or false: if every proper subsequence of a sequence $\{a_n\}$ converges, then $\{a_n\}$ converges as well.
Method 1: Tail shift and the definition of limit

Proof. The statement is true. Let $\{a_n\}$ be a sequence every proper subsequence of which converges. Define $b_k := a_{k+1}$ for $k \in \mathbb{N}$; that is, $b_k = a_{n_k}$ with $n_k := k+1$. For every $k \in \mathbb{N}$, $n_{k+1} = k+2 > k+1 = n_k$; therefore the indices are strictly increasing, and $\{b_k\}$ is a subsequence of $\{a_n\}$. Since $n_1 = 2 \neq 1$, the subsequence $\{b_k\}$ is proper. Hence, by hypothesis, $\{b_k\}$ converges; let $L := \lim b_k$.
Let $\varepsilon > 0$ be given. Since $L = \lim b_k$, there is $K \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq K$ implies $|b_k - L| < \varepsilon$. Let $N := K+1$, and let $n \in \mathbb{N}$ with $n \geq N$. Then $k := n-1$ satisfies $k \geq N-1 = K \geq 1$, so $k \in \mathbb{N}$ and $k \geq K$. Therefore

$$|a_n - L| = |a_{k+1} - L| = |b_k - L| < \varepsilon.$$

Hence, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \varepsilon$; that is, $L = \lim a_n$ (def. of limit; this is the Tail-Shift Remark with $c = 1$). Therefore $\{a_n\}$ converges. □

True.

Method 2: Cauchy Criterion

Proof. The statement is true. Let $\{a_n\}$ be a sequence every proper subsequence of which converges, and define $b_k := a_{k+1}$ for $k \in \mathbb{N}$. The indices $k+1$ are strictly increasing in k and the first one is $2 \neq 1$; therefore $\{b_k\}$ is a proper subsequence of $\{a_n\}$, and by hypothesis $\{b_k\}$ converges. Hence $\{b_k\}$ is Cauchy (Cauchy Criterion, "only if").
Let $\varepsilon > 0$ be given. Since $\{b_k\}$ is Cauchy, there is $K \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq K$ implies $|b_k - b_K| < \varepsilon$ (def. of Cauchy sequence). Let $N := K+1$, so that $a_N = a_{K+1} = b_K$. Let $n \in \mathbb{N}$ with $n \geq N$. Then $k := n-1$ satisfies $k \geq K \geq 1$, and therefore

$$|a_n - a_N| = |a_{k+1} - a_{K+1}| = |b_k - b_K| < \varepsilon.$$

Hence, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a_N| < \varepsilon$; that is, $\{a_n\}$ is Cauchy (def. of Cauchy sequence). Therefore $\{a_n\}$ converges (Cauchy Criterion, "if"). □

True.

Method 3: Divergence Criterion by contradiction

Proof. The statement is true. Let $\{a_n\}$ be a sequence every proper subsequence of which converges, and define $b_k := a_{k+1}$ for $k \in \mathbb{N}$. As in Method 1, $\{b_k\}$ is a proper subsequence of $\{a_n\}$; hence it converges by hypothesis. Let $L := \lim b_k$.
Step 1: $\{a_n\}$ is bounded. Since $\{b_k\}$ converges, it is bounded (Boundedness of Convergent Sequences): there is $M_b > 0$ such that $|b_k| \leq M_b$ for every $k \in \mathbb{N}$. Let $M := \max\{|a_1|, M_b\}$. For $n = 1$, $|a_1| \leq M$. For $n \geq 2$, $n - 1 \in \mathbb{N}$, so $|a_n| = |b_{n-1}| \leq M_b \leq M$. Therefore $|a_n| \leq M$ for every $n \in \mathbb{N}$.
Step 2: every convergent subsequence of $\{a_n\}$ has limit L. Let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$, and suppose $L' = \lim_k a_{n_k}$. For $k \in \mathbb{N}$ let $p_k := n_{k+1} - 1$. By the Subsequence Index Bound, $n_{k+1} \geq k+1$, so $p_k \geq k \geq 1$ and $p_k \in \mathbb{N}$. Also $p_{k+1} = n_{k+2} - 1 > n_{k+1} - 1 = p_k$; therefore $\{b_{p_k}\}_k$ is a subsequence of $\{b_k\}$, and

$$b_{p_k} = a_{p_k+1} = a_{n_{k+1}} \qquad \text{for every } k \in \mathbb{N}.$$

Hence $\lim_k a_{n_{k+1}} = L$ (Subsequential Limits, applied to $\{b_k\}$). On the other hand, $k \mapsto k+1$ is strictly increasing, so $\{a_{n_{k+1}}\}_k$ is a subsequence of $\{a_{n_k}\}_k$; hence $\lim_k a_{n_{k+1}} = L'$ (Subsequential Limits). Therefore $L' = L$ (Uniqueness of Limits).
Step 3: conclusion. Suppose, for contradiction, that $\{a_n\}$ diverges. By Step 1 it is bounded; therefore the Divergence Criterion gives two subsequences of $\{a_n\}$ with limits $L_1 \neq L_2$. By Step 2, $L_1 = L$ and $L_2 = L$; hence $L_1 = L_2$, which contradicts $L_1 \neq L_2$. Therefore $\{a_n\}$ converges. □

True.

3.6 Chapter Summary

Key Formula: The Chapter 3 Logical Chain

Axiom of Completeness $\Longrightarrow$ Bolzano–Weierstrass Theorem

$\Downarrow$

Monotone Convergence Theorem (via the Monotone Convergence Lemma) and Divergence Criterion, "only if" (Bolzano–Weierstrass applied twice)

- **Bolzano–Weierstrass:** bounded $\{a_n\}$ $\Longrightarrow$ some subsequence of $\{a_n\}$ converges.
- **Monotone Convergence Theorem:** eventually monotone *and* bounded $\{a_n\}$ $\Longrightarrow$ $\{a_n\}$ itself converges.
- **Divergence Criterion:** bounded $\{a_n\}$ is divergent $\Longleftrightarrow$ it has two subsequences converging to different limits. ("If": contrapositive of Subsequential Limits, no boundedness needed. "Only if": Bolzano–Weierstrass, applied twice.)

Exam Focus: Chapter 3 takeaways

- Subsequence indices must be *strictly* increasing; check this explicitly before calling something a subsequence.
- Three results in this chapter each carry a boundedness hypothesis that is easy to forget under time pressure: the Divergence Criterion, Bolzano–Weierstrass, and the Monotone Convergence Theorem. State it every time.
- To compare two given subsequences $\{a_{pk}\}$ and $\{a_{qk}\}$ (e.g. $p = 2, q = 3$), look for a common further subsequence indexed by multiples of $\operatorname{lcm}(p, q)$ (or, for the "+1" families, the residue class shared by both); Subsequential Limits plus Uniqueness of Limits then forces the two given subsequences to share a limit.
- The "proper subsequence" definition ($n_k \neq k$ for *some* k) is weak: almost every subsequence, including any single tail, is proper. Sample Midterm Question 1(c) hinges entirely on this.
- Two independent full proofs of the Monotone Convergence Theorem are examinable: via Bolzano–Weierstrass + Monotone Convergence Lemma (Method 1), and directly from the Axiom of Completeness via $\sup\{a_n : n \in \mathbb{N}\}$ (Method 2, Tutorial 3 Additional Question 4). Know both.

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4 Lectures 10–12: Series

This chapter answers three questions: (1) what does it mean for an infinite sum $a_1+a_2+a_3+\cdots$ to have a value; (2) which tools decide, quickly, whether a given series converges or diverges; and (3) how does convergence with signs (alternating, or general signs) differ from convergence of nonnegative terms. The entire chapter reduces a series to a single object already understood from Lectures 4–9: the sequence of its partial sums. Every test in this chapter – the Alternating Series Test, the Comparison Test, the Absolute Convergence Test – is, at bottom, a disguised form of the Cauchy Criterion for sequences.

4.1 Series and Partial Sums

Definition 4.1: Series

An (infinite) *series* of real numbers is a formal expression of the form

$$a_1 + a_2 + a_3 + \cdots,$$

where $\{a_n\}$ is a sequence of real numbers. The numbers in $\{a_n\}$ are called the *terms* of the series. We use the notation $\sum_{n=1}^{\infty} a_n$ (or simply $\sum a_n$) to denote the series. For each $n \in \mathbb{N}$, the number $a_1 + a_2 + \cdots + a_n$ is called the *n-th partial sum* of the series $\sum a_n$, and is written s_n; that is,

$$s_n := a_1 + a_2 + \cdots + a_n.$$

As with sequences, it is sometimes useful to start labelling a series from an integer other than 1. If a series is to start from a_0, it is denoted $\sum_{n=0}^{\infty} a_n$, and its partial sums are indexed accordingly. Both conventions appear in this chapter; the theory is identical in either case.

Definition 4.2: Convergence of Series

A series $\sum a_n$ is said to be *convergent* if its sequence of partial sums $\{s_n\}$ is convergent. In this case the limit s of $\{s_n\}$ is called the *sum* of the series, and we write

$$s = \sum_{n=1}^{\infty} a_n \qquad \left(\text{or simply } s = \sum a_n\right)$$

to mean that s is the sum of the series, and say that $\sum a_n$ *converges* to s. Otherwise the series is said to be *divergent*.

Remark (Terms and Partial Sums). It is important to keep a clear distinction between the *terms* a_n of a series and its *partial sums* $a_1+a_2+\cdots+a_n$. A series is a statement about the sequence of partial sums, not directly about the sequence of terms; the two sequences can behave very differently, as the Harmonic Series below will show.
Four series recur throughout this chapter and the exercises; each is stated here as the slides state it, and each is proved below once the tools needed to prove it have been assembled.

- Geometric series: $\displaystyle\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \cdots.$
- Harmonic series: $\displaystyle\sum \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$
- Alternating harmonic series: $\displaystyle\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.$
- p-series: $\displaystyle\sum \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ (we treat $p = 2$).

4.2 Terms, Algebra, and the Cauchy Criterion for Series

The slides state the next three theorems as immediate transports of results already proved for sequences (Lectures 4–9) to the sequence of partial sums, but do not carry out the transport in writing. We supply full proofs below, each flagged as a *QRS proof – not on the slides*, built entirely from results already proved by Lecture 9.

Theorem 4.1: Terms of Convergent Series

If a series $\sum a_n$ is convergent, then the sequence of its terms $\{a_n\}$ converges to 0.

Idea (QRS proof – not on the slides). *(Scratch work – not part of the written answer.)* Adopt the convention $s_0 := 0$, so that $a_n = s_n - s_{n-1}$ holds for every $n \in \mathbb{N}$. If $s_n \to s$, then also $s_{n-1} \to s$ (the same sequence, shifted by one index), so $a_n = s_n - s_{n-1} \to s - s = 0$ by the Algebraic Limit Theorem. Turn this into an ε-N argument with $\varepsilon/2$.

Proof. Let $\varepsilon > 0$ be given. Since $\sum a_n$ converges, by the definition of convergence of a series the sequence of partial sums $\{s_n\}$ converges to some $s \in \mathbb{R}$. By the definition of convergence of a sequence, applied to $\varepsilon/2 > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $|s_n - s| < \varepsilon/2$. Let $n \in \mathbb{N}$ with $n > N$. Then both $n \geq N$ and $n - 1 \geq N$ hold, so $|s_n - s| < \varepsilon/2$ and $|s_{n-1} - s| < \varepsilon/2$. Using the convention $s_0 := 0$, we have $a_n = s_n - s_{n-1}$ for every $n \in \mathbb{N}$; therefore, by the triangle inequality,

$$|a_n - 0| = |s_n - s_{n-1}| = |(s_n - s) - (s_{n-1} - s)| \leq |s_n - s| + |s_{n-1} - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence for all $n \in \mathbb{N}$, $n > N$ implies $|a_n - 0| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this shows $\{a_n\}$ converges to 0. □

Common slip. The converse is false: $a_n \to 0$ is *necessary* for convergence of $\sum a_n$, but not *sufficient*. The Harmonic Series below has terms tending to 0 yet diverges. The correct use of this theorem is as a quick *divergence* test, via its contrapositive: if $\{a_n\} \not\to 0$, then $\sum a_n$ diverges.

Theorem 4.2: Algebraic Limit Theorem for Series

If the series $\sum a_n$ converges to s and the series $\sum b_n$ converges to t, then

(1) for each $c \in \mathbb{R}$, $\sum ca_n$ converges to cs; and

(2) $\sum(a_n + b_n)$ converges to $s + t$.

Idea (QRS proof – not on the slides). *(Scratch work – not part of the written answer.)* The n-th partial sum of $\sum ca_n$ is c times the n-th partial sum of $\sum a_n$ (distributivity applied to a finite sum), and the n-th partial sum of $\sum(a_n + b_n)$ is the sum of the two individual partial sums (regrouping a finite sum). Apply the Algebraic Limit Theorem for sequences to these two rewritten partial-sum sequences.

Proof. Let $\{S_n\}$ and $\{T_n\}$ denote the sequences of partial sums of $\sum a_n$ and $\sum b_n$ respectively; by the definition of convergence of a series, $S_n \to s$ and $T_n \to t$.

(1) For each $n \in \mathbb{N}$, the n-th partial sum of $\sum ca_n$ is $ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n) = cS_n$, by distributivity applied to the finite sum. Since $S_n \to s$, the Algebraic Limit Theorem (for sequences) gives $cS_n \to cs$. Therefore the sequence of partial sums of $\sum ca_n$ converges to cs, so by the definition of convergence of a series, $\sum ca_n$ converges to cs.

(2) For each $n \in \mathbb{N}$, the n-th partial sum of $\sum(a_n + b_n)$ is

$$(a_1 + b_1) + (a_2 + b_2) + \cdots + (a_n + b_n) = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) = S_n + T_n,$$

where the middle equality regroups a finite sum. Since $S_n \to s$ and $T_n \to t$, the Algebraic Limit Theorem (for sequences) gives $S_n + T_n \to s + t$. Therefore $\sum(a_n + b_n)$ converges to $s + t$. □

Theorem 4.3: Cauchy Criterion for Series

A series $\sum a_n$ is convergent if and only if its sequence of partial sums is a Cauchy sequence; equivalently,

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n, m \in \mathbb{N} \quad n > m \geq N \implies |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon.$$

Idea (QRS proof – not on the slides). *(Scratch work – not part of the written answer.)* By definition $\sum a_n$ converges iff $\{s_n\}$ converges, and by the Cauchy Criterion (for sequences) $\{s_n\}$ converges iff $\{s_n\}$ is Cauchy. All that remains is the algebraic identity $s_n - s_m = a_{m+1} + \cdots + a_n$ for $n > m$, which turns the Cauchy condition on $\{s_n\}$ into the stated tail-sum condition.

Proof. By the definition of convergence of a series, $\sum a_n$ is convergent if and only if the sequence of partial sums $\{s_n\}$ is convergent. By the Cauchy Criterion (for sequences, Lecture 6), $\{s_n\}$ is convergent if and only if $\{s_n\}$ is a Cauchy sequence, that is, if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$,

$$n, m \geq N \implies |s_n - s_m| < \varepsilon.$$

It remains to identify this condition with the one stated in the theorem. Let $n, m \in \mathbb{N}$ with $n > m$. Then

$$s_n - s_m = (a_1 + a_2 + \cdots + a_n) - (a_1 + a_2 + \cdots + a_m) = a_{m+1} + a_{m+2} + \cdots + a_n,$$

since the first m terms cancel. Hence $|s_n - s_m| = |a_{m+1} + \cdots + a_n|$ whenever $n > m$, and (by symmetry of $|s_n - s_m|$ in n, m) the Cauchy condition on $\{s_n\}$ is equivalent to the stated condition with $n > m \geq N$. Combining the two equivalences, we conclude that $\sum a_n$ is convergent if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$,

$$n > m \geq N \implies |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon.$$

□

Remark (Finitely Many Terms Are Irrelevant to Convergence). The Cauchy Criterion for Series is a statement purely about *tails* $a_{m+1} + \cdots + a_n$ with $m, n \geq N$. Consequently, changing, deleting, or inserting finitely many terms of a series never changes whether it converges (although it can change the sum, if it converges). We use this freely below, for instance to compare a series with one that starts at a later index.

Exam Focus

Exam Focus: the Cauchy-tail recipe. Every test in this chapter after the Cauchy Criterion for Series (the Alternating Series Test, the Comparison Test, the Absolute Convergence Test) is proved by the same three-step recipe: (1) write the Cauchy condition for a *reference* series (or, for the Alternating Series Test, for $\{b_n\} \to 0$); (2) bound the tail sum of the *target* series by the reference tail, using $|\cdot|$, nonnegativity, or a telescoping estimate; (3) conclude via the Cauchy Criterion for Series. Recognising this pattern is the fastest way to reconstruct any of these proofs on the exam.

Theorem 4.4: Convergence of the Geometric Series

Let $a \in \mathbb{R}$ with $a \neq 0$ and $r \in \mathbb{R}$. If $|r| < 1$, the geometric series $\sum_{n=0}^{\infty} ar^n$ converges, with sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If $|r| \geq 1$, the geometric series diverges.

Idea (QRS proof – not on the slides; the slides list this only as an example). *(Scratch work – not part of the written answer.)* Compute the partial sum in closed form by the standard multiply-and-subtract trick, then let $n \to \infty$. This needs $r^n \to 0$ for $|r| < 1$, which follows from Bernoulli's Inequality $(1+x)^n \geq 1+nx$ (for $x > -1$, $n \in \mathbb{N}$, proved by induction on n, an elementary algebraic fact independent of the course's analysis theorems) and the Archimedean Property. For $|r| \geq 1$, show instead that $a_n = ar^n$ does not tend to 0, and invoke the contrapositive of the Terms of Convergent Series theorem.

Proof. *Case* $|r| < 1$. For each $n \in \mathbb{N}$, direct computation gives

$$(1-r)s_n = (1-r)\sum_{k=0}^{n-1} ar^k = a\big(1-r^n\big),$$

by telescoping the finite geometric sum; since $r \neq 1$ (as $|r| < 1$), we may divide to get $s_n = a(1-r^n)/(1-r)$ for all $n \in \mathbb{N}$. We claim $r^n \to 0$. If $r = 0$, then $r^n = 0$ for every $n \in \mathbb{N}$, so $r^n \to 0$ trivially. Otherwise $0 < |r| < 1$; let $t := 1/|r| - 1$, so that $t > 0$. By Bernoulli's Inequality, $(1+t)^n \geq 1+nt$ for all $n \in \mathbb{N}$, so

$$0 \leq |r|^n = \frac{1}{(1+t)^n} \leq \frac{1}{1+nt}.$$

Given $\varepsilon > 0$, by the Archimedean Property there exists $N \in \mathbb{N}$ with $N > (1/\varepsilon - 1)/t$, so that $1/(1+Nt) < \varepsilon$; since $1/(1+nt)$ is decreasing in n, for all $n \geq N$ we have $0 \leq |r|^n \leq 1/(1+nt) < \varepsilon$. Hence $|r|^n \to 0$, so $r^n \to 0$ (as $|r^n - 0| = |r|^n$). Now, since $s_n = a(1-r^n)/(1-r)$ for all $n \in \mathbb{N}$ and $r^n \to 0$, the Algebraic Limit Theorem gives

$$s_n = \frac{a}{1-r} - \frac{a}{1-r}r^n \longrightarrow \frac{a}{1-r} - \frac{a}{1-r}\cdot 0 = \frac{a}{1-r}.$$

Therefore $\{s_n\}$ converges to $a/(1-r)$, so by the definition of convergence of a series, $\sum_{n=0}^{\infty} ar^n$ converges to $a/(1-r)$.

Case $|r| \geq 1$. We show $a_n = ar^n$ does not converge to 0; since $\sum a_n$ has terms $a_n = ar^n$, the contrapositive of the Terms of Convergent Series theorem then gives divergence. If $|r| = 1$, then $|a_n| = |a| \cdot |r|^n = |a| > 0$ for all n, which is bounded away from 0, so $a_n \not\to 0$. If $|r| > 1$, write $|r| = 1+u$ with $u > 0$; by Bernoulli's Inequality, $|r|^n \geq 1+nu$ for all $n \in \mathbb{N}$, and by the Archimedean Property $1+nu \to \infty$ in the sense that for every $M > 0$ there is $N \in \mathbb{N}$ with $1+Nu > M$; hence $|a_n| = |a|\,(1+u)^n \geq |a|\,(1+nu)$ is unbounded, and in particular $a_n \not\to 0$ (a convergent sequence is bounded). Either way $a_n \not\to 0$, so by the contrapositive of the Terms of Convergent Series theorem, $\sum ar^n$ diverges. □

Theorem 4.5: Divergence of the Harmonic Series

The harmonic series $\sum \dfrac{1}{n}$ diverges.

Idea (QRS proof – not on the slides; the slides list this only as an example). *(Scratch work – not part of the written answer.)* Show the partial sums fail the Cauchy Criterion for Series: grouping N terms from $N+1$ to $2N$, each at least $1/(2N)$, gives a block sum of at least $1/2$, however large N is. So $\varepsilon = 1/2$ has no working N.

Proof. By the Cauchy Criterion for Series, it suffices to exhibit some $\varepsilon > 0$ for which no $N \in \mathbb{N}$ satisfies the tail condition. Take $\varepsilon = 1/2$. Let $N \in \mathbb{N}$ be arbitrary, and set $m := N$, $n := 2N$; then $n > m \geq N$. For each $k \in \{N+1, \ldots, 2N\}$ we have $k \leq 2N$, so $1/k \geq 1/(2N)$. There are exactly N such integers k, so

$$|a_{N+1} + a_{N+2} + \cdots + a_{2N}| = \sum_{k=N+1}^{2N} \frac{1}{k} \geq N \cdot \frac{1}{2N} = \frac{1}{2} = \varepsilon.$$

Thus for every $N \in \mathbb{N}$ there exist $n > m \geq N$ (namely $m = N$, $n = 2N$) with $|a_{m+1} + \cdots + a_n| \geq \varepsilon$, so the Cauchy condition of the Cauchy Criterion for Series fails for $\varepsilon = 1/2$. Therefore the sequence of

partial sums of $\sum 1/n$ is not Cauchy, so by the Cauchy Criterion for Series $\sum 1/n$ is not convergent, that is, $\sum 1/n$ diverges. □

Common Mistake

"The terms tend to 0, so the series converges." The Harmonic Series is the standard counterexample: $a_n = 1/n \to 0$ (Lecture 4–6 limit facts), yet $\sum 1/n$ diverges by the theorem just proved. The Terms of Convergent Series theorem only ever certifies *divergence* (via its contrapositive); it never certifies convergence.

4.3 The Alternating Series Test

Theorem 4.6: Alternating Series Test

If a sequence $\{b_n\}$ is decreasing and converges to 0, then the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1}b_n$$

is convergent.

Idea (Method 1: Cauchy Criterion, the slides' route). *(Scratch work – not part of the written answer.)* Group the tail sum $b_{N+1} - b_{N+2} + \cdots$ into consecutive pairs; since $\{b_n\}$ is decreasing, each pair is nonnegative, which both lower-bounds the tail sum by 0 and (grouping from the second term instead) upper-bounds it by b_{N+1}. Squeeze the tail sum into $[0, b_{N+1}]$, then use $b_n \to 0$ to make b_{N+1} as small as desired.

Proof (Method 1: via the Cauchy Criterion for Series). Let $N, n \in \mathbb{N}$ with $n > N$. Consider the block sum $b_{N+1} - b_{N+2} + b_{N+3} - \cdots + (-1)^{n-N-1}b_n$. If $n - N$ is even, we may group it as

$$(b_{N+1} - b_{N+2}) + (b_{N+3} - b_{N+4}) + \cdots + (b_{n-1} - b_n);$$

if $n - N$ is odd, we may group it as

$$(b_{N+1} - b_{N+2}) + (b_{N+3} - b_{N+4}) + \cdots + (b_{n-2} - b_{n-1}) + b_n.$$

In either case every bracketed difference is nonnegative, since $\{b_n\}$ is decreasing (so $b_k - b_{k+1} \geq 0$ for all k), and in the odd case the leftover term b_n is nonnegative. Therefore

$$b_{N+1} - b_{N+2} + b_{N+3} - \cdots + (-1)^{n-N-1}b_n \geq 0.$$

Since this quantity is nonnegative, its absolute value equals itself:

$$\left|(-1)^N b_{N+1} + (-1)^{N+1}b_{N+2} + \cdots + (-1)^{n-1}b_n\right| = b_{N+1} - b_{N+2} + b_{N+3} - \cdots + (-1)^{n-N-1}b_n.$$

Regrouping this same sum from the *second* term onward,

$$b_{N+1} - b_{N+2} + b_{N+3} - \cdots + (-1)^{n-N-1}b_n = b_{N+1} - \left(b_{N+2} - b_{N+3} + \cdots - (-1)^{n-N-1}b_n\right),$$

and the bracketed quantity is again a sum of the same telescoping shape (starting one index later), hence nonnegative by the argument above. Subtracting a nonnegative quantity from b_{N+1} gives

$$b_{N+1} - \left(b_{N+2} - b_{N+3} + \cdots - (-1)^{n-N-1}b_n\right) \leq b_{N+1}.$$

Combining the last three displays,

$$\left|(-1)^N b_{N+1} + (-1)^{N+1}b_{N+2} + \cdots + (-1)^{n-1}b_n\right| \leq b_{N+1}.$$

Now let $\varepsilon > 0$ be given. Since $\{b_n\} \to 0$ and $\{b_n\}$ is eventually nonnegative (being decreasing to 0), by the definition of convergence of a sequence there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $|b_n| < \varepsilon$; in particular $b_{N+1} < \varepsilon$. By the inequality just proved, for all $n \in \mathbb{N}$ with $n > N$,

$$\left|(-1)^N b_{N+1} + (-1)^{N+1} b_{N+2} + \cdots + (-1)^{n-1} b_n\right| \leq b_{N+1} < \varepsilon.$$

This is exactly the tail-sum condition of the Cauchy Criterion for Series applied to $\sum(-1)^{n-1}b_n$, with the roles of m, n in that theorem played by N, n here. Since $\varepsilon > 0$ was arbitrary, the Cauchy Criterion for Series is satisfied, so $\sum(-1)^{n-1}b_n$ is convergent. □

Idea (Method 2: Monotone Convergence Theorem; the slides name this route but do not carry it out). *(Scratch work – not part of the written answer.)* The odd- and even-indexed partial sums move towards each other: evens increase, odds decrease, and the gap $s_{2n-1} - s_{2n} = b_{2n} \to 0$. Get both subsequences to converge by the Monotone Convergence Theorem, show their limits coincide, then merge the two subsequences back into $\{s_n\}$.

Proof (Method 2: via the Monotone Convergence Theorem). For $n \in \mathbb{N}$,

$$s_{2n+2} - s_{2n} = b_{2n+1} - b_{2n+2} \geq 0 \quad \text{and} \quad s_{2n+1} - s_{2n-1} = -b_{2n} + b_{2n+1} = -(b_{2n} - b_{2n+1}) \leq 0,$$

since $\{b_n\}$ is decreasing; therefore $\{s_{2n}\}$ is increasing and $\{s_{2n-1}\}$ is decreasing. Moreover $s_{2n+1} - s_{2n} = b_{2n+1} \geq 0$ for all n, so $s_{2n} \leq s_{2n+1} \leq s_1$ for all n (the last step using that $\{s_{2n-1}\}$ is decreasing), which shows $\{s_{2n}\}$ is bounded above by s_1. Likewise, $s_{2n-1} - s_{2n} = b_{2n} \geq 0$ (each $b_n \geq 0$, since $\{b_n\}$ is decreasing with limit 0), so $s_{2n-1} \geq s_{2n} \geq s_2$ for all n because $\{s_{2n}\}$ is increasing; hence $\{s_{2n-1}\}$ is bounded below by s_2. Since $\{s_{2n}\}$ is increasing and bounded above, and $\{s_{2n-1}\}$ is decreasing and bounded below, the Monotone Convergence Theorem gives $L_e, L_o \in \mathbb{R}$ with $s_{2n} \to L_e$ and $s_{2n-1} \to L_o$.

We claim $L_e = L_o$. Since $\{b_n\} \to 0$, and $\{b_{2n}\}$ is a subsequence of $\{b_n\}$, the Theorem on Subsequential Limits gives $b_{2n} \to 0$ as well. Since $s_{2n-1} - s_{2n} = b_{2n}$ for all n, the Algebraic Limit Theorem gives $s_{2n-1} - s_{2n} \to 0$; but also $s_{2n-1} - s_{2n} \to L_o - L_e$ by the Algebraic Limit Theorem applied to the two convergent sequences $\{s_{2n-1}\}$ and $\{s_{2n}\}$. By the Uniqueness of Limits, $L_o - L_e = 0$, so $L_o = L_e$; call this common value L.

Finally we show $s_n \to L$. Let $\varepsilon > 0$ be given. Since $s_{2n} \to L$ there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|s_{2n} - L| < \varepsilon$; since $s_{2n-1} \to L$ there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|s_{2n-1} - L| < \varepsilon$. Let $N := \max\{2N_1, 2N_2 - 1\}$, and let $m \in \mathbb{N}$ with $m \geq N$. If m is even, write $m = 2n$ with $n = m/2 \geq N_1$, so $|s_m - L| = |s_{2n} - L| < \varepsilon$; if m is odd, write $m = 2n - 1$ with $n = (m+1)/2 \geq N_2$, so $|s_m - L| = |s_{2n-1} - L| < \varepsilon$. In either case $|s_m - L| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\{s_n\}$ converges to L, so $\sum(-1)^{n-1}b_n$ converges. □

Common slip. The hypothesis that $\{b_n\}$ is *decreasing* cannot be dropped, even if $b_n \to 0$ and $0 \leq b_n \leq 1/n$: a nonmonotone choice such as $b_n = 1/n$ for even n and $b_n = 0$ for odd n satisfies $0 \leq b_n \leq 1/n \to 0$, yet $\sum(-1)^{n-1}b_n = -\sum_{n \text{ even}} 1/n$ diverges, being a constant multiple (by the Comparison Test with the Harmonic Series, below) of a divergent series.

Example (Alternating Harmonic Series). Since $b_n = 1/n$ is decreasing (as $1/(n+1) < 1/n$ for all $n \in \mathbb{N}$) and $b_n \to 0$, the Alternating Series Test gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \text{ converges.}$$

(Its sum can be identified with calculus tools outside this course's scope; the Alternating Series Test itself only certifies convergence, not the value.)

Extension: Generalizing the Alternating Series Test (non-examinable)

An alternating series $\sum(-1)^{n-1}b_n$ has the form $\sum a_n b_n$ with $a_n = (-1)^{n-1}$. The slides pose, but do not answer, the question: under what more general conditions on $\{a_n\}$ (with $\{b_n\}$ still decreasing to 0) does $\sum a_n b_n$ remain convergent? No Dirichlet- or Abel-type test is stated or proved anywhere in Lectures 10–17; the question is left open in the deck. (Tutorial 4, Question A1 develops one such extension, *Abel's Test*, as an optional, non-examinable exercise using summation by parts.)

4.4 The Comparison Test and the Absolute Convergence Test

Theorem 4.7: Comparison Test

Let $\{a_n\}$ and $\{b_n\}$ satisfy $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$. Then

$$\sum b_n \text{ is convergent} \implies \sum a_n \text{ is convergent.}$$

Idea (Method 1: Cauchy Criterion, the slides' route). *(Scratch work – not part of the written answer.)* Nonnegativity removes the absolute value bars from the tail sum of $\sum a_n$; the hypothesis $a_n \leq b_n$ then bounds that tail sum by the tail sum of $\sum b_n$, which is small once $\sum b_n$'s Cauchy condition kicks in.

Proof (Method 1: via the Cauchy Criterion for Series). Suppose $\sum b_n$ is convergent. By the Cauchy Criterion for Series, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n > m \geq N$ implies $|b_{m+1} + b_{m+2} + \cdots + b_n| < \varepsilon$. Fix such $\varepsilon > 0$ and the corresponding N. Let $n, m \in \mathbb{N}$ with $n > m \geq N$. Since $a_k \geq 0$ for every k, the partial sum $a_{m+1} + a_{m+2} + \cdots + a_n$ is a sum of nonnegative terms, so it equals its own absolute value; and since $a_k \leq b_k$ for every k (and $b_k \geq 0$, as $b_k \geq a_k \geq 0$), term-by-term comparison of the finite sums gives

$$|a_{m+1} + a_{m+2} + \cdots + a_n| = a_{m+1} + a_{m+2} + \cdots + a_n \leq b_{m+1} + b_{m+2} + \cdots + b_n = |b_{m+1} + \cdots + b_n| < \varepsilon.$$

Thus, for the same N, $n > m \geq N$ implies $|a_{m+1} + \cdots + a_n| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, the sequence of partial sums of $\sum a_n$ satisfies the Cauchy condition, so by the Cauchy Criterion for Series, $\sum a_n$ is convergent. □

Idea (Method 2: Monotone Convergence Theorem; the slides name this route but do not carry it out). *(Scratch work – not part of the written answer.)* Nonnegative terms make both partial-sum sequences increasing; $\sum b_n$'s convergence bounds them (via Boundedness of Convergent Sequences); $a_n \leq b_n$ carries the bound over to $\sum a_n$'s partial sums; increasing and bounded gives convergence.

Proof (Method 2: via the Monotone Convergence Theorem). Let $\{A_n\}$ and $\{B_n\}$ denote the partial sums of $\sum a_n$ and $\sum b_n$. Since $a_n \geq 0$ for all n, $A_{n+1} - A_n = a_{n+1} \geq 0$, so $\{A_n\}$ is increasing; similarly $\{B_n\}$ is increasing. Suppose $\sum b_n$ converges, say $B_n \to t$. By the Boundedness of Convergent Sequences, $\{B_n\}$ is bounded; since $\{B_n\}$ is also increasing, it is bounded above by t itself: for each $n \in \mathbb{N}$ and every $m \geq n$ we have $B_m \geq B_n$, so letting $m \to \infty$ the Order Limit Theorem gives $t \geq B_n$; hence $B_n \leq t$ for all n. Since $0 \leq a_n \leq b_n$ for all n, summing over $k = 1, \ldots, n$ gives $A_n \leq B_n \leq t$ for every $n \in \mathbb{N}$, so $\{A_n\}$ is bounded above. Since $\{A_n\}$ is increasing and bounded above, the Monotone Convergence Theorem gives that $\{A_n\}$ converges, that is, $\sum a_n$ converges. □

Common Mistake

"The Comparison Test applies to any two series with $a_n \leq b_n$." The hypothesis $0 \leq a_n \leq b_n$ requires *nonnegativity*, not merely the inequality $a_n \leq b_n$. Without $a_n \geq 0$, the proof's step $|a_{m+1} + \cdots + a_n| = a_{m+1} + \cdots + a_n$ fails, since the partial sum can then be negative. (For series with mixed signs, use the Absolute Convergence Test, below, or compare $|a_n|$ against a nonnegative bound.)

Theorem 4.8: Convergence of $\sum 1/n^2$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Idea (QRS proof – not on the slides; the slides list this only as an example). *(Scratch work – not part of the written answer.)* For $n \geq 2$, $1/n^2 \leq 1/(n(n-1))$, and $\sum 1/(n(n-1))$ telescopes to a computable limit. Apply the Comparison Test on the tail $n \geq 2$, then restore the $n = 1$ term.

Proof. For $n \geq 2$, $n(n-1) \leq n \cdot n = n^2$ (since $n - 1 \leq n$), and both sides are positive, so $0 \leq \dfrac{1}{n^2} \leq \dfrac{1}{n(n-1)}$ for all $n \geq 2$. For $N \geq 2$, using the partial-fraction identity $1/(n(n-1)) = 1/(n-1) - 1/n$,

$$\sum_{n=2}^{N} \frac{1}{n(n-1)} = \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n} \right) = 1 - \frac{1}{N},$$

by telescoping. Since $1/N \to 0$ (Lecture 4–6), the Algebraic Limit Theorem gives $\sum_{n=2}^{N} 1/(n(n-1)) \to 1$ as $N \to \infty$, so $\sum_{n=2}^{\infty} 1/(n(n-1))$ converges (with sum 1). By the Comparison Test applied to $0 \leq 1/n^2 \leq 1/(n(n-1))$ for $n \geq 2$ (the Comparison Test extends immediately to hypotheses holding only for $n \geq 2$, by the Remark on finitely many terms), $\sum_{n=2}^{\infty} 1/n^2$ converges. Finally, writing the N-th partial sum of $\sum_{n=1}^{\infty} 1/n^2$ as $s_N = 1 + \sum_{n=2}^{N} 1/n^2$ and applying the Algebraic Limit Theorem to the convergent sequence $\sum_{n=2}^{N} 1/n^2$, we conclude that $\sum_{n=1}^{\infty} 1/n^2$ converges. □

Theorem 4.9: Absolute Convergence Test

If the series $\sum |a_n|$ is convergent, then the series $\sum a_n$ is convergent.

Idea (Method 1: Cauchy Criterion, the slides' route). *(Scratch work – not part of the written answer.)* The triangle inequality bounds the tail sum of $\sum a_n$ by the tail sum of $\sum |a_n|$, which is small once the latter's Cauchy condition kicks in.

Proof (Method 1: via the Cauchy Criterion for Series). Suppose $\sum |a_n|$ is convergent. By the Cauchy Criterion for Series, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n > m \geq N$ implies $\big||a_{m+1}| + |a_{m+2}| + \cdots + |a_n|\big| < \varepsilon$; since each $|a_k| \geq 0$, this reads $|a_{m+1}| + \cdots + |a_n| < \varepsilon$. Fix such $\varepsilon > 0$ and N. Let $n, m \in \mathbb{N}$ with $n > m \geq N$. By the (finite) triangle inequality,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the sequence of partial sums of $\sum a_n$ satisfies the Cauchy condition, so by the Cauchy Criterion for Series, $\sum a_n$ is convergent. □

Idea (Method 2: via the Comparison Test; the slides name this route but do not carry it out). *(Scratch work – not part of the written answer.)* Split a_n as $2 \cdot \frac{a_n + |a_n|}{2} - |a_n|$. Show $0 \leq \frac{a_n + |a_n|}{2} \leq |a_n|$, apply the Comparison Test to get convergence of $\sum \frac{a_n + |a_n|}{2}$, then recombine via the Algebraic Limit Theorem for Series.

Proof (Method 2: via the Comparison Test). For every $x \in \mathbb{R}$, $x \leq |x|$ and $-|x| \leq x$ (that is, $x \geq -|x|$), so $x + |x| \geq 0$ and $x + |x| \leq 2|x|$; dividing by 2, $0 \leq \frac{x + |x|}{2} \leq |x|$. Applying this with $x = a_n$ for each $n \in \mathbb{N}$ gives $0 \leq \frac{a_n + |a_n|}{2} \leq |a_n|$ for all n. Since $\sum |a_n|$ converges by hypothesis, the Comparison Test gives that $\sum \frac{a_n + |a_n|}{2}$ converges; call its sum u, and let $w := \sum |a_n|$. Since $a_n = 2 \cdot \frac{a_n + |a_n|}{2} - |a_n|$ for every n, the Algebraic Limit Theorem for Series gives that $\sum a_n$ converges, with sum $2u - w$. □

Definition 4.3: Absolute and Conditional Convergence

A series $\sum a_n$ is said to be *absolutely convergent* if the series $\sum |a_n|$ of absolute values converges. A series $\sum a_n$ is said to be *conditionally convergent* if it is convergent but not absolutely convergent, that is, $\sum a_n$ converges but $\sum |a_n|$ diverges.

Remark (Absolutely Convergent versus Convergent). The Absolute Convergence Test can be rephrased as: every absolutely convergent series is convergent. The converse fails in general – the Alternating Harmonic Series above is convergent (by the Alternating Series Test) but not absolutely convergent, since $\sum |(-1)^{n-1}/n| = \sum 1/n$ diverges (the Harmonic Series); so it is conditionally convergent. Consequently, every series is exactly one of: (i) absolutely convergent; (ii) conditionally convergent; or (iii) divergent.

4.5 Absolute versus Conditional Convergence, and Rearrangements

A *rearrangement* of a series $\sum a_n$ is a series $\sum b_n$ in which every term of $\{a_n\}$ appears exactly once in $\{b_n\}$; equivalently, there is a bijection $f : \mathbb{N} \to \mathbb{N}$ with $b_{f(n)} = a_n$ for all n. The slides pose, but do not resolve, the question: if $\sum a_n$ converges, must every rearrangement $\sum b_n$ also converge (to the same sum)?

Extension: Rearrangements Can Change the Sum (non-examinable; no theorem is given in this course)

The slides leave this question fully open – **no answer, example, or theorem is given anywhere in Lectures 10–17**, despite the Lecture 12 Objectives promising a condition for rearrangement-invariance. The following is presented only as motivation, not as course material to reproduce.
For a *conditionally* convergent series such as the Alternating Harmonic Series, one can show that the subseries of positive terms and the subseries of (absolute values of) negative terms *both* diverge: if either converged, the other would too, since their difference equals the convergent $\sum a_n$ and their sum equals the divergent $\sum |a_n|$ (compare via the Algebraic Limit Theorem for Series). Because each half individually diverges (sums to $+\infty$ or $-\infty$), one can rearrange the series – take enough positive terms to overshoot any target value, then enough negative terms to undershoot it, and so on – to make the rearranged partial sums converge to *any* prescribed real number, or to diverge outright. This is why the Comparison Test and the Absolute Convergence Test matter: an *absolutely* convergent series does not suffer this pathology (its sum is rearrangement-invariant), but this course does not state or prove that fact either. Both directions are left as open questions in this deck.

4.6 Worked Examples (Tutorial 4)

Worked Example (Tutorial 4, Question 1): Convergence Decisions

Determine whether each series converges or diverges.
(a) $\sum \dfrac{1}{2^n + n}$.

Method 1: Comparison Test with a geometric series

Proof. Let $n \in \mathbb{N}$. Since $n > 0$, we have $2^n + n > 2^n > 0$. Taking reciprocals of two positive numbers reverses the inequality; therefore

$$0 < \frac{1}{2^n + n} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n \quad \text{for every } n \in \mathbb{N}.$$

Let $T_n := \sum_{k=1}^{n} (1/2)^k$ be the n-th partial sum of $\sum (1/2)^n$. For each $k \in \mathbb{N}$ we have $(1/2)^{k-1} = 2 \cdot (1/2)^k$, hence $(1/2)^k = (1/2)^{k-1} - (1/2)^k$. Therefore the partial sum telescopes:

$$\begin{aligned} T_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) \\ &= 1 - \frac{1}{2^n}. \end{aligned}$$

By Bernoulli's Inequality, $2^n = (1+1)^n \geq 1 + n > n$, so $0 < (1/2)^n < 1/n$ for every $n \in \mathbb{N}$. Since $1/n \to 0$ (Archimedean), the Squeeze Theorem gives $(1/2)^n \to 0$. Hence, by the Algebraic Limit

Theorem, $T_n = 1 - (1/2)^n \to 1 - 0 = 1$. Therefore $\{T_n\}$ converges, so $\sum(1/2)^n$ converges (def. of convergence of a series). Since $0 \le 1/(2^n + n) \le (1/2)^n$ for every $n \in \mathbb{N}$ and $\sum(1/2)^n$ converges, the Comparison Test gives that $\sum 1/(2^n + n)$ converges. □

Convergent.

Method 2: Cauchy Criterion for Series with an explicit N

Proof. Let $a_k := 1/(2^k + k)$ for $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, $2^k + k > 2^k > 0$, hence $0 < a_k < (1/2)^k = (1/2)^{k-1} - (1/2)^k$. Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $N > 1/\varepsilon$. Let $n, m \in \mathbb{N}$ with $n > m \ge N$. The tail sum has positive terms, so it equals its absolute value, and

$$\begin{aligned} |a_{m+1} + \cdots + a_n| &< \sum_{k=m+1}^{n} \left(\frac{1}{2^{k-1}} - \frac{1}{2^k}\right) \\ &= \left(\frac{1}{2^m} - \frac{1}{2^{m+1}}\right) + \cdots + \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) \\ &= \frac{1}{2^m} - \frac{1}{2^n} < \frac{1}{2^m}. \end{aligned}$$

By Bernoulli's Inequality, $2^m = (1+1)^m \ge 1 + m > m \ge N > 1/\varepsilon$; therefore $1/2^m < \varepsilon$, and so $|a_{m+1} + \cdots + a_n| < \varepsilon$. Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n > m \ge N$ implies $|a_{m+1} + \cdots + a_n| < \varepsilon$. Therefore $\sum a_n$ converges (Cauchy Criterion for Series). □

Convergent.

Method 3: Monotone Convergence Theorem on the partial sums

Proof. Let $S_n := \sum_{k=1}^{n} 1/(2^k + k)$. For every $n \in \mathbb{N}$, $S_{n+1} - S_n = 1/(2^{n+1} + n + 1) > 0$; therefore $\{S_n\}$ is increasing. For each $k \in \mathbb{N}$, $2^k + k > 2^k > 0$, hence $1/(2^k + k) < (1/2)^k$. Summing over $k = 1, \ldots, n$ and telescoping $(1/2)^k = (1/2)^{k-1} - (1/2)^k$,

$$\begin{aligned} S_n &< \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) \\ &= 1 - \frac{1}{2^n} < 1. \end{aligned}$$

Hence $\{S_n\}$ is bounded above by 1. Since $\{S_n\}$ is increasing and bounded above, $\{S_n\}$ converges (MCT). Therefore $\sum 1/(2^n + n)$ converges (def. of convergence of a series). □

Convergent.

(b) $\sum \dfrac{\sin(n)}{n^2}$.

Method 1: Comparison Test followed by the Absolute Convergence Test

Proof. Let $n \in \mathbb{N}$. Since $-1 \le \sin(n) \le 1$, we have $|\sin(n)| \le 1$; since $n^2 > 0$, dividing by n^2 gives

$$0 \le \left|\frac{\sin(n)}{n^2}\right| = \frac{|\sin(n)|}{n^2} \le \frac{1}{n^2} \quad \text{for every } n \in \mathbb{N}.$$

The series $\sum 1/n^2$ converges (Convergence of $\sum 1/n^2$). Hence, by the Comparison Test with $|\sin(n)/n^2|$ in the role of a_n and $1/n^2$ in the role of b_n, the series $\sum |\sin(n)/n^2|$ converges. Therefore, by the Absolute Convergence Test, $\sum \sin(n)/n^2$ converges. □

Convergent.

Method 2: Cauchy Criterion for Series with a telescoping bound

Proof. Let $a_k := \sin(k)/k^2$ for $k \in \mathbb{N}$. Let $k \in \mathbb{N}$ with $k \geq 2$. Since $|\sin(k)| \leq 1$, $|a_k| \leq 1/k^2$. Since $0 < k-1 < k$, we have $0 < k(k-1) < k^2$, hence $1/k^2 < 1/(k(k-1))$; and $\frac{1}{k-1} - \frac{1}{k} = \frac{k-(k-1)}{k(k-1)} = \frac{1}{k(k-1)}$. Therefore

$$|a_k| < \frac{1}{k-1} - \frac{1}{k} \quad \text{for every } k \geq 2.$$

Let $\varepsilon > 0$. By the Archimedean Property there is $N \in \mathbb{N}$ with $N > 1/\varepsilon$. Let $n, m \in \mathbb{N}$ with $n > m \geq N$. Then every index $k \in \{m+1, \ldots, n\}$ satisfies $k \geq 2$, so by the triangle inequality and the bound above,

$$\begin{aligned} |a_{m+1} + \cdots + a_n| &\leq |a_{m+1}| + \cdots + |a_n| \\ &< \Big(\frac{1}{m} - \frac{1}{m+1}\Big) + \Big(\frac{1}{m+1} - \frac{1}{m+2}\Big) + \cdots + \Big(\frac{1}{n-1} - \frac{1}{n}\Big) \\ &= \frac{1}{m} - \frac{1}{n} < \frac{1}{m} \leq \frac{1}{N} < \varepsilon. \end{aligned}$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n > m \geq N$ implies $|a_{m+1} + \cdots + a_n| < \varepsilon$. Therefore $\sum \sin(n)/n^2$ converges (Cauchy Criterion for Series). □

Convergent.

(c) $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + \frac{1}{5} - \frac{1}{6^2} + \cdots$.

Method 1: Contradiction via the even-indexed partial sums

Proof. For $k \in \mathbb{N}$ let $a_k := 1/k$ if k is odd and $a_k := -1/k^2$ if k is even, so that the series is $\sum a_k$, and let $S_n := a_1 + \cdots + a_n$. Suppose, for contradiction, that $\sum a_k$ converges. Then $S_n \to S$ for some $S \in \mathbb{R}$ (def. of convergence of a series). For each $N \in \mathbb{N}$, grouping the first $2N$ terms into the N consecutive pairs $a_{2k-1} + a_{2k}$, $k = 1, \ldots, N$, gives

$$\begin{aligned} S_{2N} &= \sum_{k=1}^{N} (a_{2k-1} + a_{2k}) = \sum_{k=1}^{N} \Big(\frac{1}{2k-1} - \frac{1}{(2k)^2}\Big) \\ &= \sum_{k=1}^{N} \frac{1}{2k-1} - \sum_{k=1}^{N} \frac{1}{(2k)^2} =: P_N - Q_N. \end{aligned}$$

Step 1: $\{Q_N\}$ converges. For each $k \in \mathbb{N}$, $1/(2k)^2 = \frac{1}{4} \cdot 1/k^2$. Since $\sum 1/k^2$ converges (Convergence of $\sum 1/n^2$), the Algebraic Limit Theorem for Series with $c = \frac{1}{4}$ gives that $\sum 1/(2k)^2$ converges. Hence $Q_N \to q$ for some $q \in \mathbb{R}$.
Step 2: $\{P_N\}$ is not bounded above. If $\sum 1/(2k)$ converged, then the Algebraic Limit Theorem for Series with $c = 2$ would give that $\sum 2 \cdot \frac{1}{2k} = \sum \frac{1}{k}$ converges, which contradicts the Divergence of the Harmonic Series. Therefore $\sum 1/(2k)$ diverges. For each $k \in \mathbb{N}$, $0 < 2k-1 < 2k$, hence $0 < 1/(2k) < 1/(2k-1)$. If $\sum 1/(2k-1)$ converged, then the Comparison Test would give that $\sum 1/(2k)$ converges, which is false. Therefore $\sum 1/(2k-1)$ diverges, that is, $\{P_N\}$ does not converge. Moreover $P_{N+1} - P_N = 1/(2N+1) > 0$ for every $N \in \mathbb{N}$, so $\{P_N\}$ is increasing. If $\{P_N\}$ were bounded above, then it would converge (MCT). Hence $\{P_N\}$ is not bounded above.
Step 3: the contradiction. Since $\{2N\}$ is strictly increasing, $\{S_{2N}\}$ is a subsequence of $\{S_n\}$, so $S_{2N} \to S$ (Subsequential Limits). Since $P_N = S_{2N} + Q_N$ for every N, the Algebraic Limit Theorem gives $P_N \to S + q$. Therefore $\{P_N\}$ converges, so $\{P_N\}$ is bounded (Boundedness of Convergent Sequences). This contradicts Step 2. Hence the assumption is false, and $\sum a_k$ diverges. □

Divergent.

Method 2: Failure of the Cauchy Criterion for Series with $\varepsilon = \frac{1}{8}$

Proof. Let $a_k := 1/k$ for odd k and $a_k := -1/k^2$ for even k, as in Method 1. By the Cauchy Criterion for Series, it suffices to find $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ there are $n, m \in \mathbb{N}$ with $n > m \geq N$ and $|a_{m+1} + \cdots + a_n| \geq \varepsilon$. Take $\varepsilon = \frac{1}{8}$. Let $N \in \mathbb{N}$, and set $M := N + 1$, $m := 2M$, $n := 4M$. Then $M \geq 2$ and $n > m \geq N$. The odd indices in $\{2M+1, \ldots, 4M\}$ are $2j-1$ and the even indices are $2j$, for $j = M+1, \ldots, 2M$; each family has M members. Hence

$$a_{2M+1} + \cdots + a_{4M} = \sum_{j=M+1}^{2M} \frac{1}{2j-1} - \sum_{j=M+1}^{2M} \frac{1}{(2j)^2}.$$

Let $j \in \{M+1, \ldots, 2M\}$. Then $2j-1 \leq 4M-1 < 4M$, so $1/(2j-1) > 1/(4M)$; and $2j \geq 2M+2 > 2M$, so $(2j)^2 > 4M^2$ and $1/(2j)^2 < 1/(4M^2)$. Summing the M terms of each family,

$$\sum_{j=M+1}^{2M} \frac{1}{2j-1} > M \cdot \frac{1}{4M} = \frac{1}{4}, \qquad \sum_{j=M+1}^{2M} \frac{1}{(2j)^2} < M \cdot \frac{1}{4M^2} = \frac{1}{4M} \leq \frac{1}{8},$$

where the last inequality uses $M \geq 2$. Therefore $a_{m+1} + \cdots + a_n > \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$, so $|a_{m+1} + \cdots + a_n| \geq \frac{1}{8} = \varepsilon$. Since $N \in \mathbb{N}$ was arbitrary, the Cauchy condition fails for $\varepsilon = \frac{1}{8}$. Hence the partial sums of $\sum a_k$ are not Cauchy, and $\sum a_k$ diverges (Cauchy Criterion for Series). □

Divergent.

Worked Example (Tutorial 4, Question 2(a) = Sample Midterm Q2(a)): Two Divergent Series with Convergent Product

Statement. There are two divergent series $\sum a_n$ and $\sum b_n$ for which $\sum a_n b_n$ converges.
Answer. True.

Method 1: Witness $a_n = b_n = 1/n$ checked against the chapter's standard series

Proof. Let $a_n := 1/n$ and $b_n := 1/n$ for every $n \in \mathbb{N}$. Then $\sum a_n = \sum b_n = \sum 1/n$, and $\sum 1/n$ diverges (Divergence of the Harmonic Series); hence $\sum a_n$ and $\sum b_n$ are both divergent. For every $n \in \mathbb{N}$,

$$a_n b_n = \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}.$$

Since $\sum 1/n^2$ converges (Convergence of $\sum 1/n^2$), $\sum a_n b_n$ converges. Therefore there exist two divergent series $\sum a_n$, $\sum b_n$ for which $\sum a_n b_n$ converges, and the statement is true. □

True: $a_n = b_n = \frac{1}{n}$.

Method 2: Same witness, verified by a block-grouping estimate and the MCT

Proof. Let $a_n = b_n := 1/n$, $H_n := \sum_{k=1}^{n} 1/k$ and $Q_n := \sum_{k=1}^{n} 1/k^2$ for $n \in \mathbb{N}$.
$\sum a_n = \sum b_n$ *diverges.* Let $j \in \mathbb{N}$. The block $k = 2^{j-1}+1, \ldots, 2^j$ has $2^j - 2^{j-1} = 2^{j-1}$ terms, and each satisfies $k \leq 2^j$, hence $1/k \geq 1/2^j$. Therefore $\sum_{k=2^{j-1}+1}^{2^j} 1/k \geq 2^{j-1} \cdot \frac{1}{2^j} = \frac{1}{2}$. Splitting $\{1, \ldots, 2^J\}$ into $\{1\}$ and the blocks $j = 1, \ldots, J$ gives, for every $J \in \mathbb{N}$,

$$H_{2^J} = 1 + \sum_{j=1}^{J} \sum_{k=2^{j-1}+1}^{2^j} \frac{1}{k} \geq 1 + \frac{J}{2}.$$

Let $B > 0$. By the Archimedean Property there is $J \in \mathbb{N}$ with $J > 2B$, hence $H_{2^J} \geq 1 + J/2 > B$. Therefore $\{H_n\}$ is not bounded above, so $\{H_n\}$ does not converge (Boundedness of Convergent Sequences), that is, $\sum a_n = \sum b_n$ diverges.

$\sum a_n b_n$ *converges.* Since $a_n b_n = 1/n^2$, the partial sums of $\sum a_n b_n$ are Q_n. For every n, $Q_{n+1} - Q_n = 1/(n+1)^2 > 0$, so $\{Q_n\}$ is increasing. For $k \geq 2$, $0 < k(k-1) < k^2$, hence $1/k^2 < 1/(k(k-1)) = 1/(k-1) - 1/k$. Therefore, for $n \geq 2$,

$$\begin{aligned} Q_n &= 1 + \sum_{k=2}^{n} \frac{1}{k^2} < 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} < 2, \end{aligned}$$

and $Q_1 = 1 < 2$. Hence $\{Q_n\}$ is bounded above by 2. Since $\{Q_n\}$ is increasing and bounded above, it converges (MCT); therefore $\sum a_n b_n$ converges. Hence the statement is true. □

True: $a_n = b_n = \frac{1}{n}$.

Worked Example (Tutorial 4, Question 4): A Cauchy Estimate Without Naming the Shortcut Theorems

Problem. Prove that if the series $\sum |a_n|$ has a Cauchy sequence of partial sums and the sequence $\{b_n\}$ is bounded, then the series $\sum a_n b_n$ has a Cauchy sequence of partial sums. Do not use the Cauchy Criterion, the Comparison Test, or the Absolute Convergence Test.

Method 1: Direct ε–N estimate on the partial sums

Proof. For $n \in \mathbb{N}$ let $A_n := |a_1| + \cdots + |a_n|$ and $C_n := a_1 b_1 + \cdots + a_n b_n$. Since $\{b_n\}$ is bounded, there is $M_0 \geq 0$ such that $|b_n| \leq M_0$ for every $n \in \mathbb{N}$; let $M := M_0 + 1$. Then $M > 0$ and $|b_n| \leq M$ for every $n \in \mathbb{N}$. For $n, m \in \mathbb{N}$ with $n > m$, the first m terms cancel, so

$$A_n - A_m = |a_{m+1}| + \cdots + |a_n| \geq 0, \qquad C_n - C_m = a_{m+1} b_{m+1} + \cdots + a_n b_n.$$

Let $\varepsilon > 0$. Since $\varepsilon/M > 0$ and $\{A_n\}$ is a Cauchy sequence, there is $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n, m \geq N$ implies $|A_n - A_m| < \varepsilon/M$ (def. of Cauchy sequence). Let $n, m \in \mathbb{N}$ with $n, m \geq N$. If $n = m$, then $|C_n - C_m| = 0 < \varepsilon$. If $n > m$, then

$$\begin{aligned} |C_n - C_m| &= |a_{m+1} b_{m+1} + \cdots + a_n b_n| \\ &\leq |a_{m+1}|\,|b_{m+1}| + \cdots + |a_n|\,|b_n| && \text{(triangle inequality)} \\ &\leq M\big(|a_{m+1}| + \cdots + |a_n|\big) && (|b_k| \leq M) \\ &= M\,|A_n - A_m| < M \cdot \frac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

If $n < m$, then the case just proved with n, m interchanged gives $|C_n - C_m| = |C_m - C_n| < \varepsilon$. Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n, m \geq N$ implies $|C_n - C_m| < \varepsilon$. Therefore $\{C_n\}$, the sequence of partial sums of $\sum a_n b_n$, is a Cauchy sequence. □

Method 2: Contradiction via the negated Cauchy condition

Proof. Define A_n, C_n and $M > 0$ as in Method 1, so that $|b_n| \leq M$ for every $n \in \mathbb{N}$. Suppose, for contradiction, that $\{C_n\}$ is not a Cauchy sequence. Negating the definition, there is $\varepsilon_0 > 0$ such that for every $N \in \mathbb{N}$ there are $n, m \in \mathbb{N}$ with $n, m \geq N$ and $|C_n - C_m| \geq \varepsilon_0$. Since $|C_n - C_n| = 0 < \varepsilon_0$, such n, m are distinct, and interchanging them if necessary we may take $n > m$. Since $\varepsilon_0/M > 0$ and $\{A_n\}$ is Cauchy, there is $N_0 \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n, m \geq N_0$ implies $|A_n - A_m| < \varepsilon_0/M$. Apply the negated condition with $N = N_0$: there are $n > m \geq N_0$ with $|C_n - C_m| \geq \varepsilon_0$. By the triangle inequality and $|b_k| \leq M$,

$$\begin{aligned} \varepsilon_0 \leq |C_n - C_m| &\leq |a_{m+1}|\,|b_{m+1}| + \cdots + |a_n|\,|b_n| \\ &\leq M\big(|a_{m+1}| + \cdots + |a_n|\big) = M\,|A_n - A_m| < M \cdot \frac{\varepsilon_0}{M} = \varepsilon_0. \end{aligned}$$

Hence $\varepsilon_0 < \varepsilon_0$, a contradiction. Therefore $\{C_n\}$ is a Cauchy sequence, that is, the partial sums of $\sum a_n b_n$ form a Cauchy sequence. □

Worked Example (Tutorial 4, Question 5): Regrouping Preserves the Sum

Problem. Given $\sum a_n$ and a strictly increasing sequence $\{n_k\} \subset \mathbb{N}$ (with $n_0 := 0$), let $b_k := a_{n_{k-1}+1} + a_{n_{k-1}+2} + \cdots + a_{n_k}$ for each $k \in \mathbb{N}$. Prove that if $\sum a_n$ sums to s, then $\sum b_k$ sums to s.

Method 1: Induction on k and Subsequential Limits

Proof. Let $s_n := a_1 + \cdots + a_n$ and $t_k := b_1 + \cdots + b_k$ for $n, k \in \mathbb{N}$. We claim $t_k = s_{n_k}$ for every $k \in \mathbb{N}$, by induction on k. For $k = 1$, since $n_0 = 0$,

$$t_1 = b_1 = a_{n_0+1} + \cdots + a_{n_1} = a_1 + \cdots + a_{n_1} = s_{n_1}.$$

Suppose $t_k = s_{n_k}$ for some $k \in \mathbb{N}$. Then

$$t_{k+1} = t_k + b_{k+1} = (a_1 + \cdots + a_{n_k}) + (a_{n_k+1} + \cdots + a_{n_{k+1}}) = s_{n_{k+1}}.$$

Hence $t_k = s_{n_k}$ for every $k \in \mathbb{N}$. Since $\{n_k\}$ is a strictly increasing sequence in $\mathbb{N}$, $\{s_{n_k}\}$ is a subsequence of $\{s_n\}$. Since $\sum a_n$ sums to s, $s_n \to s$ (def. of convergence of a series); therefore $s_{n_k} \to s$ (Subsequential Limits). Since $t_k = s_{n_k}$ for every k, $t_k \to s$. Therefore $\sum b_k$ sums to s. □

Method 2: Telescoping the block sums and the definition of limit

Proof. Let s_n, t_k be as in Method 1, and let $s_0 := 0$. For each $j \in \mathbb{N}$, the first n_{j-1} terms cancel in $s_{n_j} - s_{n_{j-1}}$, so $b_j = a_{n_{j-1}+1} + \cdots + a_{n_j} = s_{n_j} - s_{n_{j-1}}$. Hence, for every $k \in \mathbb{N}$, the sum telescopes term by term:

$$t_k = (s_{n_1} - s_{n_0}) + (s_{n_2} - s_{n_1}) + \cdots + (s_{n_k} - s_{n_{k-1}}) = s_{n_k} - s_{n_0} = s_{n_k},$$

since $s_{n_0} = s_0 = 0$. Next, $n_k \geq k$ for every $k \in \mathbb{N}$ (Subsequence Index Bound), by induction: $n_1 > n_0 = 0$ gives $n_1 \geq 1$; if $n_k \geq k$, then $n_{k+1} > n_k \geq k$, and since n_{k+1} is an integer, $n_{k+1} \geq k+1$. Let $\varepsilon > 0$. Since $s_n \to s$, there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $|s_n - s| < \varepsilon$ (def. of limit). Let $k \in \mathbb{N}$ with $k \geq N$. Then $n_k \geq k \geq N$, therefore $|t_k - s| = |s_{n_k} - s| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $t_k \to s$. Therefore $\sum b_k$ sums to s. □

Worked Example (Tutorial 4, Question 3(a)): Absolute Convergence of $\sum a_n$ Implies Absolute Convergence of $\sum a_n^2$

Statement. If $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely.

Method 1: Eventual bound $|a_n| < 1$ and the Comparison Test on a tail

Proof. Suppose $\sum |a_n|$ converges. Let $A_n := |a_1| + \cdots + |a_n|$, $A_0 := 0$, $Q_n := a_1^2 + \cdots + a_n^2$, $Q_0 := 0$, and $A := \lim A_n$. By the Terms of Convergent Series applied to $\sum |a_n|$, $|a_n| \to 0$. By the definition of limit with $\varepsilon = 1$, there is $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N$ implies $|a_n| < 1$. Let $n \geq N$. Multiplying $0 \leq |a_n| < 1$ by $|a_n| \geq 0$ gives

$$0 \leq a_n^2 = |a_n| \cdot |a_n| \leq |a_n|.$$

For $j \in \mathbb{N}$ let $c_j := a_{N-1+j}^2$ and $d_j := |a_{N-1+j}|$; then $0 \leq c_j \leq d_j$ for every j. The J-th partial sum of $\sum d_j$ is $D_J = A_{N-1+J} - A_{N-1}$. Since $\{N-1+J\}_{J \in \mathbb{N}}$ is strictly increasing, $\{A_{N-1+J}\}$ is a subsequence of $\{A_n\}$, so $A_{N-1+J} \to A$ (Subsequential Limits); hence $D_J \to A - A_{N-1}$ (Algebraic Limit Theorem), and $\sum d_j$ converges. By the Comparison Test, $\sum c_j$ converges; let C_J be its partial sums and $c := \lim C_J$. For every $n \geq N$, $Q_n = Q_{N-1} + C_{n-N+1}$. Let $\varepsilon > 0$. There is $J_0 \in \mathbb{N}$ such that for all $J \geq J_0$, $|C_J - c| < \varepsilon$. Hence for all $n \geq N - 1 + J_0$ (so that $n \geq N$ and $n - N + 1 \geq J_0$), $|Q_n - (Q_{N-1} + c)| = |C_{n-N+1} - c| < \varepsilon$. Therefore $Q_n \to Q_{N-1} + c$, so $\sum a_n^2$ converges. Since $|a_n^2| = a_n^2$ for every n, $\sum |a_n^2|$ converges, that is, $\sum a_n^2$ converges absolutely. □

Method 2: Global bound by the sum and the Monotone Convergence Theorem

Proof. Suppose $\sum |a_n|$ converges. Let $A_n := |a_1| + \cdots + |a_n|$, $A_0 := 0$, $A := \lim A_n$ and $Q_n := a_1^2 + \cdots + a_n^2$. Since $A_{n+1} - A_n = |a_{n+1}| \geq 0$, $\{A_n\}$ is increasing. Fix $n \in \mathbb{N}$. For every $m \geq n$, $A_m \geq A_n$; letting $m \to \infty$, the Order Limit Theorem gives $A \geq A_n$. Hence $A_n \leq A$ for every $n \in \mathbb{N}$. Therefore, for every $n \in \mathbb{N}$,

$$|a_n| = A_n - A_{n-1} \leq A_n \leq A, \quad \text{hence} \quad a_n^2 = |a_n| \cdot |a_n| \leq A\,|a_n|\,.$$

Summing over $k = 1, \ldots, n$ gives $Q_n \leq A \cdot A_n \leq A^2$ for every $n \in \mathbb{N}$, so $\{Q_n\}$ is bounded above. Since $Q_{n+1} - Q_n = a_{n+1}^2 \geq 0$, $\{Q_n\}$ is increasing. Therefore $\{Q_n\}$ converges (MCT), so $\sum a_n^2$ converges. Since $|a_n^2| = a_n^2$ for every n, $\sum a_n^2$ converges absolutely. □

Method 3: Cauchy Criterion for Series with $N = \max\{N_1, N_2\}$

Proof. Suppose $\sum |a_n|$ converges. As in Method 1, $|a_n| \to 0$ (Terms of Convergent Series), so there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n| < 1$, and therefore $0 \leq a_n^2 \leq |a_n|$. Let $\varepsilon > 0$. By the Cauchy Criterion for Series applied to $\sum |a_n|$, there is $N_2 \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n > m \geq N_2$ implies $|a_{m+1}| + \cdots + |a_n| < \varepsilon$. Let $N := \max\{N_1, N_2\}$ and let $n, m \in \mathbb{N}$ with $n > m \geq N$. Every index $k \in \{m+1, \ldots, n\}$ satisfies $k > N_1$, so $0 \leq a_k^2 \leq |a_k|$; therefore

$$|a_{m+1}^2 + \cdots + a_n^2| = a_{m+1}^2 + \cdots + a_n^2 \leq |a_{m+1}| + \cdots + |a_n| < \varepsilon.$$

Hence for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{N}$, $n > m \geq N$ implies $|a_{m+1}^2 + \cdots + a_n^2| < \varepsilon$. Therefore $\sum a_n^2$ converges (Cauchy Criterion for Series); since $|a_n^2| = a_n^2$, $\sum a_n^2$ converges absolutely. □

4.7 Chapter Summary

Key Formula: Series Tests at a Glance

- **Terms of Convergent Series.** $\sum a_n$ convergent $\implies \{a_n\} \to 0$. (Contrapositive is a divergence test only.)
- **Algebraic Limit Theorem for Series.** $\sum a_n \to s$, $\sum b_n \to t \implies \sum ca_n \to cs$, $\sum (a_n + b_n) \to s + t$.
- **Cauchy Criterion for Series.** $\sum a_n$ convergent $\iff \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > m \geq N$, $|a_{m+1} + \cdots + a_n| < \varepsilon$.
- **Alternating Series Test.** $\{b_n\}$ decreasing, $b_n \to 0 \implies \sum (-1)^{n-1} b_n$ convergent.
- **Comparison Test.** $0 \leq a_n \leq b_n$ for all n, $\sum b_n$ convergent $\implies \sum a_n$ convergent.
- **Absolute Convergence Test.** $\sum |a_n|$ convergent $\implies \sum a_n$ convergent.
- **Geometric series.** $|r| < 1 \implies \sum_{n=0}^{\infty} ar^n = a/(1-r)$; $|r| \geq 1 \implies$ divergent.
- **Standard comparisons.** $\sum 1/n$ diverges; $\sum 1/n^2$ converges.

Exam Focus

Exam Focus: the series decision flow.

1. **Do the terms tend to 0?** If not, stop: divergent by the Terms of Convergent Series (contrapositive). If yes, continue – this alone never proves convergence.

2. **Is it alternating, with decreasing $|a_n|$?** If yes, the Alternating Series Test settles convergence immediately.

3. **Compare.** Bound a_n (or $|a_n|$) above by a series already known to converge (geometric with $|r| < 1$; $\sum 1/n^2$; a telescoping sum) via the Comparison Test, or below by a series known to diverge ($\sum 1/n$) via the contrapositive of the Comparison Test.

4. **Absolute, then conditional.** If $\sum |a_n|$ converges by any of the above, the Absolute Convergence Test gives $\sum a_n$ convergent (absolutely). If $\sum |a_n|$ can be shown divergent but $\sum a_n$ converges by the Alternating Series Test (or otherwise), the series is conditionally convergent – and its sum is not safe to rearrange.

Every one of these tests, once past the first step, is secretly the Cauchy Criterion for Series in disguise (Exam Focus box, Section 4.2): if a proof is forgotten under time pressure, rebuild it from the three-step Cauchy-tail recipe.

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5 Lectures 13–15: Limit Points, Closures and Closed Sets

This chapter answers three questions:

(1) *when does a number $a \in \mathbb{R}$ sit arbitrarily close to a set A without necessarily belonging to it?*
(2) *how do we enlarge A to a set that already contains every point it "almost" contains?*
(3) *which sets already contain everything they almost contain?*

It builds on Chapter 4 (series, and the Cauchy Criterion and Algebraic Limit Theorem for sequences that every ε-proof below quietly relies on) and prepares Chapter 6 (compact sets, which are built out of closed and bounded sets).

A note on scope. This course never defines an *open* set, an open cover, or the Heine–Borel property.

Closed sets are defined here purely in terms of limit points, closures, and sequential limits.

"Closed" is never treated as "the complement of open," and that equivalence should not be imported from other courses.

Non-circular order used in this chapter. The slides state four results in Lectures 14–15 without a written proof: the ε-Criterion for Closures, the Sequential Criterion for Closures, the Corollary on Closures of Closures, and the Closure Criterion for Closed Sets (the last is only justified by a five-step remark, not a labelled proof).

Full proofs of all four are given below, each flagged **QRS proof – not on the slides**, using only material available by that point of the course.

The order used is as follows.

(i) Limit Points and Isolated Points, with the Sequential Criterion for Limit Points proved first, since nothing later depends on anything but the raw definitions.

(ii) The Closure, with the ε-Criterion for Closures proved directly from the definitions of closure and limit point; then the Sequential Criterion for Closures proved from the ε-Criterion for Closures and the Sequential Criterion for Limit Points; then Limit Points of Closures (proved on the slides) from the ε-Criterion for Closures and the triangle inequality; then Closures of Closures as a corollary of Limit Points of Closures.

(iii) Suprema Lie in the Closure, proved from the ε-Criterion for Suprema and Infima together with either the ε-Criterion for Closures or the Sequential Criterion for Closures.

(iv) Closed Sets, with the Closure Criterion for Closed Sets proved directly from the definitions of closure and closed, and the Sequential Criterion for Closed Sets (proved on the slides) obtained by substituting the Sequential Criterion for Closures into the Closure Criterion for Closed Sets.

Each result below is used only after every item it depends on has already been established, so no circularity is possible.

5.1 Limit Points and Isolated Points

The definition of the limit of a function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ at a point $a \in \mathbb{R}$ requires points of $D \setminus \{a\}$ arbitrarily close to a.

Without this requirement, the limit statement is vacuously true for every candidate value L, so limits of functions would not even be unique.

The number a that makes this requirement meaningful is called a limit point of D; the two definitions below make this precise.

Definition 5.1: Limit Point

Let $A \subseteq \mathbb{R}$. A number $a \in \mathbb{R}$ is a *limit point* of A (also called a *cluster point* or an *accumulation point*) if for every $\varepsilon > 0$ there exists $x \in A \setminus \{a\}$ such that

$$|x - a| < \varepsilon.$$

Intuition. A limit point of A is a number with other points of A crowding arbitrarily close to it from some direction, no matter how small a window around a is chosen.

The point a itself is deliberately excluded from the witness x: otherwise every point of every nonempty set would trivially be a "limit point" of itself, and the definition would carry no information.

Definition 5.2: Isolated Point

Let $A \subseteq \mathbb{R}$. A number $a \in \mathbb{R}$ is an *isolated point* of A if $a \in A$ and a is not a limit point of A.

Intuition. An isolated point is a member of A that has a personal safety margin: some positive distance δ within which no other point of A intrudes.

Two consequences follow immediately from the two definitions.

First, a limit point of A may or may not itself belong to A: the definition only asks for other points of A nearby, never for $a \in A$.

Second, every number in A is either an isolated point or a limit point of A, and never both.

Indeed, if $a \in A$, then either a fails the limit-point condition (so a is isolated by definition) or a satisfies it (so a is a limit point); these two cases are mutually exclusive by definition.

A number outside A is never isolated, since isolation requires membership in A; it may still be a limit point of A.

Common Mistake: limit points, isolated points, and where they live

- A limit point of A *need not be in* A: e.g. 0 is a limit point of $\{1/n : n \in \mathbb{N}\}$ but $0 \notin \{1/n : n \in \mathbb{N}\}$ (Example below).
- Isolated points are always *in* A, by definition; "isolated" does not mean "outside the set."
- In the Sequential Criterion for Limit Points below, the witnessing sequence must lie in $A \setminus \{a\}$, not merely in A; a constant sequence at a proves nothing.
- This course has no notion of "open set": "F is not closed" is not the same statement as "F is open," and neither implies the other in general.

Theorem 5.1: Sequential Criterion for Limit Points

Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then a is a limit point of A if and only if there is a sequence $\{a_n\}$ contained in $A \setminus \{a\}$ with $a = \lim a_n$.

Idea. *(Scratch work – not part of the written answer.)* This proof is on the slides.

For "only if," turn the ε-statement defining a limit point into a sequence by plugging in a null sequence $\varepsilon_n = 1/n$ (the "ε_n-trick"): for each n, the definition supplies a point a_n within ε_n of a, and $\varepsilon_n \to 0$ forces $a_n \to a$.

For "if," run the trick in reverse: extract a single ε-witness $x = a_N$ from the definition of $a = \lim a_n$.

Proof. ($\Rightarrow$) Suppose a is a limit point of A.

Let $\{\varepsilon_n\}$ be the sequence of positive numbers $\varepsilon_n = 1/n$, which converges to 0 by the Archimedean Property.

Since a is a limit point of A, applying the definition of limit point with $\varepsilon = \varepsilon_n$ shows that for each $n \in \mathbb{N}$ there exists a point $a_n \in A \setminus \{a\}$ with $|a_n - a| < \varepsilon_n$.

This defines a sequence $\{a_n\}$ contained in $A \setminus \{a\}$.

Let $\varepsilon > 0$ be given.

Since $\varepsilon_n \to 0$, the tail of $\{\varepsilon_n\}$ eventually satisfies $|\varepsilon_n| < \varepsilon$, so there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon_n = |\varepsilon_n| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, this shows that $\{a_n\}$ converges to a.

($\Leftarrow$) Suppose there is a sequence $\{a_n\}$ contained in $A \setminus \{a\}$ with $a = \lim a_n$.

Let $\varepsilon > 0$ be given.

By the definition of sequence convergence, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$; in particular, the point $x = a_N$ belongs to $A \setminus \{a\}$ and satisfies $|x - a| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, we have produced, for every $\varepsilon > 0$, a point $x \in A \setminus \{a\}$ with $|x - a| < \varepsilon$.

This is exactly the definition of a being a limit point of A. □

Common slip. The witnessing sequence must avoid a itself: a constant sequence $a_n = a$ converges to a but is not contained in $A \setminus \{a\}$, and says nothing about limit points.

Do not confuse this with the closure version in Theorem 5.3 below, where a constant sequence is allowed.

5.2 The Closure of a Set

A continuous function f on $[a, b] \setminus \{0\}$ that happens to have a limit at 0 can be extended to a continuous function g on all of $[a, b]$, after which the Closed Interval Method applies to g and its extrema can be located.

This motivates enlarging a set A to include every point that A "almost reaches": the resulting set is the closure of A.

Definition 5.3: Closure of a Set

Let $A \subseteq \mathbb{R}$. The *closure* of A, written $\overline{A}$, is

$$\overline{A} := A \cup \{\text{limit points of } A\}.$$

Intuition. $\overline{A}$ patches every "gap" that A almost fills: it keeps every original point of A and throws in every number that has points of A crowding arbitrarily close to it.

Since $A \subseteq A \cup \{\text{limit points of } A\} = \overline{A}$ trivially, every set is contained in its own closure; the closure can only add points, never remove them.

Theorem 5.2: ε-Criterion for Closures

Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then $a \in \overline{A}$ if and only if for every $\varepsilon > 0$ there exists $x \in A$ with $|x - a| < \varepsilon$.

Idea. *(Scratch work – not part of the written answer.)* **QRS proof – not on the slides.**

Split on whether $a \in A$.

If $a \in A$, the point $x = a$ trivially witnesses every ε.

If a is a limit point of A but $a \notin A$, the limit-point definition already supplies an $x \in A \setminus \{a\} \subseteq A$ within ε of a.

Conversely, given an ε-witness $x \in A$ for every ε: if $a \in A$ we are done immediately; if $a \notin A$, the witness x automatically differs from a (since $x \in A$, $a \notin A$), so it also witnesses the limit-point definition.

Proof. ($\Rightarrow$) Suppose $a \in \overline{A}$.

By the definition of closure, either $a \in A$ or a is a limit point of A.

Let $\varepsilon > 0$ be given.

If $a \in A$, take $x = a \in A$; then $|x - a| = 0 < \varepsilon$.

If instead a is a limit point of A, then by the definition of limit point there exists $x \in A \setminus \{a\} \subseteq A$ with $|x - a| < \varepsilon$.

In either case there is a point $x \in A$ with $|x - a| < \varepsilon$; since $\varepsilon > 0$ was arbitrary, this holds for every $\varepsilon > 0$.

($\Leftarrow$) Suppose that for every $\varepsilon > 0$ there exists $x \in A$ with $|x - a| < \varepsilon$.

If $a \in A$, then $a \in A \subseteq \overline{A}$ by the definition of closure, and we are done.

If $a \notin A$, let $\varepsilon > 0$ be given and let $x \in A$ be a point with $|x - a| < \varepsilon$.

Since $x \in A$ and $a \notin A$, necessarily $x \neq a$, so $x \in A \setminus \{a\}$.

As $\varepsilon > 0$ was arbitrary, we have shown that for every $\varepsilon > 0$ there exists $x \in A \setminus \{a\}$ with $|x - a| < \varepsilon$, which is the definition of a being a limit point of A.

Therefore a belongs to $\{\text{limit points of } A\} \subseteq \overline{A}$. □

Common slip. Do not restrict the witness to $A \setminus \{a\}$ here: the criterion quantifies x over all of A, precisely so that a point $a \in A$ can witness itself.

Doing this over $A \setminus \{a\}$ instead reproduces the limit-point definition, not the closure criterion.

Theorem 5.3: Sequential Criterion for Closures

Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then $a \in \overline{A}$ if and only if there is a sequence $\{a_n\}$ contained in A with $a = \lim a_n$. In other words, $\overline{A} = \{\text{limits of convergent sequences in } A\}$.

Idea. *(Scratch work – not part of the written answer.)* **QRS proof – not on the slides.**
For "only if," split on whether $a \in A$: if so, use the constant sequence $a_n = a$; if a is a limit point of A, invoke the Sequential Criterion for Limit Points directly.
For "if," feed the definition of sequence convergence into the ε-Criterion for Closures just proved.

Proof. ($\Rightarrow$) Suppose $a \in \overline{A}$.
By the definition of closure, either $a \in A$ or a is a limit point of A.
If $a \in A$, the constant sequence $a_n = a$ for every $n \in \mathbb{N}$ is contained in A and converges to a.
If instead a is a limit point of A, then by the Sequential Criterion for Limit Points (Theorem 5.1) there is a sequence $\{a_n\}$ contained in $A \setminus \{a\} \subseteq A$ with $a = \lim a_n$.
In either case there is a sequence contained in A that converges to a.
($\Leftarrow$) Suppose $\{a_n\}$ is a sequence contained in A with $a = \lim a_n$.
Let $\varepsilon > 0$ be given.
By the definition of sequence convergence, there is an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$; in particular, the point $x = a_N$ belongs to A and satisfies $|x - a| < \varepsilon$.
Since $\varepsilon > 0$ was arbitrary, we have shown that for every $\varepsilon > 0$ there exists $x \in A$ with $|x - a| < \varepsilon$.
By the ε-Criterion for Closures (Theorem 5.2), $a \in \overline{A}$. □

Common slip. Unlike the Sequential Criterion for Limit Points, the sequence here is allowed to be constant and is allowed to include a itself.
Requiring $a_n \neq a$ here would wrongly exclude every point of A from its own closure.

Theorem 5.4: Limit Points of Closures

Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Then a is a limit point of $\overline{A}$ if and only if a is a limit point of A.

Idea. *(Scratch work – not part of the written answer.)* This proof is on the slides.
The easy direction only needs $A \subseteq \overline{A}$.
The hard direction takes a point $x \in \overline{A}$ close to a and, since $x \in \overline{A}$, pulls a nearby point $y \in A$ out of the ε-Criterion for Closures.
The radius fed into that criterion must be $\min\{|x - a|, \varepsilon - |x - a|\}$, both to force $y \neq a$ and to keep y inside the target ε-ball once the triangle inequality is applied.

Proof. ($\Leftarrow$) Suppose a is a limit point of A.
By definition, for every $\varepsilon > 0$ there exists $x \in A \setminus \{a\}$ with $|x - a| < \varepsilon$.
Since $A \subseteq \overline{A}$ by the definition of closure, the same point x belongs to $\overline{A} \setminus \{a\}$.
Hence, for every $\varepsilon > 0$ there exists $x \in \overline{A} \setminus \{a\}$ with $|x - a| < \varepsilon$, which is the definition of a being a limit point of $\overline{A}$.
($\Rightarrow$) Suppose a is a limit point of $\overline{A}$, so that for every $\varepsilon > 0$ there exists $x \in \overline{A} \setminus \{a\}$ with $|x - a| < \varepsilon$.
Let $\varepsilon > 0$ be given and let x be such a point.
Since $x \neq a$, both $|x - a|$ and $\varepsilon - |x - a|$ are positive, so their minimum is a valid radius.
Since $x \in \overline{A}$, the ε-Criterion for Closures applied to x with this radius gives a point $y \in A$ with

$$|y - x| < \min\{|x - a|,\ \varepsilon - |x - a|\}.$$

Because $|y - x| < |x - a|$, we must have $y \neq a$: if $y = a$ then $|a - x| = |y - x| < |x - a|$, a contradiction.
Hence $y \in A \setminus \{a\}$.
By the triangle inequality,

$$|y - a| \leq |y - x| + |x - a| < (\varepsilon - |x - a|) + |x - a| = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have shown that for every $\varepsilon > 0$ there exists $y \in A \setminus \{a\}$ with $|y - a| < \varepsilon$, which is the definition of a being a limit point of A. □

Common slip. Feeding the raw radius ε (instead of $\min\{|x-a|, \varepsilon - |x-a|\}$) into the ε-Criterion for Closures can produce a y equal to a, or a y whose distance to a exceeds ε once added to $|x-a|$; both failures break the proof.

Theorem 5.5: Corollary: Closures of Closures

Let $A \subseteq \mathbb{R}$. Then $\overline{\overline{A}} = \overline{A}$.

Idea. *(Scratch work – not part of the written answer.)* **QRS proof – not on the slides.**

Unfold $\overline{\overline{A}}$ using the definition of closure, substitute Theorem 5.4 to replace "limit points of $\overline{A}$" by "limit points of A," and note that the limit points of A already lie in $\overline{A}$ by definition.

The reverse containment is a one-line triviality.

Proof. By the definition of closure applied to the set $\overline{A}$,

$$\overline{\overline{A}} = \overline{A} \cup \{\text{limit points of } \overline{A}\}.$$

By the Theorem on Limit Points of Closures (Theorem 5.4), $\{\text{limit points of } \overline{A}\} = \{\text{limit points of } A\}$.

By the definition of closure applied to A, $\{\text{limit points of } A\} \subseteq A \cup \{\text{limit points of } A\} = \overline{A}$.

Combining these,

$$\overline{\overline{A}} = \overline{A} \cup \{\text{limit points of } A\} \subseteq \overline{A} \cup \overline{A} = \overline{A}.$$

For the reverse containment, $\overline{A} \subseteq \overline{A} \cup \{\text{limit points of } \overline{A}\} = \overline{\overline{A}}$ holds trivially, since any set is contained in its union with another set.

Therefore $\overline{\overline{A}} = \overline{A}$. □

Common slip. Both inclusions must be checked; it is not enough to observe that $\overline{A}$ "already looks closed" and skip the reverse containment $\overline{A} \subseteq \overline{\overline{A}}$.

5.3 Suprema Lie in the Closure

Lecture 16 needs the fact that $\sup A$ always lies in $\overline{A}$, in order to prove the Theorem on Extrema of Closed and Bounded Sets, but the slides leave this fact as an exercise.

Tutorial 5, Question 3 asks for exactly this proof, so it is given in full here, via the route the tutorial hint suggests, together with a second route via the Sequential Criterion for Closures.

Theorem 5.6: Suprema Lie in the Closure

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s = \sup A$. Then $s \in \overline{A}$.

Idea. *(Scratch work – not part of the written answer.)* **QRS proof – not on the slides** (Tutorial 5, Question 3; Lecture 16 leaves this as an exercise and then uses it to prove the Theorem on Extrema of Closed and Bounded Sets).

Method 1 feeds the ε-Criterion for Suprema and Infima directly into the ε-Criterion for Closures.

Method 2 builds an explicit sequence in A converging to s by taking $\varepsilon = 1/n$ and invokes the Sequential Criterion for Closures instead.

Proof (Method 1: via the ε-Criterion for Closures). Let $\varepsilon > 0$ be given.

By the ε-Criterion for Suprema and Infima (Lecture 2) applied to $s = \sup A$, there exists $a \in A$ such that $s - \varepsilon < a \leq s$.

Since $a \leq s$, we have $|a - s| = s - a$; since $s - \varepsilon < a$, we have $s - a < \varepsilon$.

Therefore $|a - s| < \varepsilon$.

Since $\varepsilon > 0$ was arbitrary, we have shown that for every $\varepsilon > 0$ there exists $a \in A$ with $|a - s| < \varepsilon$, which is exactly the statement of the ε-Criterion for Closures (Theorem 5.2).

Therefore $s \in \overline{A}$. □

Method 2: via a sequence and the Sequential Criterion for Closures. For each $n \in \mathbb{N}$, apply the ε-Criterion for Suprema and Infima with $\varepsilon = 1/n$ to obtain a point $a_n \in A$ such that $s - 1/n < a_n \leq s$.

This defines a sequence $\{a_n\}$ contained in A.

Since $1/n \to 0$ by the Archimedean Property and $s - 1/n < a_n \leq s$ for every $n \in \mathbb{N}$, the Squeeze Theorem (Lecture 5) gives $a_n \to s$.

Hence $\{a_n\}$ is a sequence contained in A converging to s, so by the Sequential Criterion for Closures (Theorem 5.3), $s \in \overline{A}$. □

Common slip. $\sup A$ need not belong to A itself, only to its closure; do not write $s \in A$ when only $s \in \overline{A}$ has been justified.

Also, the ε-Criterion for Suprema and Infima gives $a > s - \varepsilon$, not $a < s - \varepsilon$; reversing the inequality breaks the bound on $|a - s|$.

5.4 Closed Sets

The Extreme Value Theorem guarantees global extrema for a continuous function on a closed interval $[a, b]$, and this extends readily to finite unions of closed intervals.

The notion of a *closed set* generalises "closed interval" (and finite unions thereof) to the class of sets on which this extremal behaviour can be expected; Chapter 6 will show that "closed and bounded" is exactly the right generalisation.

Definition 5.4: Closed Set

A set $F \subseteq \mathbb{R}$ is *closed* if it contains all its limit points, i.e. $\{\text{limit points of } F\} \subseteq F$. Equivalently, F is closed if F has no limit points outside of F.

Intuition. A closed set never "loses" a point it is approaching: whatever number the set's own points crowd toward is already a member.

A set with no limit points at all (such as $\mathbb{N}$, in the Worked Example below) is closed *vacuously*, since the empty set of limit points is trivially contained in any F.

Theorem 5.7: Closure Criterion for Closed Sets

A set $F \subseteq \mathbb{R}$ is closed if and only if $\overline{F} \subseteq F$. Equivalently, F is closed if and only if $\overline{F} = F$.

Idea. *(Scratch work – not part of the written answer.)* **QRS proof – not on the slides** (the slides only justify this by a five-step remark, with no separately labelled proof).

Unfold $\overline{F} = F \cup \{\text{limit points of } F\}$ directly and compare with the definition of closed; no other theorem is needed.

The "equivalently" clause follows since $F \subseteq \overline{F}$ always holds.

Proof. ($\Rightarrow$) Suppose F is closed, i.e. $\{\text{limit points of } F\} \subseteq F$.

By the definition of closure, $\overline{F} = F \cup \{\text{limit points of } F\}$.

Since $F \subseteq F$ and $\{\text{limit points of } F\} \subseteq F$ by closedness, we get

$$\overline{F} = F \cup \{\text{limit points of } F\} \subseteq F \cup F = F.$$

Therefore $\overline{F} \subseteq F$.

($\Leftarrow$) Suppose $\overline{F} \subseteq F$.

By the definition of closure, $F \subseteq F \cup \{\text{limit points of } F\} = \overline{F}$ always holds; combined with the hypothesis $\overline{F} \subseteq F$, this gives $F = \overline{F}$.

Since $\{\text{limit points of } F\} \subseteq F \cup \{\text{limit points of } F\} = \overline{F} = F$, we conclude $\{\text{limit points of } F\} \subseteq F$, which is the definition of F being closed.

For the "equivalently" clause: since $F \subseteq \overline{F}$ always holds (by the same trivial containment used above), the condition $\overline{F} \subseteq F$ is equivalent to $\overline{F} = F$. □

Common slip. $F \subseteq \overline{F}$ holds for every set F, closed or not; the content of the theorem is the reverse containment $\overline{F} \subseteq F$.

Also, "closed" carries no relation to "open" in this course: it is never true, and never needs to be argued, that "F is not closed" implies "F is open."

Theorem 5.8: Sequential Criterion for Closed Sets

A set $F \subseteq \mathbb{R}$ is closed if and only if every convergent sequence contained in F has its limit in F.

Idea. *(Scratch work – not part of the written answer.)* This proof is on the slides, via an "identify-and-substitute" route rather than an ε-chase.

Let A be the set of limits of all convergent sequences contained in F; identify A with $\overline{F}$ using the Sequential Criterion for Closures, then identify "closed" with "$\overline{F} \subseteq F$" using the Closure Criterion for Closed Sets just proved.

Proof. Let $A := \{\text{limits of convergent sequences contained in } F\}$.

By the Sequential Criterion for Closures (Theorem 5.3), $\overline{F} = A$.

By the Closure Criterion for Closed Sets (Theorem 5.7), F is closed if and only if $\overline{F} \subseteq F$.

Combining these two facts, F is closed if and only if $A \subseteq F$.

Unfolding the definition of A, the statement $A \subseteq F$ says precisely that every real number that is the limit of some convergent sequence contained in F is itself a member of F; in other words, every convergent sequence contained in F has its limit in F.

Therefore F is closed if and only if every convergent sequence contained in F has its limit in F. □

Common slip. The hypothesis is that the sequence's *terms* lie in F; the criterion says nothing about a sequence with terms outside F, even if that sequence happens to converge to a point of F.

5.5 Worked Examples

Worked Example (Tutorial 5, Question 1): limit points, isolated points and closures

For each set, determine its limit points, isolated points, and closure.

(a) $A = \left\{(-1)^n + \dfrac{2}{n} : n \in \mathbb{N}\right\}$.

Write $a_n := (-1)^n + 2/n$ for $n \in \mathbb{N}$, so that $A = \{a_n : n \in \mathbb{N}\}$.

For every $n \in \mathbb{N}$ let $p_n := (-1)^n \in \{-1, 1\}$; then $a_n - p_n = 2/n$, so $|a_n - p_n| = 2/n$.

The terms split by parity: for $k \in \mathbb{N}$,

$$a_{2k} = 1 + \frac{2}{2k} = 1 + \frac{1}{k} > 1, \qquad a_{2k-1} = -1 + \frac{2}{2k-1} \in (-1, 1].$$

Hence $a_n = 1$ if and only if $n = 1$ (for n even $a_n > 1$, and for $n = 2k-1$ odd $a_n = 1 \iff 2k-1 = 1$), and $a_n \neq -1$ for every $n \in \mathbb{N}$, since $a_{2k} > 1$ and $a_{2k-1} > -1$.

Method 1: Sequential Criterion for ± 1, ε-neighbourhood exclusion elsewhere

Proof. *Step 1: 1 and −1 are limit points.* By the Algebraic Limit Theorem and $1/k \to 0$ (Archimedean),

$$\lim_{k\to\infty} a_{2k} = 1 + \lim_{k\to\infty} \frac{1}{k} = 1, \qquad \lim_{k\to\infty} a_{2k-1} = -1 + 2\lim_{k\to\infty} \frac{1}{2k-1} = -1.$$

Since $a_{2k} > 1$ for every $k \in \mathbb{N}$, the sequence $\{a_{2k}\}$ is contained in $A \setminus \{1\}$ and converges to 1; therefore 1 is a limit point of A (Sequential Criterion for Limit Points).
Since $a_{2k-1} > -1$ for every $k \in \mathbb{N}$, the sequence $\{a_{2k-1}\}$ is contained in $A \setminus \{-1\}$ and converges to -1; therefore -1 is a limit point of A (Sequential Criterion for Limit Points).
Step 2: no other number is a limit point. Let $c \in \mathbb{R}$ with $c \notin \{-1, 1\}$, and let $d := \min\{|c-1|, |c+1|\}$; then $d > 0$ and $|c - p_n| \geq d$ for every $n \in \mathbb{N}$.
By the Archimedean Property choose $N \in \mathbb{N}$ with $N > 4/d$; then for every $n \geq N$ we have $2/n \leq 2/N < d/2$.
For every $n \geq N$ the triangle inequality $|c - p_n| \leq |c - a_n| + |a_n - p_n|$ gives

$$|a_n - c| \;\geq\; |c - p_n| - |a_n - p_n| \;\geq\; d - \frac{2}{n} \;>\; d - \frac{d}{2} = \frac{d}{2}.$$

The set $G := \{|a_n - c| : 1 \leq n < N,\ a_n \neq c\}$ is a finite set of positive numbers, so

$$\varepsilon := \min\big(\{d/2\} \cup G\big) > 0.$$

Let $x \in A \setminus \{c\}$; then $x = a_n$ for some $n \in \mathbb{N}$ with $a_n \neq c$.
If $n \geq N$, then $|x - c| > d/2 \geq \varepsilon$; if $n < N$, then $|x - c| \in G$, hence $|x - c| \geq \varepsilon$.
Therefore no $x \in A \setminus \{c\}$ satisfies $|x - c| < \varepsilon$, so c is not a limit point of A (negation of def. of limit point).
Hence the set of limit points of A is exactly $\{-1, 1\}$.
Step 3: isolated points and closure. For $n \geq 2$ we have $a_n \notin \{-1, 1\}$, so $a_n \in A$ is not a limit point; therefore a_n is isolated (def. of isolated point).
The point $a_1 = 1$ belongs to A and is a limit point, so it is not isolated; hence the isolated points are $\{a_n : n \geq 2\} = A \setminus \{1\}$.
By the definition of closure, $\overline{A} = A \cup \{-1, 1\} = A \cup \{-1\}$, because $1 = a_1 \in A$; and $-1 \notin A$, so -1 is a genuinely new point. □

$$\boxed{\text{limit points } = \{-1, 1\}; \quad \text{isolated points } = A \setminus \{1\}; \quad \overline{A} = A \cup \{-1\}.}$$

Method 2: ε-definition for ± 1, contradiction via the Sequential Criterion elsewhere

Proof. *Step 1: 1 and −1 are limit points.* Let $\varepsilon > 0$. By the Archimedean Property choose $k \in \mathbb{N}$ with $k > 2/\varepsilon$; then $1/k < \varepsilon$ and $2/(2k+1) < 1/k < \varepsilon$.
The point $x := a_{2k} = 1 + 1/k$ lies in A, satisfies $x \neq 1$, and $|x - 1| = 1/k < \varepsilon$.
The point $y := a_{2k+1} = -1 + 2/(2k+1)$ lies in A, satisfies $y \neq -1$, and $|y + 1| = 2/(2k+1) < \varepsilon$.
Since $\varepsilon > 0$ was arbitrary, 1 and -1 are limit points of A (def. of limit point).
Step 2: no other number is a limit point. Suppose, for a contradiction, that $c \notin \{-1, 1\}$ is a limit point of A, and let $d := \min\{|c-1|, |c+1|\} > 0$.
By the Sequential Criterion for Limit Points there is a sequence $\{x_m\}$ contained in $A \setminus \{c\}$ with $x_m \to c$; for each $m \in \mathbb{N}$ choose $n_m \in \mathbb{N}$ with $x_m = a_{n_m}$.
Claim: for every $K \in \mathbb{N}$ there is $M \in \mathbb{N}$ such that $n_m \geq K$ for all $m \geq M$. If not, then $n_m < K$ for infinitely many m; since n_m can then take only the values $1, \ldots, K-1$, some value $q < K$ satisfies $n_m = q$ for infinitely many m, which gives a subsequence of $\{x_m\}$ that is constant equal to a_q.
That subsequence converges to a_q, and also to c (subsequences of a convergent sequence, Lecture 7); hence $a_q = c$ (uniqueness of limits), contradicting $x_m \neq c$.
By the Archimedean Property choose $K \in \mathbb{N}$ with $K > 4/d$, and by the Claim choose $M_1 \in \mathbb{N}$ with $n_m \geq K$ for all $m \geq M_1$; then $|x_m - p_{n_m}| = 2/n_m \leq 2/K < d/2$ for all $m \geq M_1$.
By the definition of $x_m \to c$ with $\varepsilon = d/2$, choose $M_2 \in \mathbb{N}$ with $|x_m - c| < d/2$ for all $m \geq M_2$.

For $m := \max\{M_1, M_2\}$ the triangle inequality gives

$$d \;\le\; |c - p_{n_m}| \;\le\; |c - x_m| + |x_m - p_{n_m}| \;<\; \frac{d}{2} + \frac{d}{2} = d,$$

which is impossible. Therefore no $c \notin \{-1, 1\}$ is a limit point, and the limit points are exactly $\{-1, 1\}$.
Step 3: isolated points and closure. The points a_n with $n \ge 2$ lie in A and are not in $\{-1, 1\}$, so they are isolated, while $a_1 = 1 \in A$ is a limit point and hence not isolated.
Therefore the isolated points form $A \setminus \{1\}$, and $\overline{A} = A \cup \{-1, 1\} = A \cup \{-1\}$ since $1 \in A$. □

$$\boxed{\text{limit points } = \{-1, 1\}; \quad \text{isolated points } = A \setminus \{1\}; \quad \overline{A} = A \cup \{-1\}.}$$

(b) $B = \{x \in \mathbb{Q} : 0 < x < 1\}$.
Method 1: ε-definition with the Density of $\mathbb{Q}$

Proof. *Step 1: every $x \in [0, 1]$ is a limit point.* Let $x \in [0, 1]$ and let $\varepsilon > 0$.
If $x < 1$, then $x < \min\{x + \varepsilon, 1\}$, so by the Density of $\mathbb{Q}$ there is $q \in \mathbb{Q}$ with $x < q < \min\{x + \varepsilon, 1\}$. Then $0 \le x < q < 1$, so $q \in B$; also $q \ne x$, and $|q - x| = q - x < \varepsilon$.
If $x = 1$, then $\max\{1 - \varepsilon, 0\} < 1$, so by the Density of $\mathbb{Q}$ there is $q \in \mathbb{Q}$ with $\max\{1 - \varepsilon, 0\} < q < 1$. Then $0 < q < 1$, so $q \in B$; also $q \ne 1$, and $|q - 1| = 1 - q < \varepsilon$.
In both cases there is $q \in B \setminus \{x\}$ with $|q - x| < \varepsilon$; since $\varepsilon > 0$ was arbitrary, x is a limit point of B (def. of limit point).
Step 2: no $x \notin [0, 1]$ is a limit point. Let $x > 1$ and put $\delta := x - 1 > 0$; every $q \in B$ satisfies $q < 1$, hence $|q - x| = x - q > x - 1 = \delta$.
Let $x < 0$ and put $\delta := -x > 0$; every $q \in B$ satisfies $q > 0$, hence $|q - x| = q - x > -x = \delta$.
In both cases no point of $B \setminus \{x\}$ lies within δ of x, so x is not a limit point of B. Therefore the limit points of B form exactly $[0, 1]$.
Step 3: isolated points and closure. Since $B \subseteq [0, 1]$, every point of B is a limit point of B, hence no point of B is isolated.
By the definition of closure, $\overline{B} = B \cup [0, 1] = [0, 1]$. □

$$\boxed{\text{limit points } = [0, 1]; \quad \text{isolated points } = \emptyset; \quad \overline{B} = [0, 1].}$$

Method 2: Sequential Criteria with the Order Limit Theorem

Proof. *Step 1: every $x \in [0, 1]$ is a limit point.* Let $x \in [0, 1]$.
If $x < 1$, for each $n \in \mathbb{N}$ use the Density of $\mathbb{Q}$ to choose $q_n \in \mathbb{Q}$ with $x < q_n < \min\{x + 1/n, 1\}$; then $q_n \in B \setminus \{x\}$ and $0 < q_n - x < 1/n$.
If $x = 1$, for each $n \in \mathbb{N}$ use the Density of $\mathbb{Q}$ to choose $q_n \in \mathbb{Q}$ with $\max\{1 - 1/n, 0\} < q_n < 1$; then $q_n \in B \setminus \{1\}$ and $0 < 1 - q_n < 1/n$.
In both cases $|q_n - x| < 1/n$ for every $n \in \mathbb{N}$, and $1/n \to 0$ (Archimedean), so $q_n \to x$ by the Squeeze Theorem.
Therefore $\{q_n\}$ is a sequence contained in $B \setminus \{x\}$ with limit x, and x is a limit point of B (Sequential Criterion for Limit Points).
Step 2: no $x \notin [0, 1]$ is a limit point. Let $\{x_n\}$ be any sequence contained in B that converges to some $x \in \mathbb{R}$.
Then $0 \le x_n \le 1$ for every $n \in \mathbb{N}$, so $0 \le x \le 1$ (Order Limit Theorem).
Hence a number $x \notin [0, 1]$ is not the limit of any sequence contained in B, in particular of none contained in $B \setminus \{x\}$; therefore x is not a limit point of B (Sequential Criterion for Limit Points).
Step 3: isolated points and closure. Steps 1 and 2 show that the limit points of B form $[0, 1] \supseteq B$, so no point of B is isolated.
By the Sequential Criterion for Closures, $\overline{B}$ is the set of limits of sequences contained in B; by Step 2 this set lies in $[0, 1]$, and by Step 1 it contains $[0, 1]$, so $\overline{B} = [0, 1]$. □

$$\boxed{\text{limit points } = [0, 1]; \quad \text{isolated points } = \emptyset; \quad \overline{B} = [0, 1].}$$

(c) $C = \{x \in \mathbb{R} : x \ne 0\}$.
Method 1: ε-definition with an explicit witness

Proof. Let $a \in \mathbb{R}$ and let $\varepsilon > 0$.
The two numbers $a + \varepsilon/2$ and $a + \varepsilon/3$ are distinct, so at most one of them equals 0; let x be one of them with $x \neq 0$.
Then $x \in C$, and $x \neq a$ because $x - a \in \{\varepsilon/2, \varepsilon/3\}$ is positive.
Moreover $|x - a| \leq \varepsilon/2 < \varepsilon$.
Hence $x \in C \setminus \{a\}$ and $|x - a| < \varepsilon$; since $\varepsilon > 0$ was arbitrary, a is a limit point of C (def. of limit point).
Since $a \in \mathbb{R}$ was arbitrary, every real number, including $0 \notin C$, is a limit point of C.
Therefore every point of C is a limit point of C, so no point of C is isolated, and $\overline{C} = C \cup \mathbb{R} = \mathbb{R}$ by the definition of closure. □

$$\boxed{\text{limit points } = \mathbb{R}; \quad \text{isolated points } = \emptyset; \quad \overline{C} = \mathbb{R}.}$$

Method 2: Sequential Criterion with a sign-preserving sequence

Proof. Let $a \in \mathbb{R}$.
If $a \geq 0$, define $x_n := a + 1/n$ for $n \in \mathbb{N}$; then $x_n > a \geq 0$, so $x_n \neq 0$ and $x_n \neq a$.
If $a < 0$, define $x_n := a - 1/n$ for $n \in \mathbb{N}$; then $x_n < a < 0$, so $x_n \neq 0$ and $x_n \neq a$.
In both cases $\{x_n\}$ is contained in $C \setminus \{a\}$, and $|x_n - a| = 1/n \to 0$ (Archimedean), so $x_n \to a$.
Therefore a is a limit point of C (Sequential Criterion for Limit Points); since $a \in \mathbb{R}$ was arbitrary, the limit points of C form all of $\mathbb{R}$.
Hence no point of C is isolated, and $\overline{C} = C \cup \mathbb{R} = \mathbb{R}$ by the definition of closure. □

$$\boxed{\text{limit points } = \mathbb{R}; \quad \text{isolated points } = \emptyset; \quad \overline{C} = \mathbb{R}.}$$

(d) $D = \left\{1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$.
Write $s_n := \sum_{k=1}^{n} 1/k^2$, so that $D = \{s_n : n \in \mathbb{N}\}$.
The sequence is strictly increasing, because $s_{n+1} - s_n = 1/(n+1)^2 > 0$ for every $n \in \mathbb{N}$.
It is bounded above: for $k \geq 2$ we have $k^2 \geq k(k-1)$, hence $\frac{1}{k^2} \leq \frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$, and the sum telescopes:

$$s_n \;\leq\; 1 + \sum_{k=2}^{n}\Big(\frac{1}{k-1} - \frac{1}{k}\Big) \;=\; 1 + \Big(1 - \frac{1}{n}\Big) \;=\; 2 - \frac{1}{n} \;<\; 2.$$

Therefore $\{s_n\}$ converges to some $S \in \mathbb{R}$ (MCT); this $S = \sum_{n=1}^{\infty} 1/n^2$.
For fixed $n \in \mathbb{N}$ we have $s_k \geq s_n$ for all $k \geq n$, so letting $k \to \infty$ gives $S \geq s_n$ (Order Limit Theorem).
Hence $s_n < s_{n+1} \leq S$ for every $n \in \mathbb{N}$, so $s_n < S$ for every $n \in \mathbb{N}$, and therefore $S \notin D$.
Method 1: Sequential Criterion for S, ε-neighbourhood exclusion elsewhere

Proof. *Step 1: S is a limit point.* Since $s_n < S$ for every $n \in \mathbb{N}$, the sequence $\{s_n\}$ is contained in $D \setminus \{S\}$, and $s_n \to S$.
Therefore S is a limit point of D (Sequential Criterion for Limit Points).
Step 2: no $c \neq S$ is a limit point. Let $c \in \mathbb{R}$ with $c \neq S$, and let $d := |c - S| > 0$.
Since $s_n \to S$, applying the definition of limit with $\varepsilon = d/2$ gives $N \in \mathbb{N}$ such that $|s_n - S| < d/2$ for all $n \geq N$.
For every $n \geq N$ the triangle inequality gives

$$|s_n - c| \;\geq\; |c - S| - |s_n - S| \;>\; d - \frac{d}{2} = \frac{d}{2}.$$

The set $G := \{|s_n - c| : 1 \leq n < N,\ s_n \neq c\}$ is a finite set of positive numbers, so $\varepsilon := \min\big(\{d/2\} \cup G\big) > 0$.
Let $x \in D \setminus \{c\}$, say $x = s_n$ with $s_n \neq c$; if $n \geq N$ then $|x - c| > d/2 \geq \varepsilon$, and if $n < N$ then $|x - c| \in G$, so $|x - c| \geq \varepsilon$.

Therefore no point of $D \setminus \{c\}$ lies within ε of c, so c is not a limit point of D; the only limit point of D is S.
Step 3: isolated points and closure. Every $s_n \in D$ satisfies $s_n \neq S$, so s_n is not a limit point of D; therefore every point of D is isolated.
By the definition of closure, $\overline{D} = D \cup \{S\}$. □

$$\boxed{\text{limit points } = \{S\},\ S = \textstyle\sum_{n=1}^{\infty} \frac{1}{n^2};\quad \text{isolated points } = D;\quad \overline{D} = D \cup \{S\}.}$$

Method 2: ε-definition for S, contradiction via the Sequential Criterion elsewhere

Proof. *Step 1: S is a limit point.* Let $\varepsilon > 0$. Since $s_n \to S$, there is $N \in \mathbb{N}$ with $|s_n - S| < \varepsilon$ for all $n \geq N$ (def. of limit).
The point $x := s_N$ lies in D, satisfies $x \neq S$ because $s_N < S$, and $|x - S| < \varepsilon$.
Since $\varepsilon > 0$ was arbitrary, S is a limit point of D (def. of limit point).
Step 2: no $c \neq S$ is a limit point. Suppose, for a contradiction, that $c \neq S$ is a limit point of D.
By the Sequential Criterion for Limit Points there is a sequence $\{x_m\}$ contained in $D \setminus \{c\}$ with $x_m \to c$; for each $m \in \mathbb{N}$ choose $n_m \in \mathbb{N}$ with $x_m = s_{n_m}$.
Claim: for every $K \in \mathbb{N}$ there is $M \in \mathbb{N}$ such that $n_m \geq K$ for all $m \geq M$. If not, then $n_m < K$ for infinitely many m, so some value $q < K$ satisfies $n_m = q$ for infinitely many m, giving a subsequence of $\{x_m\}$ that is constant equal to s_q.
That subsequence converges to s_q and also to c (subsequences, Lecture 7), so $s_q = c$ (uniqueness of limits), contradicting $x_m \neq c$.
Now let $\varepsilon > 0$; choose $N \in \mathbb{N}$ with $|s_n - S| < \varepsilon$ for all $n \geq N$, and by the Claim choose $M \in \mathbb{N}$ with $n_m \geq N$ for all $m \geq M$.
Then $|x_m - S| = |s_{n_m} - S| < \varepsilon$ for all $m \geq M$; since $\varepsilon > 0$ was arbitrary, $x_m \to S$.
Hence $x_m \to c$ and $x_m \to S$, so $c = S$ (uniqueness of limits), contradicting $c \neq S$. Therefore S is the only limit point of D.
Step 3: isolated points and closure. Every point of D differs from S, so it is not a limit point and is therefore isolated; and $\overline{D} = D \cup \{S\}$ by the definition of closure. □

$$\boxed{\text{limit points } = \{S\},\ S = \textstyle\sum_{n=1}^{\infty} \frac{1}{n^2};\quad \text{isolated points } = D;\quad \overline{D} = D \cup \{S\}.}$$

(e) $E = \left\{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} : n \in \mathbb{N}\right\}$.
Write $t_n := \sum_{k=1}^{n} 1/k$, so that $E = \{t_n : n \in \mathbb{N}\}$; the sequence is strictly increasing, because $t_{n+1} - t_n = 1/(n+1) > 0$.
It is unbounded above: for $j \in \mathbb{N}$ the block $\sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i}$ has 2^{j-1} terms, each at least $\frac{1}{2^j}$, so the block is at least $\frac{1}{2}$, and therefore for every $K \in \mathbb{N}$

$$t_{2^K} \;=\; 1 + \sum_{j=1}^{K} \sum_{i=2^{j-1}+1}^{2^j} \frac{1}{i} \;\geq\; 1 + \frac{K}{2}.$$

Method 1: ε-neighbourhood exclusion using the finitely many small terms

Proof. Let $c \in \mathbb{R}$. By the Archimedean Property choose $K \in \mathbb{N}$ with $K > 2\,|c|$, and put $N := 2^K$.
Then $t_N \geq 1 + K/2 > 1 + |c| \geq c + 1$, and since $\{t_n\}$ is increasing, $t_n \geq t_N > c + 1$ for every $n \geq N$.
Hence $|t_n - c| = t_n - c > 1$ for every $n \geq N$.
The set $G := \{|t_n - c| : 1 \leq n < N,\ t_n \neq c\}$ is a finite set of positive numbers, so $\varepsilon := \min(\{1\} \cup G) > 0$.
Let $x \in E \setminus \{c\}$, say $x = t_n$ with $t_n \neq c$; if $n \geq N$ then $|x - c| > 1 \geq \varepsilon$, and if $n < N$ then $|x - c| \in G$, so $|x - c| \geq \varepsilon$.
Therefore no point of $E \setminus \{c\}$ lies within ε of c, so c is not a limit point of E; since $c \in \mathbb{R}$ was arbitrary, E has no limit points.

Hence every point of E belongs to E and is not a limit point, so every point of E is isolated, and $\overline{E} = E \cup \emptyset = E$ by the definition of closure. □

$$\boxed{\text{limit points } = \emptyset; \quad \text{isolated points } = E; \quad \overline{E} = E.}$$

Method 2: Contradiction via the Sequential Criterion

Proof. Suppose, for a contradiction, that some $c \in \mathbb{R}$ is a limit point of E.
By the Sequential Criterion for Limit Points there is a sequence $\{x_m\}$ contained in $E \setminus \{c\}$ with $x_m \to c$; for each $m \in \mathbb{N}$ choose $n_m \in \mathbb{N}$ with $x_m = t_{n_m}$.
Claim: for every $K \in \mathbb{N}$ there is $M \in \mathbb{N}$ such that $n_m \geq K$ for all $m \geq M$. If not, then $n_m < K$ for infinitely many m, so some value $q < K$ satisfies $n_m = q$ for infinitely many m, giving a subsequence of $\{x_m\}$ that is constant equal to t_q.
That subsequence converges to t_q and also to c (subsequences, Lecture 7), so $t_q = c$ (uniqueness of limits), contradicting $x_m \neq c$.
By the Archimedean Property choose $L \in \mathbb{N}$ with $L > 2|c|$; then $t_{2^L} \geq 1 + L/2 > c + 1$, so $t_n > c + 1$ for every $n \geq 2^L$ because $\{t_n\}$ is increasing.
By the Claim with $K = 2^L$ there is $M \in \mathbb{N}$ such that $n_m \geq 2^L$ for all $m \geq M$, hence $|x_m - c| = t_{n_m} - c > 1$ for all $m \geq M$.
But $x_m \to c$ gives, with $\varepsilon = 1$, some $M' \in \mathbb{N}$ with $|x_m - c| < 1$ for all $m \geq M'$; at $m = \max\{M, M'\}$ both $|x_m - c| > 1$ and $|x_m - c| < 1$ hold, which is impossible.
Therefore E has no limit points, every point of E is isolated, and $\overline{E} = E$ by the definition of closure. □

$$\boxed{\text{limit points } = \emptyset; \quad \text{isolated points } = E; \quad \overline{E} = E.}$$

Worked Example: a limit point proved directly from the definition

Show, directly from the definition of limit point (not the Sequential Criterion), that 0 is a limit point of $A = \{1/n : n \in \mathbb{N}\}$ but $0 \notin A$, and that every point of A is isolated.
Method 1: Direct ε-witness and explicit gaps

Proof that 0 is a limit point. Let $\varepsilon > 0$. By the Archimedean Property choose $N \in \mathbb{N}$ with $N > 1/\varepsilon$; hence $1/N < \varepsilon$.
The point $x := 1/N$ belongs to A, and $x \neq 0$ because $1/N > 0$; therefore $x \in A \setminus \{0\}$ and $|x - 0| = 1/N < \varepsilon$.
Since $\varepsilon > 0$ was arbitrary, for every $\varepsilon > 0$ there is $x \in A \setminus \{0\}$ with $|x - 0| < \varepsilon$, so 0 is a limit point of A (def. of limit point).
Finally $1/n > 0$ for every $n \in \mathbb{N}$, hence $1/n \neq 0$ for every $n \in \mathbb{N}$, and therefore $0 \notin A$. □

Proof that every point of A is isolated. Fix $m \in \mathbb{N}$. If $m \geq 2$, let $g_m := \min\{\frac{1}{m} - \frac{1}{m+1}, \frac{1}{m-1} - \frac{1}{m}\}$; if $m = 1$, let $g_1 := 1 - \frac{1}{2}$. In both cases $g_m > 0$.
Let $n \in \mathbb{N}$ with $n \neq m$.
If $n > m$, then $n \geq m + 1$, so $1/n \leq 1/(m+1)$, and therefore $|1/n - 1/m| = 1/m - 1/n \geq 1/m - 1/(m+1) \geq g_m$.
If $n < m$ (possible only when $m \geq 2$), then $n \leq m - 1$, so $1/n \geq 1/(m-1)$, and therefore $|1/n - 1/m| = 1/n - 1/m \geq 1/(m-1) - 1/m \geq g_m$.
Hence $|1/n - 1/m| \geq g_m$ for every $n \in \mathbb{N}$ with $n \neq m$, so no point of $A \setminus \{1/m\}$ lies within g_m of $1/m$.
Therefore $1/m$ is not a limit point of A; since $1/m \in A$, the point $1/m$ is isolated (def. of isolated point). □
Method 2: Contradiction with the Archimedean Property and the identity $|1/n - 1/m| = |m - n|/(mn)$

Proof that 0 is a limit point. Suppose, for a contradiction, that 0 is not a limit point of A.

Negating the definition, there is $\varepsilon_0 > 0$ such that every $x \in A \setminus \{0\}$ satisfies $|x - 0| \geq \varepsilon_0$.
Since $1/n \in A \setminus \{0\}$ for every $n \in \mathbb{N}$, this gives $1/n \geq \varepsilon_0$, that is $n \leq 1/\varepsilon_0$, for every $n \in \mathbb{N}$.
Hence $1/\varepsilon_0$ is an upper bound for $\mathbb{N}$, which contradicts the Archimedean Property; therefore 0 is a limit point of A.
Since $1/n > 0$ for every $n \in \mathbb{N}$, we have $0 \notin A$. □

Proof that every point of A is isolated. Fix $m \in \mathbb{N}$ and suppose, for a contradiction, that $1/m$ is a limit point of A.
Applying the definition of limit point with $\varepsilon := 1/(2m^2)$, there is $n \in \mathbb{N}$ with $n \neq m$ and $|1/n - 1/m| < 1/(2m^2)$.
Since $1/(2m^2) \leq 1/(2m)$, this gives $1/n > 1/m - 1/(2m) = 1/(2m)$, hence $n < 2m$.
Since $n \neq m$ are integers, $|m - n| \geq 1$, and therefore

$$\left|\frac{1}{n} - \frac{1}{m}\right| = \frac{|m-n|}{mn} \geq \frac{1}{mn} > \frac{1}{m \cdot 2m} = \frac{1}{2m^2},$$

which contradicts $|1/n - 1/m| < 1/(2m^2)$.
Therefore $1/m$ is not a limit point of A; since $1/m \in A$, the point $1/m$ is isolated (def. of isolated point). □

Worked Example (Sample Midterm, Question 2(b)): true or false

Claim: $C = \{x \in \mathbb{R} : x \neq 0\}$ is closed.
Method 1: A limit point outside C (def. of closed set)

Proof. We show that 0 is a limit point of C. Let $\varepsilon > 0$ and put $x := \varepsilon/2$.
Then $x \neq 0$, so $x \in C \setminus \{0\}$, and $|x - 0| = \varepsilon/2 < \varepsilon$.
Since $\varepsilon > 0$ was arbitrary, for every $\varepsilon > 0$ there is $x \in C \setminus \{0\}$ with $|x - 0| < \varepsilon$; therefore 0 is a limit point of C (def. of limit point).
But $0 \notin C$ by the definition of C; hence $\{\text{limit points of } C\} \not\subseteq C$.
Therefore C does not contain all its limit points, so C is not closed (def. of closed set), and the claim is false. □
Answer. False.
Method 2: Sequential Criterion for Closed Sets

Proof. Define $x_n := 1/n$ for $n \in \mathbb{N}$. Since $1/n > 0$, we have $x_n \neq 0$, so $\{x_n\}$ is a sequence contained in C.
By the Archimedean Property, $x_n = 1/n \to 0$, so $\{x_n\}$ is a convergent sequence contained in C with limit 0.
But $0 \notin C$, so not every convergent sequence contained in C has its limit in C.
Therefore C is not closed (Sequential Criterion for Closed Sets), and the claim is false. □
Answer. False.

Worked Example (Tutorial 5, Question 4): closedness under union and arbitrary intersection

(a) If F_1 and F_2 are closed sets, then $F_1 \cup F_2$ is closed.
Method 1: Sequential Criterion for Closed Sets with a subsequence

Proof. Let $\{x_n\}$ be a sequence contained in $F_1 \cup F_2$ with $x_n \to x$ for some $x \in \mathbb{R}$.
For every $n \in \mathbb{N}$ we have $x_n \in F_1$ or $x_n \in F_2$; since there are infinitely many indices n, at least one of the two index sets $\{n : x_n \in F_1\}$, $\{n : x_n \in F_2\}$ is infinite.
Relabelling F_1 and F_2 if necessary, assume $\{n : x_n \in F_1\}$ is infinite, and list its elements as $n_1 < n_2 < n_3 < \cdots$.
Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ contained in F_1, and $x_{n_k} \to x$ because every subsequence of a

convergent sequence converges to the same limit (Lecture 7).
Since F_1 is closed, every convergent sequence contained in F_1 has its limit in F_1 (Sequential Criterion for Closed Sets); hence $x \in F_1 \subseteq F_1 \cup F_2$.
Since $\{x_n\}$ was an arbitrary convergent sequence contained in $F_1 \cup F_2$, every such sequence has its limit in $F_1 \cup F_2$.
Therefore $F_1 \cup F_2$ is closed (Sequential Criterion for Closed Sets). □
Method 2: Contradiction via the limit-point definition with $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$

Proof. Let x be a limit point of $F_1 \cup F_2$, and suppose, for a contradiction, that $x \notin F_1 \cup F_2$, so $x \notin F_1$ and $x \notin F_2$.
Since F_1 is closed and $x \notin F_1$, the point x is not a limit point of F_1 (def. of closed set); negating the definition of limit point, there is $\varepsilon_1 > 0$ such that every $y \in F_1 \setminus \{x\}$ satisfies $|y - x| \geq \varepsilon_1$.
In the same way, since F_2 is closed and $x \notin F_2$, there is $\varepsilon_2 > 0$ such that every $y \in F_2 \setminus \{x\}$ satisfies $|y - x| \geq \varepsilon_2$.
Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\} > 0$, and let $y \in (F_1 \cup F_2) \setminus \{x\}$.
If $y \in F_1$, then $|y - x| \geq \varepsilon_1 \geq \varepsilon$; if $y \in F_2$, then $|y - x| \geq \varepsilon_2 \geq \varepsilon$.
Hence no point of $(F_1 \cup F_2) \setminus \{x\}$ lies within ε of x, so x is not a limit point of $F_1 \cup F_2$, which contradicts the choice of x.
Therefore every limit point of $F_1 \cup F_2$ belongs to $F_1 \cup F_2$, and $F_1 \cup F_2$ is closed (def. of closed set). □
(b) If $\{F_t\}_{t \in T}$ is an arbitrary collection of closed sets, then $\bigcap_{t \in T} F_t$ is closed.
Method 1: Sequential Criterion for Closed Sets

Proof. Let $\{x_n\}$ be a sequence contained in $\bigcap_{t \in T} F_t$ with $x_n \to x$ for some $x \in \mathbb{R}$, and fix $t \in T$.
Since $\bigcap_{s \in T} F_s \subseteq F_t$, the sequence $\{x_n\}$ is contained in F_t.
Since F_t is closed and $\{x_n\}$ is a convergent sequence contained in F_t, its limit satisfies $x \in F_t$ (Sequential Criterion for Closed Sets).
Since $t \in T$ was arbitrary, $x \in F_t$ for every $t \in T$, that is, $x \in \bigcap_{t \in T} F_t$.
Therefore every convergent sequence contained in $\bigcap_{t \in T} F_t$ has its limit in $\bigcap_{t \in T} F_t$, so the intersection is closed (Sequential Criterion for Closed Sets). □
Method 2: Limit-point definition

Proof. Let x be a limit point of $\bigcap_{t \in T} F_t$, and fix $t \in T$.
Let $\varepsilon > 0$. By the definition of limit point there is $y \in \left(\bigcap_{s \in T} F_s\right) \setminus \{x\}$ with $|y - x| < \varepsilon$.
Since $\bigcap_{s \in T} F_s \subseteq F_t$, this y belongs to $F_t \setminus \{x\}$ and satisfies $|y - x| < \varepsilon$.
Since $\varepsilon > 0$ was arbitrary, x is a limit point of F_t (def. of limit point); since F_t is closed, $x \in F_t$ (def. of closed set).
Since $t \in T$ was arbitrary, $x \in F_t$ for every $t \in T$, hence $x \in \bigcap_{t \in T} F_t$.
Therefore $\bigcap_{t \in T} F_t$ contains all its limit points, so it is closed (def. of closed set). □

Worked Example: $\mathbb{N}$ is closed, with no limit points at all

Show that $\mathbb{N} \subseteq \mathbb{R}$ is closed.
Method 1: Sequential Criterion for Closed Sets via the Cauchy Criterion

Proof. Let $\{x_n\}$ be a sequence contained in $\mathbb{N}$ with $x_n \to x$ for some $x \in \mathbb{R}$.
Since $\{x_n\}$ converges, it is a Cauchy sequence (Cauchy Criterion, Lecture 6); applying the definition of Cauchy sequence with $\varepsilon = \frac{1}{2}$ gives $N_0 \in \mathbb{N}$ such that $|x_n - x_m| < \frac{1}{2}$ for all $n, m \geq N_0$.
Two distinct natural numbers differ by at least 1; hence, for $n, m \geq N_0$, the inequality $|x_n - x_m| < \frac{1}{2} < 1$ forces $x_n = x_m$.
Therefore $x_n = k := x_{N_0}$ for all $n \geq N_0$, where $k \in \mathbb{N}$; the sequence is eventually constant, so $x = \lim x_n = k \in \mathbb{N}$.
Since $\{x_n\}$ was an arbitrary convergent sequence contained in $\mathbb{N}$, every convergent sequence contained in $\mathbb{N}$ has its limit in $\mathbb{N}$.
Therefore $\mathbb{N}$ is closed (Sequential Criterion for Closed Sets). □

Method 2: No limit points, so closed vacuously (def. of closed set)

Proof. Let $c \in \mathbb{R}$. If $n, n' \in \mathbb{N}$ both satisfy $|n-c| < \frac{1}{2}$ and $|n'-c| < \frac{1}{2}$, then $|n-n'| \leq |n-c| + |c-n'| < 1$, hence $n = n'$, because distinct natural numbers differ by at least 1.
Therefore at most one natural number lies within $\frac{1}{2}$ of c.
If no natural number other than possibly c itself lies within $\frac{1}{2}$ of c, put $\varepsilon := \frac{1}{2}$.
Otherwise there is exactly one $n_0 \in \mathbb{N}$ with $n_0 \neq c$ and $|n_0 - c| < \frac{1}{2}$; put $\varepsilon := |n_0 - c|$, so that $0 < \varepsilon < \frac{1}{2}$.
Let $y \in \mathbb{N} \setminus \{c\}$. In the first case $|y-c| \geq \frac{1}{2} = \varepsilon$; in the second case either $y = n_0$, so $|y-c| = \varepsilon$, or $y \neq n_0$, so $|y-c| \geq \frac{1}{2} > \varepsilon$. Hence $|y-c| \geq \varepsilon$ in every case.
Therefore no point of $\mathbb{N} \setminus \{c\}$ lies within ε of c, so c is not a limit point of $\mathbb{N}$; since $c \in \mathbb{R}$ was arbitrary, $\mathbb{N}$ has no limit points.
Hence $\{\text{limit points of } \mathbb{N}\} = \emptyset \subseteq \mathbb{N}$, and $\mathbb{N}$ is closed (def. of closed set). □

5.6 Chapter Summary

Key Formula: equivalent characterisations

For $A \subseteq \mathbb{R}$, $a \in \mathbb{R}$, and $F \subseteq \mathbb{R}$:

$$\begin{aligned} a \text{ is a limit point of } A &\iff \forall \varepsilon > 0\ \exists x \in A \setminus \{a\},\ |x-a| < \varepsilon \\ &\iff \exists\, \{a_n\} \subseteq A \setminus \{a\},\ a = \lim a_n. \end{aligned}$$

$$\overline{A} := A \cup \{\text{limit points of } A\}; \qquad \begin{aligned} a \in \overline{A} &\iff \forall \varepsilon > 0\ \exists x \in A,\ |x-a| < \varepsilon \\ &\iff \exists\, \{a_n\} \subseteq A,\ a = \lim a_n. \end{aligned}$$

$$\overline{\overline{A}} = \overline{A}. \qquad A \text{ nonempty, bdd. above} \implies \sup A \in \overline{A}.$$

$$\begin{aligned} F \text{ is closed} &\iff \{\text{limit points of } F\} \subseteq F \iff \overline{F} \subseteq F \iff \overline{F} = F \\ &\iff \text{every convergent sequence in } F \text{ has its limit in } F. \end{aligned}$$

Exam Focus: Chapter 5 takeaways

- There is no "open set" in this course: never answer a closedness question by discussing complements or open covers.
- To disprove "F is closed," exhibit *one* limit point of F outside F (the fastest route on true/false questions, as in the Sample Midterm example above).
- To prove "F is closed," the Sequential Criterion for Closed Sets is usually faster than chasing the raw limit-point definition: take an arbitrary convergent sequence in F and show its limit is in F.
- A limit point need not be in the set; an isolated point must be. Every point of a set is exactly one of the two.
- $\sup A \in \overline{A}$ always (for A nonempty and bounded above), but $\sup A$ need not be in A itself; this is the fact Lecture 16 needs (and leaves as an exercise) for the Theorem on Extrema of Closed and Bounded Sets.
- Closure is idempotent: $\overline{\overline{A}} = \overline{A}$. Applying $\overline{\cdot}$ a second time never adds anything new.

6 Lectures 16–17: Compact Sets

This chapter answers three questions: (1) *when does a nonempty set of real numbers actually attain a largest and a smallest element, rather than merely a supremum and an infimum?* (2) *what single sequential property of a set turns out to be exactly equivalent to being both closed and bounded?* (3) *why must an infinite nested chain of closed and bounded sets always share a common point?* It builds on Chapter 5 (limit points, closures, and closed sets, Section 5) and prepares the midterm synthesis chapter, which reviews Lectures 1–17 as a whole.

Throughout, "compact" always means *(sequentially) compact* in the sense defined below; this is the only notion of compactness used in this course, and it is defined entirely in terms of sequences, without reference to open sets or open covers.

6.1 Extrema of Closed and Bounded Sets

A nonempty, bounded-above set always has a supremum, by the Axiom of Completeness; but a supremum need not belong to the set, so a set need not have a maximum. This lecture identifies exactly the extra ingredient – closedness – that upgrades a supremum into an honest maximum.

Extension: Why Boundedness Matters (Motivation)

The Extreme Value Theorem (Lectures 18–20, out of scope for the midterm) asserts that a *continuous* function on a closed and bounded interval attains a maximum and a minimum. Boundedness of the domain is essential there too: the function $f : x \in \mathbb{R} \mapsto \dfrac{1}{1+x^2}$ is continuous and bounded below by 0, but has no global minimum on any closed interval $[a, \infty)$, since $f(x) \to 0$ as $x \to \infty$ without f ever equalling 0. This chapter proves the analogous, purely set-theoretic fact – that closedness and boundedness of a *set* (not yet a function) already guarantee attained extrema – which is one of the two ingredients the Extreme Value Theorem itself will later combine with continuity. No continuity or function-limit result from Lectures 18–20 is used anywhere below.

Recall (Maximum versus Supremum). An upper bound for a set A that also belongs to A is called the maximum of A, written $\max A$; the supremum $\sup A$ is the least upper bound for A, which may or may not belong to A. Consequently $\max A$, when it exists, always equals $\sup A$, and conversely $\sup A$ is the maximum of A if and only if $\sup A \in A$. The theorem below identifies exactly when this last condition holds.

Theorem 6.1: Extrema of Closed and Bounded Sets

A nonempty, closed, and bounded set $K \subseteq \mathbb{R}$ has a maximum and a minimum.

By the identity $\min K = -\max(-K)$, where $-K := \{-a : a \in K\}$ is again nonempty, closed, and bounded whenever K is, it suffices to prove that every nonempty closed and bounded set has a maximum; the proof below does exactly this and then transfers the conclusion to the minimum.

Idea. *(Scratch work – not part of the written answer.)* We already have almost every ingredient from Chapter 5. The Axiom of Completeness gives $s := \sup K$. The Closure Criterion for Closed Sets says that a closed set contains its own closure, $\overline{K} \subseteq K$, so it suffices to show $s \in \overline{K}$: once that holds, s is an upper bound for K that also lies in K, i.e. the maximum. The one missing ingredient is the fact that a supremum always lies in the closure of the set that produced it (Theorem 5.6, Suprema Lie in the Closure); the slides leave this as an exercise, so it is a QRS proof – not on the slides. For self-containedness we reprove it here directly from the ε-Criterion for Suprema: given $\varepsilon > 0$, that criterion hands us a point of K strictly above $s - \varepsilon$, and since s is an upper bound, that same point cannot exceed s, so it is automatically within ε of s on both sides – exactly the ε-Criterion for Closures.

Proof. Let $K \subseteq \mathbb{R}$ be nonempty, closed, and bounded. Since K is nonempty and bounded above, the Axiom of Completeness asserts that $s := \sup K$ exists.

We first show that $s \in \overline{K}$. This fact – that the supremum of any nonempty, bounded-above set lies in the closure of that set – was proved directly in Chapter 5 (Section 5, "Suprema lie in the closure"); since the slides leave it as an exercise at exactly this point in the course, we reprove it here from first principles so that this chapter is self-contained. Let $\varepsilon > 0$ be given. Since $s = \sup K$ is an upper bound for K that is no greater than any other upper bound, the ε-Criterion for Suprema and Infima asserts that there is a number $a \in K$ with $s - \varepsilon < a$. Because s is an upper bound for K and $a \in K$, also $a \leq s < s + \varepsilon$. Combining the two inequalities gives $s - \varepsilon < a < s + \varepsilon$, i.e. $|a - s| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this produces, for every $\varepsilon > 0$, a number $a \in K$ with $|a - s| < \varepsilon$; this is exactly the ε-Criterion for Closures for the statement $s \in \overline{K}$. Therefore $s \in \overline{K}$.

Since K is closed, the Closure Criterion for Closed Sets asserts that $\overline{K} \subseteq K$. Therefore the upper bound $s = \sup K \in \overline{K} \subseteq K$ is an upper bound for K that also belongs to K; that is, $s = \max K$. This proves that K has a maximum.

For the minimum, apply the maximum case just proved to the set $-K$. Since K is nonempty, so is $-K$. Since K is bounded, so is $-K$, because $a \in K$ satisfies $|a| \leq B$ for some bound B exactly when $-a \in -K$ satisfies $|-a| \leq B$. Since K is closed, so is $-K$: if $\{b_n\}$ is a convergent sequence contained in $-K$ with limit b, then $\{-b_n\}$ is a convergent sequence contained in K with limit $-b$ (negation is compatible with limits by the Algebraic Limit Theorem), so the Sequential Criterion for Closed Sets applied to K gives $-b \in K$, hence $b \in -K$; since $\{b_n\}$ was arbitrary, the Sequential Criterion for Closed Sets asserts that $-K$ is closed. Thus $-K$ is nonempty, closed, and bounded, so by the maximum case, $-K$ has a maximum $\max(-K)$. Therefore $\min K := -\max(-K)$ exists. □

Common slip. A supremum need not belong to its set; it is closedness, via the Closure Criterion for Closed Sets, that forces $\sup K$ into K. Do not write "K is bounded above, so it has a maximum" – boundedness alone only gives a supremum (Axiom of Completeness); closedness is the separate, necessary ingredient that upgrades it to a maximum. A bounded but non-closed set, such as $(0, 1)$, has a supremum (1) that is not a maximum.

6.2 (Sequentially) Compact Sets

The Extrema theorem suggests that closed and bounded sets behave unusually well with sequences: every sequence they contain is, in a sense to be made precise, "trapped" inside the set. The next definition isolates exactly this sequential behaviour, independently of closedness or boundedness, and the theorem after it shows the two notions coincide.

Definition 6.1: (Sequentially) Compact Set

A set $K \subseteq \mathbb{R}$ is said to be **(sequentially) compact** if every sequence contained in the set has a convergent subsequence converging to a limit in the set.

Remark (Comparison with Sequential Criterion for Closedness). The two notions differ in exactly one place – where "subsequence" appears:

- **Closedness:** every *convergent* sequence in K has its limit in K.
- **Sequential compactness:** every sequence in K has a *convergent subsequence* with limit in K.

Compactness is the stronger requirement: it does not assume the original sequence converges at all, only that *some* subsequence of it does, and that this subsequence's limit lies in K. Every compact set is closed (this is proved as part of the next theorem), but as Worked Example 1(b) below shows, not every closed set is compact.

Extension: Compactness via Open Covers (not used in this course)

Other textbooks define compactness topologically, via open covers (a set is compact if every open cover admits a finite subcover), and then prove the Heine–Borel Theorem that this coincides with "closed and bounded" for subsets of $\mathbb{R}$. This course never uses open covers or the Heine–Borel Theorem by name; the sequential definition above is the only definition of compactness used here, and it is proved equivalent to "closed and bounded" directly, without any open-cover machinery, in the next theorem.

6.3 Sequential Compactness of Closed and Bounded Sets

Theorem 6.2: Sequential Compactness of Closed and Bounded Sets

A set $K \subseteq \mathbb{R}$ is closed and bounded if and only if every sequence $\{a_n\}$ contained in the set has a convergent subsequence converging to a limit in the set (i.e. if and only if K is (sequentially) compact).

Idea. *(Scratch work – not part of the written answer.)* The theorem has two directions, and the "if" direction itself splits into two independent parts, exactly as the slides present it.

- ($\Longrightarrow$) Closed and bounded implies compact: take any sequence in K; it is bounded because K is; Bolzano–Weierstrass gives a convergent subsequence; closedness of K (Sequential Criterion) forces the limit into K.
- ($\Longleftarrow$, closedness) Compact implies closed: take any *convergent* sequence in K; compactness gives it a convergent subsequence with limit in K; but a convergent sequence and its subsequences all share the same limit (Subsequential Limits), so the original sequence's limit is that same point of K.
- ($\Longleftarrow$, boundedness) Compact implies bounded, proved by contrapositive: if K were unbounded, build a sequence $a_n \in K$ with $|a_n|$ exceeding an unbounded control sequence M_n; then no subsequence of $\{a_n\}$ can be bounded, hence none can converge, contradicting compactness.

Proof. Let $K \subseteq \mathbb{R}$.

Part 1 ($\Longrightarrow$): if K is closed and bounded, then K is compact. Suppose K is closed and bounded. Let $\{a_n\}$ be an arbitrary sequence contained in K. Since K is bounded, the sequence $\{a_n\}$ is bounded. Hence the Bolzano–Weierstrass Theorem asserts that $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$. Since $\{a_{n_k}\}$ is a convergent sequence contained in the closed set K, the Sequential Criterion for Closed Sets asserts that its limit lies in K. Since $\{a_n\}$ was an arbitrary sequence contained in K, every sequence contained in K has a convergent subsequence with limit in K; that is, K is (sequentially) compact.

Part 2 ($\Longleftarrow$, closedness): if K is compact, then K is closed. Suppose K is compact. Let $\{a_n\}$ be an arbitrary convergent sequence contained in K, with limit $a := \lim a_n$. Since K is compact, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ with a limit in K. The theorem on Subsequential Limits asserts that the convergent sequence $\{a_n\}$ shares the same limit a as each of its subsequences, so $\{a_{n_k}\}$ also converges to a, and therefore $a \in K$. Since $\{a_n\}$ was an arbitrary convergent sequence contained in K, the Sequential Criterion for Closed Sets asserts that K is closed.

Part 3 ($\Longleftarrow$, boundedness, by contrapositive): if K is compact, then K is bounded. We prove the contrapositive: if K is unbounded, then K is not compact. Suppose K is unbounded. Let $\{M_n\}$ denote a positive, increasing, and unbounded sequence (for instance $M_n := n$). For each $n \in \mathbb{N}$, negating the definition of boundedness of K produces a number $a_n \in K$ with $|a_n| > M_n$; carrying this out for every $n \in \mathbb{N}$ forms a sequence $\{a_n\}$ contained in K.

Let $\{a_{n_k}\}$ denote an arbitrary subsequence of $\{a_n\}$; we show it diverges. Let $M > 0$ be given. Since $\{M_n\}$ is unbounded and positive, negating the definition of its boundedness produces an index $n \in \mathbb{N}$ with $M_n > M$. Set $k := n$. Since $\{a_{n_k}\}$ is a subsequence, its index sequence satisfies $n_k \geq k$ for every $k \in \mathbb{N}$; applying this at $k = n$ gives $n_k \geq n$. Since $\{M_n\}$ is increasing, $n_k \geq n$ implies $M_{n_k} \geq M_n$. By construction of $\{a_n\}$, the term at index n_k satisfies $|a_{n_k}| > M_{n_k}$. Chaining these three facts,

$$|a_{n_k}| > M_{n_k} \geq M_n > M.$$

Since $M > 0$ was arbitrary, this produces, for every $M > 0$, an index k (namely $k = n$) with $|a_{n_k}| > M$; this is exactly the negation of the definition of boundedness for $\{a_{n_k}\}$, so $\{a_{n_k}\}$ is unbounded. The contrapositive of the theorem on Boundedness of Convergent Sequences then asserts that $\{a_{n_k}\}$ diverges. Since $\{a_{n_k}\}$ was an arbitrary subsequence of $\{a_n\}$, every subsequence of $\{a_n\}$ diverges, so in particular no subsequence of $\{a_n\}$ converges to a limit in K. Hence $\{a_n\}$ is a sequence contained in K with no convergent subsequence with limit in K, so K is not compact.

Combining Parts 1–3: K is closed and bounded if and only if K is (sequentially) compact. □

Common slip. In Part 3, the argument needs $\{M_n\}$ to be *increasing*, not merely unbounded: this is exactly what lets an index n witnessing $M_n > M$ transfer to the later index $n_k \geq n$ via $M_{n_k} \geq M_n$. More generally, when disproving compactness, the witnessing sequence must have *no* subsequence converging to a limit inside the set – it is not enough for the sequence itself to diverge, or for it to be merely unbounded "on average"; every single subsequence must fail. Finally, remember compactness demands the subsequential limit lie *in the set*: a bounded sequence with a convergent subsequence whose limit lies just outside the set is not a counterexample to unboundedness, but it *is* a valid witness against compactness (see Worked Example 3).

6.4 Nested Set Property

Theorem 6.3: Nested Set Property

If $K_n \subseteq \mathbb{R}$ is a nonempty, closed, and bounded set for each $n \in \mathbb{N}$, and $K_n \supseteq K_{n+1}$ for every $n \in \mathbb{N}$, then the resulting nested sequence of closed and bounded sets

$$K_1 \supseteq K_2 \supseteq K_3 \supseteq K_4 \supseteq \cdots \supsetneq \emptyset$$

has a nonempty intersection; that is, there is a number $a \in \mathbb{R}$ with $a \in K_n$ for every $n \in \mathbb{N}$.

Remark (Comparison with the Nested Interval Property). The Nested Interval Property (Chapter 1) is the special case of the Nested Set Property in which each closed and bounded set K_n is a closed and bounded interval $[a_n, b_n]$. The proof below replaces that theorem's interval-specific "supremum of left endpoints" argument with a sequential argument that works for an arbitrary nested chain of closed and bounded sets, using the Sequential Compactness of Closed and Bounded Sets in place of interval-specific reasoning.

Idea. *(Scratch work – not part of the written answer.)* Pick one point $a_n \in K_n$ from each set; the whole sequence $\{a_n\}$ lands inside the outermost set K_1, since the sets are nested. Because K_1 is nonempty, closed, and bounded, it is compact, so $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ with limit $a \in K_1$. Nesting also forces every sufficiently late term a_{n_k} (once $n_k \geq n$) into any fixed K_n, so the *tail* of the subsequence, from that point on, is a convergent sequence sitting entirely inside the closed set K_n – and that tail converges to the very same limit a, by the theorem on Subsequential Limits applied to the tail (which is itself a subsequence). Hence $a \in K_n$, and this argument repeats for every n.

Proof. For each $n \in \mathbb{N}$, since K_n is nonempty, there is a number $a_n \in K_n$; carrying this out for every $n \in \mathbb{N}$ forms a sequence $\{a_n\}$. Since $K_1 \supseteq K_n$ for every $n \in \mathbb{N}$ (the sets are nested with K_1 outermost), each term satisfies $a_n \in K_n \subseteq K_1$, so $\{a_n\}$ is a sequence contained in K_1. Since K_1 is nonempty, closed, and bounded, the Sequential Compactness of Closed and Bounded Sets asserts that K_1 is (sequentially) compact; applying the definition of compactness to $\{a_n\}$ gives a convergent subsequence $\{a_{n_k}\}$ with limit $a := \lim a_{n_k} \in K_1$.

We now show that $a \in K_n$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. For every $k \in \mathbb{N}$ with $k \geq n$, the property $n_k \geq k$ of subsequence indices gives $n_k \geq k \geq n$; since the sets K_n are nested and decreasing, $n_k \geq n$ implies $K_{n_k} \subseteq K_n$, and therefore $a_{n_k} \in K_{n_k} \subseteq K_n$. Since this holds for every $k \geq n$, the tail $\{a_{n_k}\}_{k=n}^{\infty}$ is a sequence contained in K_n. Moreover, this tail is itself a subsequence of the convergent sequence $\{a_{n_k}\}$, so the theorem on Subsequential Limits asserts that it converges to the same limit $a = \lim a_{n_k}$. Thus $\{a_{n_k}\}_{k=n}^{\infty}$ is a convergent sequence contained in the closed set K_n, so the Sequential Criterion for Closed Sets asserts that its limit a lies in K_n.

Since $n \in \mathbb{N}$ was arbitrary, $a \in K_n$ for every $n \in \mathbb{N}$; that is, $a \in \bigcap_{n=1}^{\infty} K_n$. Hence the intersection is nonempty. □

Common slip. It is tempting to argue as though the *whole* subsequence $\{a_{n_k}\}$ lies in a fixed K_n; only a *tail* of it does, from the point where the subsequence's index n_k has caught up to n. Early terms a_{n_k}, for small k, are guaranteed only to lie in the outermost set K_1. The fix is that a tail of a convergent sequence is itself a subsequence and so converges to the very same limit (Subsequential Limits); this is what lets the argument recover the single limit a inside every K_n, rather than producing a different candidate point for each n.

6.5 Worked Examples

Worked Example (Tutorial 5, Question 2): Deciding Compactness

Determine if each of the following sets is (sequentially) compact.

$$A = [0,1] \cup [5,6], \quad B = \mathbb{R}_0^+, \quad C = \{x \in \mathbb{R} : 0 \le x \le 1 \text{ and } x \text{ is irrational}\},$$

$$D = \left\{1 + \tfrac{1}{4} + \tfrac{1}{9} + \cdots + \tfrac{1}{n^2} : n \in \mathbb{N}\right\}, \quad E = \left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n} : n \in \mathbb{N}\right\}.$$

(a) $A = [0,1] \cup [5,6]$. Method 1: Closed-and-bounded characterisation.

Proof. For every $a \in A$, either $0 \le a \le 1$ or $5 \le a \le 6$; in both cases $-6 \le 0 \le a \le 6$, so $|a| \le 6$. Therefore A is bounded, and A is nonempty because $0 \in A$.
Every closed bounded interval $[c,d]$ is closed. Indeed, let $\{x_n\}$ be a convergent sequence contained in $[c,d]$, with $x := \lim x_n$. Then $c \le x_n \le d$ for every $n \in \mathbb{N}$, so the Order Limit Theorem gives $c \le x \le d$, that is, $x \in [c,d]$. Hence every convergent sequence contained in $[c,d]$ has its limit in $[c,d]$, and the Sequential Criterion for Closed Sets asserts that $[c,d]$ is closed. In particular $[0,1]$ and $[5,6]$ are closed. Since the union of two closed sets is closed (Tutorial 5, Question 4(a)), $A = [0,1] \cup [5,6]$ is closed.
Therefore A is nonempty, closed and bounded, so the Sequential Compactness of Closed and Bounded Sets (Theorem 6.2) asserts that A is (sequentially) compact. □

A is compact.

Method 2: Sequential compactness definition with Bolzano–Weierstrass.

Proof. Let $\{a_n\}$ be an arbitrary sequence contained in A. Let $I_1 := \{n \in \mathbb{N} : a_n \in [0,1]\}$ and $I_2 := \{n \in \mathbb{N} : a_n \in [5,6]\}$. Since every a_n lies in $[0,1]$ or in $[5,6]$, we have $\mathbb{N} = I_1 \cup I_2$. Since $\mathbb{N}$ is infinite and a union of two finite sets is finite, at least one of I_1, I_2 is infinite. Let I be an infinite one of them, and let $[c,d]$ be the corresponding interval ($[0,1]$ for I_1, $[5,6]$ for I_2). List the elements of I in increasing order as $m_1 < m_2 < m_3 < \cdots$. Then $\{a_{m_j}\}$ is a subsequence of $\{a_n\}$, and $c \le a_{m_j} \le d$ for every $j \in \mathbb{N}$.
Hence $\{a_{m_j}\}$ is bounded, so the Bolzano–Weierstrass Theorem asserts that it has a convergent subsequence $\{a_{m_{j_k}}\}$; let $a := \lim_{k\to\infty} a_{m_{j_k}}$. Since $c \le a_{m_{j_k}} \le d$ for every $k \in \mathbb{N}$, the Order Limit Theorem gives $c \le a \le d$. Therefore $a \in [c,d] \subseteq A$. The indices $m_{j_1} < m_{j_2} < \cdots$ are strictly increasing, so $\{a_{m_{j_k}}\}$ is a subsequence of $\{a_n\}$. Hence $\{a_n\}$ has a convergent subsequence with limit in A. Since $\{a_n\}$ was arbitrary, every sequence contained in A has a convergent subsequence with limit in A; that is, A is (sequentially) compact. □

A is compact.

(b) $B = \mathbb{R}_0^+ = [0,\infty)$. Method 1: Witnessing sequence with no convergent subsequence.

Proof. Let $b_n := n$ for every $n \in \mathbb{N}$. Since $n \geq 1 > 0$, $b_n \in B$ for every $n \in \mathbb{N}$, so $\{b_n\}$ is a sequence contained in B. Let $\{b_{n_k}\}$ be an arbitrary subsequence of $\{b_n\}$. The Subsequence Index Bound gives $n_k \geq k$ for every $k \in \mathbb{N}$; hence

$$b_{n_k} = n_k \geq k \quad \text{for every } k \in \mathbb{N}.$$

Let $M > 0$ be given. The Archimedean Property gives $K \in \mathbb{N}$ with $K > M$, and therefore $|b_{n_K}| = n_K \geq K > M$. Since $M > 0$ was arbitrary, no $M > 0$ satisfies $|b_{n_k}| \leq M$ for every $k \in \mathbb{N}$; hence $\{b_{n_k}\}$ is unbounded. The contrapositive of Boundedness of Convergent Sequences then asserts that $\{b_{n_k}\}$ diverges. Since $\{b_{n_k}\}$ was an arbitrary subsequence, no subsequence of $\{b_n\}$ converges, so in particular none converges to a limit in B. Therefore B is not (sequentially) compact. □

B is not compact.

Method 2: Contradiction via the closed-and-bounded characterisation.

Proof. Suppose, for contradiction, that B is (sequentially) compact. Then the Sequential Compactness of Closed and Bounded Sets (Theorem 6.2) asserts that B is closed and bounded. Hence there is $M > 0$ such that $|x| \leq M$ for every $x \in B$. Since $M + 1 \geq 0$, we have $M + 1 \in B$, and $|M + 1| = M + 1 > M$. This contradicts $|x| \leq M$ for every $x \in B$. Therefore B is not (sequentially) compact. □

B is not compact.

(c) $C = \{x \in [0, 1] : x$ irrational$\}$. Method 1: Witnessing sequence converging to a point outside the set.

Proof. Let $x_n := \dfrac{\sqrt{2}}{n+1}$ for every $n \in \mathbb{N}$. Each x_n is irrational: if $x_n = q$ for some $q \in \mathbb{Q}$, then $\sqrt{2} = (n + 1)q \in \mathbb{Q}$, which contradicts the irrationality of $\sqrt{2}$ (Lecture 1). Since $1 < \sqrt{2} < 2$ and $n + 1 \geq 2$,

$$0 < \frac{\sqrt{2}}{n+1} \leq \frac{\sqrt{2}}{2} < 1 \quad \text{for every } n \in \mathbb{N}.$$

Therefore $x_n \in C$ for every $n \in \mathbb{N}$, so $\{x_n\}$ is a sequence contained in C.
The Archimedean Property gives $\frac{1}{n} \to 0$; the sequence $\left\{\frac{1}{n+1}\right\}$ is the subsequence with $n_k = k + 1$, so $\frac{1}{n+1} \to 0$ (Subsequential Limits). Hence the Algebraic Limit Theorem gives

$$\lim x_n = \sqrt{2} \cdot \lim \frac{1}{n+1} = \sqrt{2} \cdot 0 = 0.$$

Let $\{x_{n_k}\}$ be an arbitrary subsequence of $\{x_n\}$. Subsequential Limits asserts that $x_{n_k} \to 0$, and by Uniqueness of Limits, 0 is its only limit. Since $0 = \frac{0}{1} \in \mathbb{Q}$, the number 0 is not irrational, so $0 \notin C$. Hence no subsequence of $\{x_n\}$ converges to a limit in C. Therefore C is not (sequentially) compact. □

C is not compact.

Method 2: Contradiction via Extrema of Closed and Bounded Sets.

Proof. Suppose, for contradiction, that C is (sequentially) compact. Then C is closed and bounded (Theorem 6.2). Also C is nonempty: $\frac{\sqrt{2}}{2}$ is irrational (if $\frac{\sqrt{2}}{2} = q \in \mathbb{Q}$, then $\sqrt{2} = 2q \in \mathbb{Q}$), and $0 < \frac{\sqrt{2}}{2} < 1$, so $\frac{\sqrt{2}}{2} \in C$. Hence the Extrema of Closed and Bounded Sets (Theorem 6.1) asserts that $m := \min C$ exists; that is, $m \in C$ and $m \leq x$ for every $x \in C$.
Since $m \in C$, m is irrational and $0 \leq m \leq 1$; since $0 \in \mathbb{Q}$, $m \neq 0$, so $0 < m \leq 1$. If $\frac{m}{2} = q$ for some $q \in \mathbb{Q}$, then $m = 2q \in \mathbb{Q}$, a contradiction; hence $\frac{m}{2}$ is irrational. Moreover $0 < \frac{m}{2} < m \leq 1$. Therefore $\frac{m}{2} \in C$ and $\frac{m}{2} < m$, which contradicts $m \leq x$ for every $x \in C$. Hence C is not (sequentially) compact. □

C is not compact.

(d) $D = \left\{1 + \frac{1}{4} + \cdots + \frac{1}{n^2} : n \in \mathbb{N}\right\}$. **Method 1: Witnessing sequence converging to the sum of the series.**

Proof. Let $d_n := \sum_{k=1}^{n} \frac{1}{k^2}$ for every $n \in \mathbb{N}$. Then $d_n \in D$ for every $n \in \mathbb{N}$, so $\{d_n\}$ is a sequence contained in D. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (Convergence of $\sum 1/n^2$), so by the definition of convergence of a series its partial sums converge:

$$d_n \to L := \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

For every $n \in \mathbb{N}$, $d_{n+1} - d_n = \frac{1}{(n+1)^2} > 0$, so $\{d_n\}$ is strictly increasing. Fix $n \in \mathbb{N}$. Then $d_m \geq d_{n+1}$ for every $m \geq n+1$, so the Order Limit Theorem (letting $m \to \infty$) gives

$$L \geq d_{n+1} = d_n + \frac{1}{(n+1)^2} > d_n.$$

Since $n \in \mathbb{N}$ was arbitrary, $L \neq d_n$ for every $n \in \mathbb{N}$; that is, $L \notin D$.
Let $\{d_{n_k}\}$ be an arbitrary subsequence of $\{d_n\}$. Subsequential Limits asserts that $d_{n_k} \to L$, and by Uniqueness of Limits, L is its only limit. Since $L \notin D$, the subsequence does not converge to a limit in D. Hence $\{d_n\}$ is a sequence contained in D with no subsequence converging to a limit in D. Therefore D is not (sequentially) compact. □

$\boxed{D \text{ is not compact.}}$

Method 2: Contradiction via Extrema of Closed and Bounded Sets.

Proof. Let $d_n := \sum_{k=1}^{n} \frac{1}{k^2}$, so that $D = \{d_n : n \in \mathbb{N}\}$. Suppose, for contradiction, that D is (sequentially) compact. Then D is closed and bounded (Theorem 6.2), and D is nonempty because $d_1 = 1 \in D$. Hence the Extrema of Closed and Bounded Sets (Theorem 6.1) asserts that $M := \max D$ exists; that is, $M \in D$ and $x \leq M$ for every $x \in D$. Since $M \in D$, there is $m \in \mathbb{N}$ with $M = d_m$. But $d_{m+1} \in D$ and

$$d_{m+1} = d_m + \frac{1}{(m+1)^2} > d_m = M,$$

which contradicts $x \leq M$ for every $x \in D$. Therefore D is not (sequentially) compact. □

$\boxed{D \text{ is not compact.}}$

(e) $E = \left\{1 + \frac{1}{2} + \cdots + \frac{1}{n} : n \in \mathbb{N}\right\}$. **Method 1: Witnessing sequence with unbounded subsequences.**

Proof. Let $e_n := \sum_{k=1}^{n} \frac{1}{k}$ for every $n \in \mathbb{N}$. Then $e_n \in E$ for every $n \in \mathbb{N}$, so $\{e_n\}$ is a sequence contained in E. For every $n \in \mathbb{N}$, $e_{n+1} - e_n = \frac{1}{n+1} > 0$, so $\{e_n\}$ is strictly increasing; hence $m \geq k$ implies $e_m \geq e_k$. If $\{e_n\}$ were bounded, the Monotone Convergence Theorem would assert that $\{e_n\}$ converges, that is, that $\sum_{n=1}^{\infty} \frac{1}{n}$ converges; this contradicts the Divergence of the Harmonic Series. Hence $\{e_n\}$ is unbounded. Since $e_n \geq e_1 = 1 > 0$ for every $n \in \mathbb{N}$, this means: for every $M > 0$ there is $K \in \mathbb{N}$ with $e_K > M$.
Let $\{e_{n_k}\}$ be an arbitrary subsequence of $\{e_n\}$. The Subsequence Index Bound gives $n_k \geq k$, so $e_{n_k} \geq e_k$ for every $k \in \mathbb{N}$. Let $M > 0$ be given, and choose $K \in \mathbb{N}$ with $e_K > M$. Then $|e_{n_K}| = e_{n_K} \geq e_K > M$. Since $M > 0$ was arbitrary, $\{e_{n_k}\}$ is unbounded, so the contrapositive of Boundedness of Convergent Sequences asserts that $\{e_{n_k}\}$ diverges. Since $\{e_{n_k}\}$ was an arbitrary subsequence, no subsequence of $\{e_n\}$ converges to a limit in E. Therefore E is not (sequentially) compact. □

$\boxed{E \text{ is not compact.}}$

Method 2: Contradiction via Extrema of Closed and Bounded Sets.

Proof. Let $e_n := \sum_{k=1}^{n} \frac{1}{k}$, so that $E = \{e_n : n \in \mathbb{N}\}$. Suppose, for contradiction, that E is (sequentially) compact. Then E is closed and bounded (Theorem 6.2), and E is nonempty because $e_1 = 1 \in E$. Hence the Extrema of Closed and Bounded Sets (Theorem 6.1) asserts that $M := \max E$ exists; that is, $M \in E$ and $x \leq M$ for every $x \in E$. Since $M \in E$, there is $m \in \mathbb{N}$ with $M = e_m$. But $e_{m+1} \in E$ and

$$e_{m+1} = e_m + \frac{1}{m+1} > e_m = M,$$

which contradicts $x \leq M$ for every $x \in E$. Therefore E is not (sequentially) compact. □

E is not compact.

Worked Example (Tutorial 5, Question 5): A Closed Subset of a Compact Set is Compact

Use the definition of (sequential) compactness to prove that a closed subset of a (sequentially) compact set is (sequentially) compact; i.e., for any compact set $K \subseteq \mathbb{R}$ and any subset $A \subseteq K$, prove that A is compact if it is closed. Do not use the Sequential Compactness of Closed and Bounded Sets or the Bolzano–Weierstrass Theorem.

Method 1: Compactness definition with the Sequential Criterion for Closed Sets.

Proof. Let $K \subseteq \mathbb{R}$ be compact, and let $A \subseteq K$ be closed. Let $\{a_n\}$ be an arbitrary sequence contained in A. Since $a_n \in A$ and $A \subseteq K$, we have $a_n \in K$ for every $n \in \mathbb{N}$; therefore $\{a_n\}$ is a sequence contained in K. Since K is compact, the definition of (sequential) compactness asserts that $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\}$ with limit $a := \lim a_{n_k} \in K$. Since $a_{n_k} \in A$ for every $k \in \mathbb{N}$, $\{a_{n_k}\}$ is a convergent sequence contained in A. Since A is closed, the Sequential Criterion for Closed Sets asserts that the limit of $\{a_{n_k}\}$ lies in A; that is, $a \in A$. Hence $\{a_n\}$ has a convergent subsequence with limit in A. Since $\{a_n\}$ was arbitrary, every sequence contained in A has a convergent subsequence with limit in A. Therefore A is (sequentially) compact. □

Method 2: ε-Criterion for Closures with the Closure Criterion for Closed Sets.

Proof. Let $K \subseteq \mathbb{R}$ be compact, and let $A \subseteq K$ be closed. Let $\{a_n\}$ be an arbitrary sequence contained in A. Since $A \subseteq K$, $\{a_n\}$ is a sequence contained in K, so the definition of (sequential) compactness of K gives a convergent subsequence $\{a_{n_k}\}$ with limit $a := \lim a_{n_k} \in K$.

We show that $a \in \overline{A}$. Let $\varepsilon > 0$ be given. By the definition of $a = \lim a_{n_k}$, there is $N \in \mathbb{N}$ such that $|a_{n_k} - a| < \varepsilon$ for every $k \geq N$. In particular, $x := a_{n_N}$ satisfies $x \in A$ and $|x - a| < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, for every $\varepsilon > 0$ there is $x \in A$ with $|x - a| < \varepsilon$; hence the ε-Criterion for Closures asserts that $a \in \overline{A}$.

Since A is closed, the Closure Criterion for Closed Sets asserts that $\overline{A} \subseteq A$. Therefore $a \in \overline{A} \subseteq A$, so $\{a_n\}$ has a convergent subsequence with limit in A. Since $\{a_n\}$ was arbitrary, every sequence contained in A has a convergent subsequence with limit in A; that is, A is (sequentially) compact. □

Worked Example (Sample Midterm, Question 2(c)): Compact Minus Closed

True or false: if K is a compact set and F is a closed set, then $K \setminus F$ is a compact set.

Answer. False.

Method 1: Counterexample with a witnessing sequence converging outside the set.

Proof. Let $K := [0, 1]$ and $F := \{0\}$. The set K is nonempty, since $0 \in K$, and bounded, since $|x| \leq 1$ for every $x \in K$. The set K is closed: if $\{x_n\}$ is a convergent sequence contained in $[0, 1]$ with limit x, then $0 \leq x_n \leq 1$ for every $n \in \mathbb{N}$, so the Order Limit Theorem gives $0 \leq x \leq 1$, that is, $x \in K$; hence the Sequential Criterion for Closed Sets asserts that K is closed. Therefore K is nonempty, closed and bounded, and Theorem 6.2 asserts that K is compact.

The set F is closed: if $\{y_n\}$ is a convergent sequence contained in $\{0\}$, then $y_n = 0$ for every $n \in \mathbb{N}$, so $|y_n - 0| = 0 < \varepsilon$ for every $\varepsilon > 0$ and every $n \in \mathbb{N}$; hence $\lim y_n = 0 \in F$ (def. of limit, Uniqueness of Limits), and the Sequential Criterion for Closed Sets asserts that F is closed.
Now $K \setminus F = \{x \in [0,1] : x \neq 0\} = (0,1]$. Let $a_n := \frac{1}{n}$ for every $n \in \mathbb{N}$. Since $0 < \frac{1}{n} \leq 1$ for every $n \in \mathbb{N}$, $\{a_n\}$ is a sequence contained in $(0,1]$, and $a_n \to 0$ (Archimedean). Let $\{a_{n_k}\}$ be an arbitrary subsequence of $\{a_n\}$. Subsequential Limits asserts that $a_{n_k} \to 0$, and by Uniqueness of Limits, 0 is its only limit. Since $0 \notin (0,1]$, the subsequence does not converge to a limit in $K \setminus F$. Hence $\{a_n\}$ has no subsequence converging to a limit in $K \setminus F$, so $K \setminus F$ is not compact. Therefore K is compact, F is closed, and $K \setminus F$ is not compact, so the statement is false. □

FALSE; counterexample $K = [0,1],\ F = \{0\},\ K \setminus F = (0,1]$ is not compact.

Method 2: Definitions for the hypotheses, contradiction via Extrema of Closed and Bounded Sets.

Proof. Let $K := [0,1]$ and $F := \{0\}$.
The set K is compact by the definition. Let $\{x_n\}$ be an arbitrary sequence contained in $[0,1]$. Since $|x_n| \leq 1$ for every $n \in \mathbb{N}$, $\{x_n\}$ is bounded, so the Bolzano–Weierstrass Theorem asserts that it has a convergent subsequence $\{x_{n_k}\}$; let $x := \lim x_{n_k}$. Since $0 \leq x_{n_k} \leq 1$ for every $k \in \mathbb{N}$, the Order Limit Theorem gives $0 \leq x \leq 1$, so $x \in K$. Since $\{x_n\}$ was arbitrary, every sequence contained in K has a convergent subsequence with limit in K; hence K is (sequentially) compact.
The set F is closed by the definition of closed set: we show that F has no limit points. Let $c \in \mathbb{R}$. If $c = 0$, then $F \setminus \{c\} = \emptyset$, so with $\varepsilon := 1$ there is no $x \in F \setminus \{c\}$ with $|x - c| < \varepsilon$. If $c \neq 0$, then $F \setminus \{c\} = \{0\}$; with $\varepsilon := |c| > 0$, the only candidate $x = 0$ satisfies $|0 - c| = |c| = \varepsilon$, which is not less than ε. In both cases c is not a limit point of F. Hence the set of limit points of F is $\emptyset \subseteq F$, and F is closed.
Now $K \setminus F = (0,1]$. Suppose, for contradiction, that $(0,1]$ is compact. Then $(0,1]$ is closed and bounded (Theorem 6.2), and it is nonempty since $1 \in (0,1]$. Hence the Extrema of Closed and Bounded Sets (Theorem 6.1) asserts that $m := \min(0,1]$ exists; that is, $m \in (0,1]$ and $m \leq x$ for every $x \in (0,1]$. Since $m \in (0,1]$, $0 < m \leq 1$, so $0 < \frac{m}{2} < m \leq 1$. Therefore $\frac{m}{2} \in (0,1]$ and $\frac{m}{2} < m$, which contradicts $m \leq x$ for every $x \in (0,1]$. Hence $K \setminus F$ is not compact, while K is compact and F is closed; so the statement is false. □

FALSE; counterexample $K = [0,1],\ F = \{0\},\ K \setminus F = (0,1]$ is not compact.

Worked Example: An Application of the Nested Set Property

Let $K_n := \left[0, \frac{1}{n}\right] \cup [2,3]$ for each $n \in \mathbb{N}$. Verify that $\{K_n\}$ satisfies the hypotheses of the Nested Set Property, and determine $\bigcap_{n=1}^{\infty} K_n$.

Proof (hypotheses). Let $n \in \mathbb{N}$. Every closed bounded interval $[c,d]$ is closed: if $\{x_m\}$ is a convergent sequence contained in $[c,d]$ with limit x, then $c \leq x_m \leq d$ for every $m \in \mathbb{N}$, so the Order Limit Theorem gives $c \leq x \leq d$; hence the Sequential Criterion for Closed Sets asserts that $[c,d]$ is closed. Therefore $\left[0, \frac{1}{n}\right]$ and $[2,3]$ are closed, and K_n is closed because a union of two closed sets is closed (Tutorial 5, Question 4(a)). For every $x \in K_n$, either $0 \leq x \leq \frac{1}{n} \leq 1$ or $2 \leq x \leq 3$; hence $0 \leq x \leq 3$ and $|x| \leq 3$, so K_n is bounded. Since $0 \leq 0 \leq \frac{1}{n}$, we have $0 \in \left[0, \frac{1}{n}\right] \subseteq K_n$, so K_n is nonempty.
Since $n < n+1$, we have $\frac{1}{n+1} < \frac{1}{n}$. Hence, if $x \in \left[0, \frac{1}{n+1}\right]$, then $0 \leq x \leq \frac{1}{n+1} < \frac{1}{n}$, so $x \in \left[0, \frac{1}{n}\right]$; that is, $\left[0, \frac{1}{n+1}\right] \subseteq \left[0, \frac{1}{n}\right]$. Since the second piece $[2,3]$ is the same for every n,

$$K_{n+1} = \left[0, \tfrac{1}{n+1}\right] \cup [2,3] \subseteq \left[0, \tfrac{1}{n}\right] \cup [2,3] = K_n.$$

Since $n \in \mathbb{N}$ was arbitrary, each K_n is nonempty, closed and bounded, and $K_{n+1} \subseteq K_n$ for every

$n \in \mathbb{N}$. Therefore the Nested Set Property (Theorem 6.3) asserts that $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. □

Method 1: Double inclusion with contradiction via the Archimedean Property.

Proof. We prove $\bigcap_{n=1}^{\infty} K_n = \{0\} \cup [2,3]$ by double inclusion.
($\supseteq$) Let $x \in \{0\} \cup [2,3]$. If $x \in [2,3]$, then $x \in [2,3] \subseteq K_n$ for every $n \in \mathbb{N}$. If $x = 0$, then $0 \leq x \leq \frac{1}{n}$, so $x \in \left[0, \frac{1}{n}\right] \subseteq K_n$ for every $n \in \mathbb{N}$. In both cases $x \in K_n$ for every $n \in \mathbb{N}$, so $x \in \bigcap_{n=1}^{\infty} K_n$.
($\subseteq$) Let $x \in \bigcap_{n=1}^{\infty} K_n$, and suppose, for contradiction, that $x \notin \{0\} \cup [2,3]$. Since $x \in K_1 = [0,1] \cup [2,3]$ and $x \notin [2,3]$, we have $x \in [0,1]$; since $x \neq 0$, we have $0 < x \leq 1$. Since $x > 0$, the Archimedean Property gives $n \in \mathbb{N}$ with $\frac{1}{n} < x$. Then $x \notin \left[0, \frac{1}{n}\right]$ because $x > \frac{1}{n}$, and $x \notin [2,3]$ because $x \leq 1 < 2$. Hence $x \notin K_n$, which contradicts $x \in \bigcap_{n=1}^{\infty} K_n$. Therefore $x \in \{0\} \cup [2,3]$.
Combining both inclusions, $\bigcap_{n=1}^{\infty} K_n = \{0\} \cup [2,3]$. □

$$\boxed{\bigcap_{n=1}^{\infty} K_n = \{0\} \cup [2,3].}$$

Method 2: Distributing the intersection and the Order Limit Theorem.

Proof. First we show $\bigcap_{n=1}^{\infty} \left(\left[0, \frac{1}{n}\right] \cup [2,3] \right) = \left(\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right] \right) \cup [2,3]$. Let x lie in the left-hand side. If $x \in [2,3]$, then x lies in the right-hand side. If $x \notin [2,3]$, then for every $n \in \mathbb{N}$, $x \in \left[0, \frac{1}{n}\right] \cup [2,3]$ and $x \notin [2,3]$ give $x \in \left[0, \frac{1}{n}\right]$; hence $x \in \bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right]$, which is contained in the right-hand side. Conversely, let x lie in the right-hand side. If $x \in [2,3]$, then $x \in K_n$ for every $n \in \mathbb{N}$; if $x \in \bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right]$, then $x \in \left[0, \frac{1}{n}\right] \subseteq K_n$ for every $n \in \mathbb{N}$. In both cases x lies in the left-hand side, so the two sets are equal.
Next we show $\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right] = \{0\}$. Since $0 \leq 0 \leq \frac{1}{n}$ for every $n \in \mathbb{N}$, 0 lies in every $\left[0, \frac{1}{n}\right]$, so $\{0\} \subseteq \bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right]$. Conversely, let $x \in \bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right]$; then $0 \leq x \leq \frac{1}{n}$ for every $n \in \mathbb{N}$. The constant sequence $\{x\}$ converges to x, the sequence $\left\{\frac{1}{n}\right\}$ converges to 0 (Archimedean), and $x \leq \frac{1}{n}$ for every $n \in \mathbb{N}$; hence the Order Limit Theorem gives $x \leq 0$. Combined with $0 \leq x$, this gives $x = 0$. Therefore $\bigcap_{n=1}^{\infty} \left[0, \frac{1}{n}\right] = \{0\}$.
Substituting into the first identity, $\bigcap_{n=1}^{\infty} K_n = \{0\} \cup [2,3]$. □

$$\boxed{\bigcap_{n=1}^{\infty} K_n = \{0\} \cup [2,3].}$$

Worked Example: Extrema of a Compact Set, Attained Versus Unattained

Let $K := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$. Determine whether K is compact, and if so, find $\max K$ and $\min K$. Contrast the result with the set D of Worked Example 1(d), whose supremum is not attained.

Method 1: Closed-and-bounded characterisation, with closedness from limit points.

Proof. For every $n \in \mathbb{N}$, $0 < \frac{1}{n} \leq 1$, and also $0 \leq 0 \leq 1$; hence $0 \leq x \leq 1$ for every $x \in K$. Therefore $|x| \leq 1$ for every $x \in K$, so K is bounded, and K is nonempty because $0 \in K$.
We show that every limit point of K equals 0. Let $c \in \mathbb{R}$ with $c \neq 0$.
Case $c < 0$. Let $\varepsilon := -c > 0$. For every $x \in K$, $x \geq 0$, so $|x - c| = x - c \geq -c = \varepsilon$. Hence no $x \in K \setminus \{c\}$ satisfies $|x - c| < \varepsilon$, and c is not a limit point of K.
Case $c > 0$. The Archimedean Property gives $N \in \mathbb{N}$ with $\frac{1}{N} < \frac{c}{2}$. If $x \in K$ satisfies $|x - c| < \frac{c}{2}$, then $x > c - \frac{c}{2} = \frac{c}{2} > \frac{1}{N} > 0$, so $x \neq 0$ and $x = \frac{1}{m}$ for some $m \in \mathbb{N}$ with $\frac{1}{m} > \frac{1}{N}$, that is, $m < N$. Let

$$\varepsilon := \min\left(\left\{ \tfrac{c}{2} \right\} \cup \left\{ \left| \tfrac{1}{m} - c \right| : m \in \mathbb{N},\ m < N,\ \tfrac{1}{m} \neq c \right\} \right),$$

the minimum of finitely many positive numbers, so $\varepsilon > 0$. Suppose $x \in K \setminus \{c\}$ satisfies $|x - c| < \varepsilon$.

Then $|x-c| < \frac{c}{2}$, so $x = \frac{1}{m}$ with $m < N$ and $\frac{1}{m} \neq c$; hence $|x-c| \geq \varepsilon$ by the choice of ε, a contradiction. Hence no $x \in K \setminus \{c\}$ satisfies $|x-c| < \varepsilon$, and c is not a limit point of K.
Therefore $\{\text{limit points of } K\} \subseteq \{0\} \subseteq K$, so K is closed by the definition of closed set. Since K is nonempty, closed and bounded, Theorem 6.2 asserts that K is compact, and Theorem 6.1 asserts that $\max K$ and $\min K$ exist. Since $x \leq 1$ for every $x \in K$ and $1 = \frac{1}{1} \in K$, the number 1 is an upper bound for K that belongs to K; hence $\max K = 1$. Since $0 \leq x$ for every $x \in K$ and $0 \in K$, the number 0 is a lower bound for K that belongs to K; hence $\min K = 0$. □

$$\boxed{K \text{ is compact}, \quad \max K = 1, \quad \min K = 0.}$$

Method 2: Sequential compactness definition with witnessing subsequences.

Proof. Let $\{x_n\}$ be an arbitrary sequence contained in K.
Case 1: some $v \in K$ satisfies $x_n = v$ for infinitely many $n \in \mathbb{N}$. List these indices in increasing order as $n_1 < n_2 < n_3 < \cdots$. Then $\{x_{n_k}\}$ is a subsequence with $x_{n_k} = v$ for every $k \in \mathbb{N}$, so $|x_{n_k} - v| = 0 < \varepsilon$ for every $\varepsilon > 0$ and every $k \in \mathbb{N}$. Hence $x_{n_k} \to v$ (def. of limit), and $v \in K$.
Case 2: every $v \in K$ satisfies $x_n = v$ for only finitely many $n \in \mathbb{N}$. For each $k \in \mathbb{N}$, let $F_k := \{n \in \mathbb{N} : x_n \in \{0, 1, \frac{1}{2}, \ldots, \frac{1}{k}\}\}$; it is a union of $k+1$ finite sets, hence finite. Since F_1 is finite and $\mathbb{N}$ is infinite, there is $n_1 \in \mathbb{N}$ with $n_1 \notin F_1$. Having chosen n_k, since F_{k+1} is finite and $\{n \in \mathbb{N} : n > n_k\}$ is infinite, there is $n_{k+1} > n_k$ with $n_{k+1} \notin F_{k+1}$. Then $n_1 < n_2 < \cdots$, so $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$. For each $k \in \mathbb{N}$, $x_{n_k} \in K$ and $x_{n_k} \notin \{0, 1, \frac{1}{2}, \ldots, \frac{1}{k}\}$, so $x_{n_k} = \frac{1}{m}$ for some $m > k$; hence

$$0 < x_{n_k} < \frac{1}{k} \quad \text{for every } k \in \mathbb{N}.$$

Since the constant sequence $\{0\}$ converges to 0 and $\frac{1}{k} \to 0$ (Archimedean), the Squeeze Theorem gives $x_{n_k} \to 0$, and $0 \in K$.
In both cases $\{x_n\}$ has a convergent subsequence with limit in K. Since $\{x_n\}$ was arbitrary, K is (sequentially) compact by definition.
For the extrema, every $x \in K$ satisfies $x \leq 1$ (since $\frac{1}{n} \leq 1$ and $0 < 1$), and $1 \in K$; hence 1 is an upper bound for K that belongs to K, so $\max K = 1$ by the definition of maximum. Every $x \in K$ satisfies $x \geq 0$, and $0 \in K$; hence 0 is a lower bound for K that belongs to K, so $\min K = 0$ by the definition of minimum. □

$$\boxed{K \text{ is compact}, \quad \max K = 1, \quad \min K = 0.}$$

Contrast with D. Let $L := \sum_{n=1}^{\infty} \frac{1}{n^2}$ and $d_n := \sum_{k=1}^{n} \frac{1}{k^2}$. By Worked Example 1(d), Method 1, $\{d_n\}$ is a sequence contained in D with $d_n \to L$ and $L \notin D$; hence, by the Sequential Criterion for Closed Sets, D is not closed. Moreover $L = \sup D$: $L > d_n$ for every $n \in \mathbb{N}$, so L is an upper bound for D; and for every $\varepsilon > 0$ the convergence $d_n \to L$ gives $n \in \mathbb{N}$ with $L - \varepsilon < d_n$, so the ε-Criterion for Suprema applies. Since $\sup D = L \notin D$, the supremum of D is not attained, and D has no maximum (Worked Example 1(d), Method 2). By contrast, K contains its only limit point 0, so K is closed, and $\sup K = \max K = 1 \in K$.

6.6 Summary of Sets of Real Numbers

This section reproduces the deck's own cumulative review of Lectures 13–17: the theory of limit points, closures, closed sets, and closed-and-bounded (compact) sets, in one place.

Limit Points (Chapter 5, Section 5).

- A limit point a of a set A is a number for which there are other numbers in the set arbitrarily close to it.
- Equivalently, a is not isolated from the rest of A: there is no $\varepsilon > 0$ for which A lies completely outside $(a-\varepsilon, a) \cup (a, a+\varepsilon)$.

- An isolated point is a number in the set that is isolated from the rest of the set; that is, it is in the set but is not a limit point.
- Limit points are important in the definition of limits of functions; functions may only have limits at limit points of their domains.
- Limit points are precisely the limits of sequences in the set whose terms are not the limit point itself; this is the Sequential Criterion for Limit Points.

Closure (Chapter 5, Section 5).

- When all limit points of a set A are added to the set, the result is the closure $\overline{A}$ of the set.
- When a continuous function has limits at all limit points of its domain D, the domain can be extended to $\overline{D}$ while maintaining continuity.
- The closure of a set is precisely the set of all limits of convergent sequences in the set; this Sequential Criterion for Closures is a straightforward consequence of the Sequential Criterion for Limit Points.
- Taking the closure of a set does not introduce new limit points; that is, $\overline{\overline{A}} = \overline{A}$.

Closed Sets (Chapter 5, Section 5).

- The closure of a set is an example of a closed set, which is a set containing all its limit points.
- Since a closed set contains all its limit points, it coincides with its closure.
- Closed sets generalise closed intervals.
- Closed sets are precisely those containing all limits of convergent sequences in the set; this is the Sequential Criterion for Closed Sets.

Closed and Bounded (Compact) Sets (this chapter).

- All closed and bounded sets have maxima and minima; this is deduced directly from the Axiom of Completeness together with the Closure Criterion for Closed Sets (Theorem 6.1).
- A set is closed and bounded precisely when it is (sequentially) compact: every sequence in the set has a subsequence converging to a limit in the set. This is the Sequential Compactness of Closed and Bounded Sets (Theorem 6.2), a consequence of the Bolzano–Weierstrass Theorem.
- Closed and bounded sets generalise closed and bounded intervals; for example, the Nested Set Property (Theorem 6.3) generalises the Nested Interval Property from closed and bounded intervals to nonempty, closed, and bounded sets in general.

Sequential Criteria for Sets. Two pairs of criteria run in exact parallel, differing only in whether a subsequence is needed:

$$a \text{ is a limit point of } A \iff \exists\, \{a_n\} \text{ in } A \setminus \{a\} \text{ with } a = \lim a_n,$$

$$a \text{ is in } \overline{A} \iff \exists\, \{a_n\} \text{ in } A \text{ with } a = \lim a_n,$$

$$A \text{ is closed} \iff \text{for every } \{a_n\} \text{ in } A,\ \{a_n\} \text{ convergent} \implies \lim a_n \in A,$$

$$A \text{ is compact} \iff \text{for every } \{a_n\} \text{ in } A,\ \exists \text{ convergent subsequence } \{a_{n_k}\} \text{ with } \lim a_{n_k} \in A.$$

6.7 Chapter Summary

Key Formula: Chapter 6 Recap

For a nonempty set $K \subseteq \mathbb{R}$,

$$K \text{ closed and bounded} \iff K \text{ (sequentially) compact} \iff \max K,\ \min K \text{ both exist.}$$

Nested nonempty closed and bounded sets $K_1 \supseteq K_2 \supseteq \cdots$ always satisfy $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ (Nested Set Property), generalising the Nested Interval Property from intervals to arbitrary closed and bounded sets.

Exam Focus: Chapter 6 Takeaways

- **To show a set is compact:** verify it is nonempty, closed, and bounded, then cite the Sequential Compactness of Closed and Bounded Sets (Theorem 6.2) – do *not* re-derive closedness or boundedness from scratch if either was already established (e.g. via Tutorial 5, Question 4, for unions/intersections of closed sets).
- **To show a set is *not* compact:** exhibit one explicit sequence contained in the set with no subsequence converging to a limit in the set. Two standard recipes: (i) an *unbounded* witness sequence (e.g. $a_n = n$), whose every subsequence is itself unbounded and hence divergent; (ii) a *convergent* witness sequence whose limit lies just outside the set (e.g. $1/n \to 0$ for $(0, 1]$), so every subsequence shares that same outside limit, by the theorem on Subsequential Limits.
- Compactness is strictly stronger than closedness alone: $\mathbb{R}_0^+$ is closed but unbounded, hence not compact (Worked Example 1(b)).
- Tutorial 5, Question 5's restricted method (definition of compactness plus the Sequential Criterion for Closed Sets, *without* Bolzano–Weierstrass or Theorem 6.2) is a favourite source of exam instructions of the form "prove using the definition only."
- Set operations do not all preserve compactness: unions and intersections of closed sets stay closed (Tutorial 5, Question 4), but $K \setminus F$ for compact K and closed F need *not* be compact (Worked Example 3) – removing points can destroy closedness even when both original sets were well-behaved.

7 Midterm Summary: Lectures 1–17

This chapter compresses Lectures 1–17 – the real numbers and the Axiom of Completeness, sequences and their limits, subsequences, the Bolzano–Weierstrass and Monotone Convergence Theorems, series, and the topology of sets of real numbers – for the Midterm Test. It introduces no new material; it compares, connects, and drills. The Midterm follows a fixed format: four questions worth six marks each (twenty-four marks in total), to be completed in sixty minutes under closed-book conditions (calculators are allowed). Questions 1 and 2 are true/false items, each with three parts, for which no justification is required. Questions 3 and 4 each ask the student to write down a definition and use that definition to show a stated result, and each typically bans one or more named theorems from the proof, forcing a first-principles argument. Students may cite any result proved in the lectures or tutorials, unless the question asks them to prove that very result.

7.1 Comparison of Central Results

The six tables below list every named Axiom, Theorem, Lemma, or Lecture Definition from Lectures 1–17, grouped by topic block, together with the hypotheses it needs, the result it is proved from, and where it was tested (lecture, tutorial question, or sample Midterm question).

Real numbers and completeness (Lectures 1–3).

Result	Statement (brief)	Hypotheses	Proved from	Tested
Bounds & Extrema (Def.)	$s = \sup A$: upper bound; $\exists$ no smaller upper bound. $m = \max A$: $m \in A$ and upper bound.	$A \subseteq \mathbb{R}$	definitions only	L2; T1 Q1–5; Sample Q3
Axiom of Completeness	Nonempty, bounded-above A has a supremum.	$A \neq \emptyset$, bdd. above	axiom (no proof)	L2; used everywhere
ε-Criterion (Sup, Inf)	$s = \sup A$ iff s upper bound and $\forall \varepsilon > 0\ \exists a \in A$: $s - \varepsilon < a$.	–	unpacks Def. of sup	L2; T1 Q5; T5 Q3
Nested Interval Property	Nested closed bdd. $I_n = [a_n, b_n]$: $\bigcap I_n \neq \emptyset$.	I_n closed, bdd., nested	Axiom of Completeness	L3; T1 QA2
Archimedean Property	$\forall a \in \mathbb{R}\ \exists n \in \mathbb{N}$: $n > a$ (equiv. $\forall b > 0\ \exists n$: $\frac{1}{n} < b$).	–	Axiom of Completeness	L3; used everywhere

Sequences, limit theorems, and Cauchy sequences (Lectures 4–6).

Result	Statement (brief)	Hypotheses	Proved from	Tested
Uniqueness of Limits	A convergent sequence has a unique limit.	$\{a_n\}$ convergent	ε-chase + triangle inequality	L4
Boundedness of Conv. Seq.	Convergent $\Longrightarrow$ bounded.	$\{a_n\}$ convergent	the $\varepsilon = 1$ trick	L4; L16 (contrapositive)
Algebraic Limit Thm.	Limits respect $+, -, \times, \div$. Parts 1–3 proved; part 4 (reciprocal) stated only.	$a_n \to a$, $b_n \to b$	ε-splitting ($\varepsilon/2$, etc.)	L5; T2 Q4
Order Limit Thm.	Limits respect $\leq$. Part 1 proved; parts 2–3 stated only.	eventually $a_n \geq 0$ (etc.)	definition of limit	L5
Squeeze Thm.	$a_n \leq b_n \leq c_n$, same limit l on outer two $\Longrightarrow$ $b_n \to l$.	eventual sandwich	Def. of limit (not OLT)	L5; T2 Q3
Cauchy Criterion	Convergent iff Cauchy.	–	"only if": triangle ineq.; "if": NIP	L6; T2 Q5; Sample Q4

Subsequences and the big convergence theorems (Lectures 7–9).

Result	Statement (brief)	Hypotheses	Proved from	Tested
Subsequential Limits	Every subsequence of a convergent sequence shares its limit.	$a_n \to a$	$n_k \geq k$ + Def. of limit	L7; T3 Q3, Q4
Divergence Criterion	Bounded seq. divergent iff two subseq. with different limits.	$\{a_n\}$ bounded	"if": Subseq. Limits; "only if": B-W (twice)	L7–8
Bolzano–Weierstrass Thm.	Every bounded sequence has a convergent subsequence.	$\{a_n\}$ bounded	Axiom of Completeness + ε-Criterion	L8; T3 QA3; Sample Q1(b)
Monotone Conv. Thm. (+Lemma)	Eventually monotone + bounded $\Longrightarrow$ convergent.	–	B-W Thm. + Monotone Conv. Lemma	L9; T3 QA1, QA2, QA4

Series and series tests (Lectures 10–12).

Result	Statement (brief)	Hypotheses	Proved from	Tested
Convergence of Series (Def.)	$\sum a_n$ convergent iff partial sums $\{s_n\} \to s$.	–	definition	L10
Terms of Conv. Series	$\sum a_n$ convergent $\implies$ $a_n \to 0$.	–	not shown on slides (cite as known)	L10; T4 Q3(c)
Alg. Limit Thm. for Series	$\sum a_n = s, \sum b_n = t \implies \sum ca_n = cs$, $\sum(a_n + b_n) = s + t$.	–	not shown (transport of ALT)	L10; T4 Q2(c), Q5
Cauchy Criterion for Series	$\sum a_n$ convergent iff partial sums Cauchy.	–	Cauchy Crit. (seq.) on $\{s_n\}$	L10; T4 Q4, QA1, QA2, QA3
Alternating Series Test	$\{b_n\}$ decreasing, $\to 0$ $\implies$ $\sum(-1)^{n-1}b_n$ convergent.	$b_n \geq 0$, decr., $\to 0$	Cauchy Criterion for Series	L11; T4 Q1(c), Q2(d)
Comparison Test	$0 \leq a_n \leq b_n$, $\sum b_n$ conv. $\implies$ $\sum a_n$ conv.	–	Cauchy Criterion for Series	L12; T4 Q1(a),(b), Q3(a)
Absolute Conv. Test	$\sum \lvert a_n \rvert$ conv. $\implies$ $\sum a_n$ conv.	–	Cauchy Crit. for Series + triangle ineq.	L12; T4 Q3(a)
Abs./Cond. Conv. (Def.)	Absolutely: $\sum \lvert a_n \rvert$ conv. Conditionally: conv. but not absolutely.	–	definition	L12; Sample Q2(a)

Limit points and closure (Lectures 13–14).

Result	Statement (brief)	Hypotheses	Proved from	Tested
Limit Point / Isolated Pt. (Def.)	a limit pt. of A: $\forall \varepsilon > 0$ $\exists x \in A \setminus \{a\}$: $\lvert x - a \rvert < \varepsilon$. Isolated: $a \in A$, not a limit pt.	–	definition	L13; T5 Q1
Seq. Criterion for Limit Pts.	a limit pt. of A iff $\exists \{a_n\} \subset A \setminus \{a\}$: $a_n \to a$.	–	Def. of limit (both directions)	L13; T5 Q1
Closure (Def.) + ε-Criterion	$\overline{A} := A \cup \{\text{limit pts. of } A\}$; $a \in \overline{A}$ iff $\forall \varepsilon > 0$ $\exists x \in A$: $\lvert x - a \rvert < \varepsilon$.	–	Def. (ε-Crit. not shown)	L14; T5 Q3
Seq. Criterion for Closures	$a \in \overline{A}$ iff $\exists \{a_n\} \subset A$: $a_n \to a$.	–	transport of Seq. Crit. for LP	L14
Limit Pts. of Closures / $\overline{\overline{A}} = \overline{A}$	a limit pt. of $\overline{A}$ iff limit pt. of A; hence $\overline{\overline{A}} = \overline{A}$.	–	ε-Crit. for Closures + triangle ineq.	L14

Closed sets and compactness (Lectures 15–17).

Result	Statement (brief)	Hypotheses	Proved from	Tested
Closed Set (Def.) + Closure Crit.	F closed: contains all limit pts.; equiv. $\overline{F} \subseteq F$.	–	Limit Pts. of Closures	L15; Sample Q2(b)
Seq. Criterion for Closed Sets	F closed iff every conv. seq. in F has limit in F.	–	Seq. Crit. Closures + Closure Crit.	L15; T5 Q4
Extrema of Closed & Bdd. Sets	Nonempty closed bdd. K has a max and a min.	K closed, bdd., $\neq \emptyset$	Axiom of Completeness + $\sup A \in \overline{A}$ (exercise)	L16; T5 Q3
Sequential Compactness	K closed and bounded iff K is (sequentially) compact.	–	B-W Thm. + Seq. Crit. Closed Sets	L16; T5 Q2, Q5; Sample Q2(c)
Nested Set Property	Nested nonempty closed bdd. $\{K_n\}$: $\bigcap K_n \neq \emptyset$.	K_n nested, closed, bdd., $\neq \emptyset$	Seq. Compactness + Subseq. Limits + Seq. Crit. Closed	L17

7.2 Standard Problem Types and Solution Patterns

Method Pattern: prove $s = \sup A$

Two routes are exam-legal; the sample Midterm (Q3) explicitly bans the ε-Criterion route, so both must be drilled.
Method A (raw definition).

1. State the goal: show (i) s is an upper bound for A, and (ii) s is not greater than any other upper bound for A.

2. For (i), let $a \in A$ be arbitrary and show $a \leq s$ directly.

3. For (ii), let b be an arbitrary upper bound for A and show $s \leq b$, typically by supposing $b < s$ and constructing an $a' \in A$ with $a' > b$, contradicting that b is an upper bound.

4. Conclude, by the definition of supremum, that $s = \sup A$.

Skeleton A. Proof. We show that s is an upper bound for A and that s is not greater than any other upper bound for A. Let $a \in A$ be arbitrary. [Algebra showing $a \leq s$.] Therefore s is an upper bound for A. Now let b be an arbitrary upper bound for A, and suppose, for contradiction, that $b < s$. [Construct $a' \in A$ with $a' > b$.] Since $a' \in A$ and $a' > b$, b is not an upper bound for A, which is a contradiction. Therefore $s \leq b$. Since b was an arbitrary upper bound, s is an upper bound for A that is no greater than any other upper bound for A. Hence, by the definition of supremum, $s = \sup A$. □
Method B (ε-Criterion, when not banned).

1. Show s is an upper bound for A (as in Method A).

2. Fix $\varepsilon > 0$ arbitrary. Construct $a \in A$ with $s - \varepsilon < a$.

3. Conclude, by the ε-Criterion for Suprema, that $s = \sup A$.

Skeleton B. Proof. [Upper bound, as before.] Let $\varepsilon > 0$ be given. [Construct $a \in A$ with $a > s - \varepsilon$.] Therefore $\forall \varepsilon > 0\ \exists a \in A$ with $s - \varepsilon < a$. Since s is an upper bound for A satisfying this condition, the ε-Criterion for Suprema asserts that $s = \sup A$. □

Method Pattern: prove a set has no maximum

1. Recall: $m = \max A$ means $m \in A$ and m is an upper bound for A.
2. Let $m \in A$ be arbitrary (a "candidate" maximum).
3. Construct an explicit $a' \in A$, depending on m, with $a' > m$ (often $a' = m + \delta(m)$ for a small $\delta(m) > 0$ chosen so a' stays in A).
4. Conclude m is not an upper bound for A, hence not the maximum. Since $m \in A$ was arbitrary, A has no maximum.

Skeleton. Proof. Let $m \in A$ be arbitrary. [Construct $a' \in A$ with $a' > m$.] Since $a' \in A$ and $a' > m$, m is not an upper bound for A. Therefore m is not the maximum of A. Since $m \in A$ was arbitrary, therefore A has no maximum. □

Method Pattern: ε-N proof of a limit

1. **(Idea, scratch only.)** Start from the target $|a_n - a| < \varepsilon$ and simplify algebraically until a clean bound of the form $|a_n - a| \le \dfrac{C}{n^p}$ (constants $C, p > 0$) appears.
2. Solve $\dfrac{C}{n^p} < \varepsilon$ for n to see how large n must be; this suggests the choice of N.
3. Use the Archimedean Property to produce an $N \in \mathbb{N}$ with $\dfrac{1}{N} < \dfrac{\varepsilon}{C}$ (or the p-th-root analogue).
4. **(Proof, write-up.)** State "Let $\varepsilon > 0$ be given," invoke the Archimedean Property for N, then take arbitrary $n \ge N$ and chain the inequalities down to $|a_n - a| < \varepsilon$.

Skeleton. Proof. Let $\varepsilon > 0$ be given. By the Archimedean Property, there exists $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \dfrac{\varepsilon}{C}$. Let $n \in \mathbb{N}$ with $n \ge N$. Then $|a_n - a| \le \dfrac{C}{n} \le \dfrac{C}{N} < \varepsilon$. Therefore, $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \ge N$: $|a_n - a| < \varepsilon$, which is the definition of $a = \lim a_n$. □

Method Pattern: ε-arguments with absolute values

1. Triangle inequality is the default tool: $|x + y| \le |x| + |y|$, and its extension $|x - z| \le |x - y| + |y - z|$ for inserting a midpoint.
2. Add-and-subtract a convenient middle term to split a product or quotient difference into pieces each controlled by one hypothesis (e.g. $a_n b_n - ab = (a_n - a)b_n + a(b_n - b)$).
3. To bound a denominator away from 0 (needed for quotients/reciprocals): use the hypothesis's own convergence with a small fixed ε' (e.g. $\varepsilon' = |a|/2$) to show that eventually $|a_n| \ge |a|/2 > 0$, then bound $1/|a_n| \le 2/|a|$.
4. Split a target ε into pieces that sum back to ε: either equal pieces ($\varepsilon/2 + \varepsilon/2$) or hypothesis-shaped pieces (e.g. $\varepsilon/(2(|a| + |b| + 1))$) so that each controllable error contributes a fair share.

Skeleton (bounding a denominator). Proof. Since $a_n \to a$ and $a \neq 0$, taking $\varepsilon' = |a|/2 > 0$ in the definition of the limit gives some $N_1 \in \mathbb{N}$ such that $n \ge N_1 \implies |a_n - a| < |a|/2$. Therefore, for $n \ge N_1$, $|a_n| \ge |a| - |a_n - a| > |a| - |a|/2 = |a|/2 > 0$, so $|a_n| \ge |a|/2$ for all $n \ge N_1$. □

Method Pattern: Cauchy sequence from the definition (bound first)

1. If the target involves a product or square of the terms (e.g. $\{a_n^2\}$), first show $\{a_n\}$ is bounded: a Cauchy sequence is bounded by the same $\varepsilon = 1$ trick as Boundedness of Convergent Sequences (take $\varepsilon = 1$ in the Cauchy definition to bound all terms from some N onward, then take the max over the finitely many terms before N).
2. Let $M > 0$ denote this bound.
3. Factor the target difference, e.g. $|a_n^2 - a_m^2| = |a_n - a_m|\,|a_n + a_m| \le 2M\,|a_n - a_m|$.
4. Given $\varepsilon > 0$, apply the Cauchy definition of $\{a_n\}$ with $\varepsilon' = \varepsilon/(2M)$ to control $|a_n - a_m|$, then chain back to the target.

Skeleton. Proof. Since $\{a_n\}$ is Cauchy, it is bounded: [$\varepsilon = 1$ argument] there is $M > 0$ with $|a_n| \le M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ be given. Since $\{a_n\}$ is Cauchy, there is $N \in \mathbb{N}$ such that $n, m \ge N \implies |a_n - a_m| < \dfrac{\varepsilon}{2M}$. Therefore, for $n, m \ge N$, $|a_n^2 - a_m^2| = |a_n - a_m|\,|a_n + a_m| \le |a_n - a_m|\,(|a_n| + |a_m|) \le 2M\,|a_n - a_m| < 2M \cdot \dfrac{\varepsilon}{2M} = \varepsilon$. Hence $\{a_n^2\}$ is Cauchy. □

Method Pattern: induction

1. State the proposition $P(n)$ to be shown for all $n \in \mathbb{N}$ (or $n \ge n_0$).
2. **Base case:** verify $P(1)$ (or $P(n_0)$) directly.
3. **Inductive step:** assume $P(k)$ holds for some $k \in \mathbb{N}$ (the induction hypothesis), and show $P(k+1)$ follows.
4. Conclude, by (mathematical) induction, that $P(n)$ holds for every $n \in \mathbb{N}$.

Skeleton. Proof. We show $P(n)$ for all $n \in \mathbb{N}$ by induction. **Base case.** [Verify $P(1)$.] **Inductive step.** Suppose $P(k)$ holds for some $k \in \mathbb{N}$. [Use this to derive $P(k+1)$.] Therefore $P(k) \implies P(k+1)$. By induction, therefore $P(n)$ holds for every $n \in \mathbb{N}$. □

Method Pattern: Archimedean arguments

1. Two interchangeable forms: $\forall a \in \mathbb{R}\ \exists n \in \mathbb{N}$: $n > a$; and $\forall b > 0\ \exists n \in \mathbb{N}$: $\frac{1}{n} < b$.
2. Use the second form whenever the target inequality already has the shape "something $< \varepsilon$" (the most common exam use, inside an ε-N proof).
3. Use the first form when the target is an unboundedness-flavoured claim (e.g. showing a set of natural numbers is nonempty, as in the Archimedean Property's own proof).

Skeleton. Proof. Let $\varepsilon > 0$ be given. By the Archimedean Property, there exists $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \varepsilon$. [Continue the ε-N argument using this N.] □

Method Pattern: subsequence arguments / Divergence Criterion

1. **Check boundedness first.** The Divergence Criterion as stated in this course applies only to bounded sequences; for an unbounded sequence, use a different route (e.g. exhibit an unbounded subsequence directly, or use the contrapositive of Boundedness of Convergent Sequences).
2. Exhibit two explicit index maps $n_k, m_k : \mathbb{N} \to \mathbb{N}$ (strictly increasing) defining two subsequences.
3. Compute or bound each subsequence's limit (typically via the Algebraic Limit Theorem or Squeeze Theorem).
4. Show the two limits differ.
5. Conclude, by the Divergence Criterion, that the (bounded) sequence diverges.

Skeleton. Proof. For each $n \in \mathbb{N}$, $|a_n| \leq M$ [bound], so $\{a_n\}$ is bounded. Consider the subsequences $\{a_{n_k}\}$ and $\{a_{m_k}\}$ [define n_k, m_k]. [Compute each limit.] Then $\lim_k a_{n_k} = L_1$ and $\lim_k a_{m_k} = L_2$ with $L_1 \neq L_2$. Since $\{a_n\}$ is bounded and has two subsequences converging to different limits, the Divergence Criterion asserts that $\{a_n\}$ is divergent. □

Method Pattern: series – which test?

A short decision procedure for "is $\sum a_n$ convergent?":

1. **Terms test first.** If $a_n \not\to 0$, then $\sum a_n$ diverges immediately (contrapositive of Terms of Convergent Series). If $a_n \to 0$, this alone proves nothing – proceed.
2. **Alternating?** If $a_n = (-1)^{n-1} b_n$ (or $(-1)^n b_n$) with $b_n \geq 0$ eventually decreasing and $b_n \to 0$: the Alternating Series Test gives convergence directly.
3. **Nonnegative terms, comparable to a known series?** Use the Comparison Test: find $0 \leq a_n \leq b_n$ with $\sum b_n$ known convergent (gives $\sum a_n$ convergent), or $0 \leq b_n \leq a_n$ with $\sum b_n$ known divergent (gives $\sum a_n$ divergent, by the contrapositive).
4. **Mixed signs, not a clean alternating pattern?** Test absolute convergence: does $\sum |a_n|$ converge (often via Comparison)? If yes, the Absolute Convergence Test gives $\sum a_n$ convergent. If $\sum |a_n|$ diverges but the Alternating Series Test still applies, the series is conditionally convergent.
5. **No clean comparison available?** Fall back to the Cauchy Criterion for Series directly: bound (or bound below) the tail sum $|a_{N+1} + \cdots + a_n|$.

Method Pattern: limit points / closure / closed sets via sequences

1. To show a is a limit point of A: exhibit a sequence $\{a_n\} \subset A \setminus \{a\}$ with $a_n \to a$ (Sequential Criterion for Limit Points).
2. To show $a \in \overline{A}$: exhibit a sequence $\{a_n\} \subset A$ (the constant sequence $a_n = a$ is allowed if $a \in A$) with $a_n \to a$ (Sequential Criterion for Closures).
3. To show a set F is closed: take an *arbitrary* convergent sequence $\{a_n\} \subset F$ with limit a, and show $a \in F$ (Sequential Criterion for Closed Sets). To disprove closedness: exhibit one convergent sequence in F whose limit lies outside F.
4. To show F is *not* closed via limit points instead: exhibit a limit point of F that is not in F.

Skeleton (proving F closed). Proof. Let $\{a_n\}$ be an arbitrary convergent sequence contained in F, with limit a. [Show $a \in F$, typically by identifying the form of a.] Therefore $a \in F$. Since $\{a_n\}$ was an arbitrary convergent sequence in F, by the Sequential Criterion for Closed Sets, F is closed. □

Method Pattern: compactness – prove and disprove

To prove K is (sequentially) compact, either

1. show K is closed and bounded, then cite the Sequential Compactness of Closed and Bounded Sets; or
2. work directly from the definition: take an arbitrary sequence in K, extract a convergent subsequence (e.g. via Bolzano–Weierstrass), and show its limit lies in K.

To disprove compactness, exhibit one sequence in K with *no* subsequence converging to a limit in K. In practice this comes from exactly one of two failures:

(a) K is unbounded: the witnessing sequence is itself unbounded (e.g. $a_n = n$ when $K \supseteq \mathbb{N}$), so no subsequence is even bounded, hence none converges.

(b) K is bounded but not closed: the witnessing sequence lies in K and converges (in $\mathbb{R}$) to a point *outside* K; since every subsequence of a convergent sequence shares its limit (Subsequential Limits), no subsequence can converge to a limit inside K either.

Method Pattern: answering True/False items fast

1. No justification is required on the paper, but decide correctly under time pressure by checking against a small mental library of stock (counter)examples: $a_n = (-1)^n$; $a_n = n$; $a_n = 1/n$; $a_n = 1$ if n odd, 0 if n even; the harmonic series $\sum 1/n$; the alternating harmonic series $\sum (-1)^{n-1}/n$; $A = (0,1)$, $A = \mathbb{N}$, $A = \{1/n : n \in \mathbb{N}\}$.
2. For a claimed **True** statement: try to recall or instantly construct a witnessing example; if one comes to mind within a few seconds, it is very likely True.
3. For a claimed **False** statement: recall the *exact* correct theorem and see which hypothesis, quantifier, or strict/non-strict distinction the false statement has quietly dropped or reversed (see the Traps below – this is precisely how most False items in Tutorials 1–5 and the sample Midterm are built).
4. If genuinely unsure, sketch the construction on scratch paper first; a five-line justification, while not required, prevents careless flips of True/False under time pressure.

7.3 Common Mistakes and Exam Traps

Trap 1: sup vs. max

The supremum of a set need not belong to the set (e.g. $\sup(0,1) = 1 \notin (0,1)$), so "$s = \sup A$" alone never lets you write "$s \in A$." To prove $m = \max A$ you must separately exhibit $m \in A$ *and* show m is an upper bound; to prove A has *no* maximum, do not merely note $\sup A \notin A$ – if $\sup A$ exists but is not attained, that already shows no maximum exists, but if A is unbounded above there is no supremum either, and both cases must be handled explicitly.

Trap 2: quantifier order

In every ε-N definition, N is chosen *after*, and may depend on, ε – never the reverse. Writing "let $N \in \mathbb{N}$, then for all $\varepsilon > 0$..." reverses the quantifiers and proves a strictly stronger (usually false) statement. Likewise, N may not depend on n: choosing "$N = n$" or "N depending on the specific n being tested" invalidates the $\forall n \geq N$ clause that follows.

Trap 3: strict vs. non-strict inequalities under limits

The Order Limit Theorem only preserves $\leq$, never $<$: even if $\{a_n\}$ eventually satisfies the strict inequality $a_n > 0$, the limit may only satisfy $\lim a_n \geq 0$ (e.g. $a_n = 1/n > 0$ but $\lim a_n = 0$). Never upgrade a non-strict conclusion "$\lim a_n \geq 0$" back to "$\lim a_n > 0$," and never assume a hypothesis stated with $\leq$ can be sharpened to $<$ after taking limits.

Trap 4: $a_n \to 0$ does not give convergence of $\sum a_n$

The Theorem on Terms of Convergent Series is a *necessary*, not sufficient, condition: $a_n \to 0$ never by itself proves $\sum a_n$ converges. The standard counterexample is the harmonic series $\sum 1/n$, whose terms $\to 0$ but which diverges (provable directly via the Cauchy Criterion for Series: the block sum from $N+1$ to $2N$ is always at least $\frac{1}{2}$, so the partial sums are never Cauchy). Only the contrapositive direction ($a_n \not\to 0 \implies \sum a_n$ diverges) is a valid one-line test.

Trap 5: the Divergence Criterion needs boundedness

As stated in this course, the Divergence Criterion is an "if and only if" for *bounded* sequences only; an unbounded sequence with two subsequences going to different limits (or to $\pm\infty$) is not covered by this exact theorem, and citing it without first establishing boundedness is an incomplete proof. For unbounded sequences, argue divergence directly (e.g. exhibit an unbounded subsequence, contradicting Boundedness of Convergent Sequences).

Trap 6: a limit point need not be in the set

a can be a limit point of A without $a \in A$ (e.g. $A = (0, 1)$ has limit point $0 \notin A$). Conversely, points of A need not be limit points – they may be isolated. Do not conflate "limit point of A" with "element of A," and remember every point of A is exactly one of isolated or a limit point (never both, never neither).

Trap 7: closed *and* bounded are both needed

Neither hypothesis alone gives extrema or compactness. $\mathbb{N}$ is closed (it has no limit points at all) but unbounded, and has no maximum. $(0, 1)$ is bounded but not closed, and has neither a maximum nor a minimum. Only a set that is *both* closed and bounded is guaranteed a max and min (Extrema of Closed and Bounded Sets) and is guaranteed compact (Sequential Compactness of Closed and Bounded Sets); when disproving compactness, always identify *which* of the two properties fails.

Trap 8: using a banned theorem

Every Q3/Q4-style proof question in this course typically forbids one or more named theorems (e.g. Sample Q3 bans the ε-Criterion for Suprema and Infima; Sample Q4 and Tutorial 2 Q5(b) ban the Cauchy Criterion and the Algebraic Limit Theorem). Read the restriction line *before* drafting the proof: a mathematically correct proof that cites a banned theorem earns little to no credit, since the entire point of the question is to test the raw definition. Always re-check the final write-up against the stated restrictions before moving on.

Trap 9: sentence fragments and missing "therefore"

A bare chain of symbols, even if mathematically correct, does not satisfy the course's marking rubric. Every inferential step needs an explicit connective ("therefore," "hence," "so," or $\therefore$), and every quantifier needs words around it: never write a bare "$\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \ldots$" string inside a Proof paragraph – write "Let $\varepsilon > 0$ be given. There exists $N \in \mathbb{N}$ such that ...." Compare the bad fragment "$|a| \leq b.\ -b \leq a \leq b.$" with the required "$|a| \leq b.$ Therefore $-b \leq a \leq b.$"

7.4 Integrated Worked Review Examples

Integrated Example 1: a recursive sequence (Lectures 1–3, 5, 9)

Define $a_1 := 1$ and $a_{n+1} := \sqrt{2 + a_n}$ for each $n \in \mathbb{N}$. Show that $\{a_n\}$ converges, and find its limit.
Method 1: Induction for bounds and monotonicity, then the Monotone Convergence Theorem

Proof. **Bounded.** Let $P(n)$ be the statement $0 < a_n < 2$. We show that $P(n)$ holds for every $n \in \mathbb{N}$ by induction. Base case: $a_1 = 1$ and $0 < 1 < 2$. Therefore $P(1)$ holds. Inductive step: suppose $P(k)$ holds for some $k \in \mathbb{N}$, that is, $0 < a_k < 2$. Adding 2 gives $2 < 2 + a_k < 4$. For real numbers $0 < u < v$ we have $\sqrt{v} - \sqrt{u} = \dfrac{v - u}{\sqrt{v} + \sqrt{u}} > 0$, so taking square roots preserves strict inequalities between positive numbers. Hence

$$\sqrt{2} < \sqrt{2 + a_k} < \sqrt{4} \implies \sqrt{2} < a_{k+1} < 2 \implies 0 < a_{k+1} < 2.$$

Therefore $P(k) \implies P(k+1)$, and by induction $0 < a_n < 2$ for every $n \in \mathbb{N}$.
Increasing. Let $n \in \mathbb{N}$. Since $0 < a_n < 2$, we have $a_n - 2 < 0$ and $a_n + 1 > 0$. Therefore

$$(a_n - 2)(a_n + 1) < 0 \implies a_n^2 - a_n - 2 < 0 \implies a_n^2 < 2 + a_n = a_{n+1}^2.$$

Hence $0 < a_{n+1}^2 - a_n^2 = (a_{n+1} - a_n)(a_{n+1} + a_n)$. Since $a_{n+1} + a_n > 0$, therefore $a_{n+1} - a_n > 0$, that is, $a_n < a_{n+1}$. Since $n \in \mathbb{N}$ was arbitrary, $\{a_n\}$ is increasing.
Convergent. $\{a_n\}$ is increasing, and $a_n < 2$ for every $n \in \mathbb{N}$. Therefore $\{a_n\}$ is monotone and bounded, hence convergent (MCT). Let $L := \lim_{n\to\infty} a_n$.
Limit. The sequence $\{a_{n+1}\}$ is the subsequence $\{a_{n_k}\}$ with the strictly increasing indices $n_k = k+1$. Therefore $\lim_{n\to\infty} a_{n+1} = L$ (Subsequential Limits). Squaring the recursion gives $a_{n+1}^2 = 2 + a_n$ for every $n \in \mathbb{N}$. By the Algebraic Limit Theorem (product rule and sum rule),

$$\lim_{n\to\infty} a_{n+1}^2 = L \cdot L = L^2 \qquad \text{and} \qquad \lim_{n\to\infty} (2 + a_n) = 2 + L.$$

Since $\{a_{n+1}^2\}$ and $\{2 + a_n\}$ are the same sequence, uniqueness of limits gives

$$L^2 = 2 + L \implies L^2 - L - 2 = 0 \implies (L - 2)(L + 1) = 0 \implies L = 2 \text{ or } L = -1.$$

For every $n \in \mathbb{N}$, $a_n > 0$, so $a_n \geq 0$. Therefore $L \geq 0$ (Order Limit Theorem). Hence $L \neq -1$, and therefore $L = 2$. □

$$\boxed{\lim_{n\to\infty} a_n = 2.}$$

Method 2: Contraction estimate and the definition of limit

Proof. **Positivity.** We show $a_n > 0$ for every $n \in \mathbb{N}$ by induction. Base case: $a_1 = 1 > 0$. Inductive step: if $a_k > 0$ for some $k \in \mathbb{N}$, then $2 + a_k > 2 > 0$, so $a_{k+1} = \sqrt{2 + a_k} > 0$. Therefore $a_n > 0$ for every $n \in \mathbb{N}$, and every $a_{n+1} = \sqrt{2 + a_n}$ is defined.
One-step estimate. Let $n \in \mathbb{N}$. Since $a_n > 0$, we have $\sqrt{2 + a_n} + 2 > 2$. Multiplying and dividing by $\sqrt{2 + a_n} + 2$,

$$|a_{n+1} - 2| = \left|\sqrt{2 + a_n} - 2\right| = \frac{|(2 + a_n) - 4|}{\sqrt{2 + a_n} + 2} = \frac{|a_n - 2|}{\sqrt{2 + a_n} + 2} \leq \frac{|a_n - 2|}{2}.$$

Geometric bound. We show $|a_n - 2| \leq \dfrac{1}{2^{n-1}}$ for every $n \in \mathbb{N}$ by induction. Base case: $|a_1 - 2| = 1 = \dfrac{1}{2^0}$. Inductive step: if $|a_k - 2| \leq \dfrac{1}{2^{k-1}}$ for some $k \in \mathbb{N}$, then the one-step estimate gives $|a_{k+1} - 2| \leq \dfrac{1}{2} \cdot \dfrac{1}{2^{k-1}} = \dfrac{1}{2^k}$. Therefore the bound holds for every $n \in \mathbb{N}$.
Next, $2^{n-1} \geq n$ for every $n \in \mathbb{N}$, by induction: $2^0 = 1 \geq 1$, and if $2^{k-1} \geq k$ then $2^k = 2 \cdot 2^{k-1} \geq 2k \geq k + 1$. Therefore

$$|a_n - 2| \leq \frac{1}{2^{n-1}} \leq \frac{1}{n} \qquad \text{for every } n \in \mathbb{N}.$$

Limit. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $N \in \mathbb{N}$ such that $\dfrac{1}{N} < \varepsilon$. Let $n \in \mathbb{N}$ with $n \geq N$. Then

$$|a_n - 2| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore $\forall \varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - 2| < \varepsilon$. Hence $\{a_n\}$ converges and $\lim a_n = 2$ (def. of limit). □

$$\boxed{\lim_{n\to\infty} a_n = 2.}$$

Integrated Example 2: a Cauchy sequence generates a telescoping series (Lectures 6, 7, 5, 10)

Let $\{a_n\}$ be a Cauchy sequence, and define $b_n := a_{n+1} - a_n$ for each $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} b_n$ is convergent, and find its sum.
Method 1: Telescoping partial sums, Cauchy Criterion, Algebraic Limit Theorem

Proof. **Partial sums.** For $N \in \mathbb{N}$ let $s_N := b_1 + b_2 + \cdots + b_N$. We show $s_N = a_{N+1} - a_1$ for every $N \in \mathbb{N}$ by induction. Base case: $s_1 = b_1 = a_2 - a_1$. Inductive step: if $s_k = a_{k+1} - a_1$ for some $k \in \mathbb{N}$, then

$$s_{k+1} = s_k + b_{k+1} = (a_{k+1} - a_1) + (a_{k+2} - a_{k+1}) = a_{k+2} - a_1.$$

Therefore $s_N = a_{N+1} - a_1$ for every $N \in \mathbb{N}$ (telescoping).
Convergence of $\{a_n\}$. $\{a_n\}$ is Cauchy. Therefore $\{a_n\}$ is convergent (Cauchy Criterion). Let $L := \lim_{n\to\infty} a_n$. The sequence $\{a_{N+1}\}$ is the subsequence of $\{a_n\}$ with the strictly increasing indices $n_k = k + 1$. Therefore $\lim_{N\to\infty} a_{N+1} = L$ (Subsequential Limits).
Sum. The constant sequence $\{a_1\}$ converges to a_1. Hence, by the Algebraic Limit Theorem (difference of two convergent sequences),

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} (a_{N+1} - a_1) = L - a_1.$$

Therefore the sequence of partial sums $\{s_N\}$ converges to $L - a_1$, so $\sum b_n$ is convergent with sum $L - a_1$ (def. of convergence of a series). □

$$\boxed{\sum_{n=1}^{\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} a_n - a_1.}$$

Method 2: Cauchy Criterion for Series, then an index shift

Proof. **Block sums.** Let $m, n \in \mathbb{N}$ with $n > m$. In the sum

$$b_{m+1} + b_{m+2} + \cdots + b_n = (a_{m+2} - a_{m+1}) + (a_{m+3} - a_{m+2}) + \cdots + (a_{n+1} - a_n)$$

each a_j with $m+2 \le j \le n$ appears once with sign $+$ and once with sign $-$. Therefore $b_{m+1} + \cdots + b_n = a_{n+1} - a_{m+1}$. In the same way, $s_N := b_1 + \cdots + b_N = a_{N+1} - a_1$ for every $N \in \mathbb{N}$.
Convergence. Let $\varepsilon > 0$ be given. Since $\{a_n\}$ is Cauchy, there is $N \in \mathbb{N}$ such that $|a_p - a_q| < \varepsilon$ for all $p, q \ge N$. Let $n > m \ge N$. Then $n+1 > m+1 \ge N$, so

$$|b_{m+1} + \cdots + b_n| = |a_{n+1} - a_{m+1}| < \varepsilon.$$

Therefore $\forall \varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\forall n > m \ge N$, $|b_{m+1} + \cdots + b_n| < \varepsilon$. Hence $\sum b_n$ is convergent (Cauchy Criterion for Series). Let $s := \lim_{N\to\infty} s_N$.
Sum. For every $N \in \mathbb{N}$, $a_{N+1} = s_N + a_1$. Therefore $a_{N+1} \to s + a_1$ as $N \to \infty$ (Algebraic Limit Theorem). Let $\varepsilon > 0$ be given. There is $N_1 \in \mathbb{N}$ such that $|a_{N+1} - (s + a_1)| < \varepsilon$ for every $N \ge N_1$. Let $n \ge N_1 + 1$. Then $n = (n-1) + 1$ with $n - 1 \ge N_1$, so $|a_n - (s + a_1)| < \varepsilon$. Therefore $a_n \to s + a_1$ (def. of limit), that is, $\lim_{n\to\infty} a_n = s + a_1$. Hence $s = \lim_{n\to\infty} a_n - a_1$. □

$$\boxed{\sum_{n=1}^{\infty} (a_{n+1} - a_n) = \lim_{n\to\infty} a_n - a_1.}$$

Integrated Example 3: divergence of a bounded sequence via the Divergence Criterion (Lectures 3, 5, 7–8)

Define $c_n := (-1)^n + \dfrac{(-1)^n}{n}$ for each $n \in \mathbb{N}$. Show that $\{c_n\}$ is divergent.
Method 1: Boundedness and two subsequences with different limits (Divergence Criterion)

Proof. **Bounded.** Let $n \in \mathbb{N}$. Then $c_n = (-1)^n \left(1 + \dfrac{1}{n}\right)$, so

$$|c_n| = |(-1)^n| \cdot \left(1 + \frac{1}{n}\right) = 1 + \frac{1}{n} \le 1 + 1 = 2.$$

Therefore $|c_n| \le 2$ for every $n \in \mathbb{N}$, and $\{c_n\}$ is bounded.
The sequence $\{1/k\}$. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $N \in \mathbb{N}$ with $\dfrac{1}{N} < \varepsilon$. For every $k \ge N$, $\left|\dfrac{1}{k} - 0\right| = \dfrac{1}{k} \le \dfrac{1}{N} < \varepsilon$. Therefore $\dfrac{1}{k} \to 0$ (def. of limit).
Even and odd subsequences. The index maps $n_k = 2k$ and $m_k = 2k - 1$ are strictly increasing, so $\{c_{2k}\}$ and $\{c_{2k-1}\}$ are subsequences of $\{c_n\}$. For each $k \in \mathbb{N}$,

$$c_{2k} = 1 + \frac{1}{2} \cdot \frac{1}{k}, \qquad c_{2k-1} = -1 - \frac{1}{2k-1}.$$

Since $\frac{1}{k} \to 0$, the Algebraic Limit Theorem gives $c_{2k} \to 1 + \frac{1}{2} \cdot 0 = 1$. For the odd terms, $2k - 1 \ge k$, so $0 < \dfrac{1}{2k-1} \le \dfrac{1}{k}$ for every $k \in \mathbb{N}$. Since the constant sequence 0 and the sequence $\frac{1}{k}$ both tend to 0, the Squeeze Theorem gives $\dfrac{1}{2k-1} \to 0$. Hence $c_{2k-1} \to -1 - 0 = -1$ (Algebraic Limit Theorem).
Conclusion. $\{c_n\}$ is bounded, and it has two subsequences with limits 1 and -1, where $1 \ne -1$. Therefore $\{c_n\}$ is divergent (Divergence Criterion). □

$$\boxed{\{c_n\} \text{ is divergent (bounded, with subsequential limits 1 and } -1).}$$

Method 2: Contradiction from the definition of limit with $\varepsilon = 1$

Proof. **Bounded.** For each $n \in \mathbb{N}$, $|c_n| = |(-1)^n|\left(1 + \frac{1}{n}\right) = 1 + \frac{1}{n} \leq 2$. Therefore $\{c_n\}$ is bounded.
Subsequential limits. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $K \in \mathbb{N}$ with $\frac{1}{K} < \varepsilon$. Let $k \geq K$. Then

$$|c_{2k} - 1| = \frac{1}{2k} < \frac{1}{k} \leq \frac{1}{K} < \varepsilon, \qquad |c_{2k-1} - (-1)| = \frac{1}{2k-1} \leq \frac{1}{k} \leq \frac{1}{K} < \varepsilon.$$

Therefore $c_{2k} \to 1$ and $c_{2k-1} \to -1$ (def. of limit).
Divergence. Suppose, for contradiction, that $\{c_n\}$ converges to some $L \in \mathbb{R}$. Taking $\varepsilon = 1$ in the definition of limit, there is $N \in \mathbb{N}$ such that $|c_n - L| < 1$ for every $n \geq N$. Both $2N \geq N$ and $2N + 1 \geq N$, and

$$c_{2N} - c_{2N+1} = \left(1 + \frac{1}{2N}\right) - \left(-1 - \frac{1}{2N+1}\right) = 2 + \frac{1}{2N} + \frac{1}{2N+1} > 2.$$

On the other hand, by the triangle inequality,

$$|c_{2N} - c_{2N+1}| \leq |c_{2N} - L| + |L - c_{2N+1}| < 1 + 1 = 2.$$

Hence $2 < c_{2N} - c_{2N+1} \leq |c_{2N} - c_{2N+1}| < 2$, which is impossible. Therefore $\{c_n\}$ does not converge, that is, $\{c_n\}$ is divergent. □

$$\boxed{\{c_n\} \text{ is divergent (bounded, with subsequential limits 1 and } -1).}$$

Integrated Example 4: limit points, closure, closedness, and compactness of one set (Lectures 5, 13–16)

Define $A := \left\{(-1)^n \frac{n}{n+1} : n \in \mathbb{N}\right\}$. Find the limit points of A, find $\overline{A}$, and determine whether A is closed or (sequentially) compact.
Method 1: Sequential Criterion for Limit Points and a tail-gap argument

Proof. Let $a_n := (-1)^n \frac{n}{n+1}$ for $n \in \mathbb{N}$, so $A = \{a_n : n \in \mathbb{N}\}$. For each $k \in \mathbb{N}$,

$$a_{2k} = \frac{2k}{2k+1} = 1 - \frac{1}{2k+1}, \qquad a_{2k-1} = -\frac{2k-1}{2k} = -1 + \frac{1}{2k}.$$

Bounds. For every $n \in \mathbb{N}$, $0 < \frac{n}{n+1} < 1$. Hence $|a_n| = \frac{n}{n+1} < 1$, that is, $-1 < a_n < 1$. Therefore A is bounded, $1 \notin A$ and $-1 \notin A$.
1 and -1 are limit points. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $K \in \mathbb{N}$ with $\frac{1}{K} < \varepsilon$. For every $k \geq K$,

$$|a_{2k} - 1| = \frac{1}{2k+1} < \frac{1}{K} < \varepsilon, \qquad |a_{2k-1} - (-1)| = \frac{1}{2k} < \frac{1}{K} < \varepsilon.$$

Hence $a_{2k} \to 1$ and $a_{2k-1} \to -1$ (def. of limit). Since $a_{2k} \neq 1$ and $a_{2k-1} \neq -1$ for every $k \in \mathbb{N}$, $\{a_{2k}\}$ lies in $A \setminus \{1\}$ and $\{a_{2k-1}\}$ lies in $A \setminus \{-1\}$. Therefore 1 and -1 are limit points of A (Sequential Criterion for Limit Points).

No other limit points. Let $x \in \mathbb{R}$ with $x \neq 1$ and $x \neq -1$, and let $d := \frac{1}{2}\min\{|x-1|, |x+1|\}$. Then $d > 0$, $|x-1| \geq 2d$ and $|x+1| \geq 2d$. By the Archimedean Property there is $K \in \mathbb{N}$ with $\dfrac{1}{K} < d$. Let $n \in \mathbb{N}$ with $n \geq 2K$. If $n = 2k$ is even, then $k \geq K$ and $|a_n - 1| = \dfrac{1}{2k+1} < \dfrac{1}{K} < d$, so by the triangle inequality

$$|a_n - x| \geq |x-1| - |a_n - 1| > 2d - d = d.$$

If $n = 2k-1$ is odd, then $2k - 1 \geq 2K$ gives $k > K$, so $|a_n + 1| = \dfrac{1}{2k} < \dfrac{1}{K} < d$ and

$$|a_n - x| \geq |x+1| - |a_n + 1| > 2d - d = d.$$

Hence $|a_n - x| > d$ for every $n \geq 2K$. The set $E := \{|a_n - x| : 1 \leq n < 2K,\ a_n \neq x\}$ is a finite set of positive numbers, so $\delta := \min(\{d\} \cup E) > 0$. Let $y \in A \setminus \{x\}$. Then $y = a_n$ for some $n \in \mathbb{N}$ with $a_n \neq x$. If $n \geq 2K$, then $|y - x| > d \geq \delta$; if $n < 2K$, then $|y - x| \in E$, so $|y - x| \geq \delta$. Therefore no $y \in A \setminus \{x\}$ satisfies $|y - x| < \delta$, and x is not a limit point of A (def. of limit point). Hence the set of limit points of A is exactly $\{-1, 1\}$.

Closure and closedness. By definition, $\overline{A} = A \cup \{\text{limit points of } A\} = A \cup \{-1, 1\}$. Since $1 \in \overline{A}$ and $1 \notin A$, $\overline{A} \not\subseteq A$. Therefore A is not closed (Closure Criterion).

Compactness. The sequence $\{a_{2k}\}$ lies in A and converges to 1. Every subsequence of $\{a_{2k}\}$ converges to 1 (Subsequential Limits), and by uniqueness of limits it converges to no other number. Since $1 \notin A$, no subsequence of $\{a_{2k}\}$ converges to a limit in A. Therefore A is not (sequentially) compact (def. of compactness). Together with the bounds, A is bounded but not closed. □

Limit points of $A = \{-1, 1\}$; $\quad \overline{A} = A \cup \{-1, 1\}$; A is bounded but not closed, hence not compact.

Method 2: ε-definition with a case split on the position of x, then Sequential Compactness

Proof. For $k \in \mathbb{N}$ let $p_k := \dfrac{2k}{2k+1} = 1 - \dfrac{1}{2k+1}$ and $q_k := -\dfrac{2k-1}{2k} = -1 + \dfrac{1}{2k}$; these are the terms of A with even and with odd index, so $A = P \cup Q$ with $P := \{p_k : k \in \mathbb{N}\}$ and $Q := \{q_k : k \in \mathbb{N}\}$. Since $\dfrac{1}{2k+1}$ and $\dfrac{1}{2k}$ decrease as k increases, $\{p_k\}$ is increasing and $\{q_k\}$ is decreasing. Hence, for every $k \in \mathbb{N}$,

$$\tfrac{2}{3} = p_1 \leq p_k < 1 \qquad \text{and} \qquad -1 < q_k \leq q_1 = -\tfrac{1}{2}.$$

Therefore $P \subseteq [\frac{2}{3}, 1)$ and $Q \subseteq (-1, -\frac{1}{2}]$. Hence $A \subseteq (-1, 1)$, so A is bounded and $\pm 1 \notin A$, and A has no point in $(-\frac{1}{2}, \frac{2}{3})$. Let $x \in \mathbb{R}$.

Case $x = 1$ or $x = -1$. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $k \in \mathbb{N}$ with $\dfrac{1}{k} < \varepsilon$. Then $p_k \in A \setminus \{1\}$ with $|p_k - 1| = \dfrac{1}{2k+1} < \varepsilon$, and $q_k \in A \setminus \{-1\}$ with $|q_k + 1| = \dfrac{1}{2k} < \varepsilon$. Therefore 1 and -1 are limit points of A (def. of limit point).

Case $x > 1$ or $x < -1$. If $x > 1$, let $\varepsilon := x - 1 > 0$; every $y \in A$ satisfies $y < 1$, so $|y - x| = x - y > x - 1 = \varepsilon$. If $x < -1$, let $\varepsilon := -1 - x > 0$; every $y \in A$ satisfies $y > -1$, so $|y - x| = y - x > -1 - x = \varepsilon$. In both cases no $y \in A \setminus \{x\}$ satisfies $|y - x| < \varepsilon$, so x is not a limit point of A.

Case $-\frac{1}{2} < x < \frac{2}{3}$. Let $\varepsilon := \min\{x + \frac{1}{2},\ \frac{2}{3} - x\} > 0$. Each $y \in A$ satisfies $y \leq -\frac{1}{2}$ or $y \geq \frac{2}{3}$. If $y \leq -\frac{1}{2}$, then $|y - x| = x - y \geq x + \frac{1}{2} \geq \varepsilon$; if $y \geq \frac{2}{3}$, then $|y - x| = y - x \geq \frac{2}{3} - x \geq \varepsilon$. Therefore x is not a limit point of A.

Case $\frac{2}{3} \le x < 1$. Since $1 - x > 0$, the Archimedean Property gives $K \in \mathbb{N}$ with $\dfrac{1}{K} < 1 - x$, so $p_K = 1 - \dfrac{1}{2K+1} > 1 - \dfrac{1}{K} > x$. Let

$$\varepsilon := \min\big(\{p_K - x,\ x + \tfrac{1}{2}\} \cup \{|p_k - x| : 1 \le k < K,\ p_k \ne x\}\big),$$

a minimum of finitely many positive numbers, so $\varepsilon > 0$. Let $y \in A \setminus \{x\}$. If $y \in Q$, then $y \le -\frac{1}{2}$ and $|y - x| = x - y \ge x + \frac{1}{2} \ge \varepsilon$. If $y = p_k$ with $k \ge K$, then $y \ge p_K > x$ and $|y - x| \ge p_K - x \ge \varepsilon$. If $y = p_k$ with $k < K$, then $p_k \ne x$ and $|y - x| \ge \varepsilon$. Therefore x is not a limit point of A.

Case $-1 < x \le -\frac{1}{2}$. Since $x + 1 > 0$, the Archimedean Property gives $K \in \mathbb{N}$ with $\dfrac{1}{K} < x + 1$, so $q_K = -1 + \dfrac{1}{2K} < -1 + \dfrac{1}{K} < x$. Let

$$\varepsilon := \min\big(\{x - q_K,\ \tfrac{2}{3} - x\} \cup \{|q_k - x| : 1 \le k < K,\ q_k \ne x\}\big) > 0.$$

Let $y \in A \setminus \{x\}$. If $y \in P$, then $y \ge \frac{2}{3}$ and $|y - x| = y - x \ge \frac{2}{3} - x \ge \varepsilon$. If $y = q_k$ with $k \ge K$, then $y \le q_K < x$ and $|y - x| \ge x - q_K \ge \varepsilon$. If $y = q_k$ with $k < K$, then $q_k \ne x$ and $|y - x| \ge \varepsilon$. Therefore x is not a limit point of A.

The five cases cover every $x \in \mathbb{R}$. Therefore the set of limit points of A is exactly $\{-1, 1\}$, and $\overline{A} = A \cup \{-1, 1\}$ (def. of closure).

Not closed. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $N \in \mathbb{N}$ with $\dfrac{1}{N} < \varepsilon$. For every $k \ge N$, $|p_k - 1| = \dfrac{1}{2k+1} < \dfrac{1}{k} \le \dfrac{1}{N} < \varepsilon$. Therefore $\{p_k\}$ is a sequence in A with $p_k \to 1$ (def. of limit), and $1 \notin A$. Hence A is not closed (Sequential Criterion for Closed Sets).

Not compact. A set of real numbers is (sequentially) compact if and only if it is closed and bounded (Sequential Compactness). A is bounded but not closed. Therefore A is not compact. □

Limit points of $A = \{-1, 1\}$; $\quad \overline{A} = A \cup \{-1, 1\}$; A is bounded but not closed, hence not compact.

Integrated Example 5: the Nested Set Property in action (Lectures 3, 15, 16–17)

For each $n \in \mathbb{N}$, define $K_n := \left[-\dfrac{1}{n}, \dfrac{1}{n}\right] \cup \{5\}$. Show $\{K_n\}$ satisfies the hypotheses of the Nested Set Property, and identify $\bigcap_{n=1}^{\infty} K_n$.

Method 1: Limit-point definition for each piece, union of closed sets, Archimedean Property

Proof. **Closed.** Fix $n \in \mathbb{N}$ and write $I_n := \left[-\frac{1}{n}, \frac{1}{n}\right]$, so $K_n = I_n \cup \{5\}$.

I_n is closed. Let $a \in \mathbb{R}$ with $a \notin I_n$. If $a > \frac{1}{n}$, let $\varepsilon := a - \frac{1}{n} > 0$; every $x \in I_n$ satisfies $x \le \frac{1}{n}$, so $|x - a| = a - x \ge a - \frac{1}{n} = \varepsilon$. If $a < -\frac{1}{n}$, let $\varepsilon := -\frac{1}{n} - a > 0$; every $x \in I_n$ satisfies $x \ge -\frac{1}{n}$, so $|x - a| = x - a \ge -\frac{1}{n} - a = \varepsilon$. In both cases no $x \in I_n \setminus \{a\}$ satisfies $|x - a| < \varepsilon$, so a is not a limit point of I_n. Therefore every limit point of I_n lies in I_n, and I_n is closed (def. of closed set).

$\{5\}$ is closed. Let $a \in \mathbb{R}$ with $a \ne 5$, and let $\varepsilon := \frac{1}{2}|5 - a| > 0$. The only element of $\{5\} \setminus \{a\}$ is 5, and $|5 - a| = 2\varepsilon > \varepsilon$. Therefore no $x \in \{5\} \setminus \{a\}$ satisfies $|x - a| < \varepsilon$, so a is not a limit point of $\{5\}$. Hence every limit point of $\{5\}$ lies in $\{5\}$, and $\{5\}$ is closed.

The union of two closed sets is closed (Tutorial 5 Q4(a)). Therefore $K_n = I_n \cup \{5\}$ is closed.

Bounded and nonempty. Let $x \in K_n$. Then $x = 5$ or $-\frac{1}{n} \le x \le \frac{1}{n}$; since $\frac{1}{n} \le 1$, in both cases $-1 \le x \le 5$, so $|x| \le 5$. Therefore K_n is bounded. Also $0 \in I_n \subseteq K_n$, so $K_n \ne \emptyset$.

Nested. Let $n \in \mathbb{N}$ and $x \in K_{n+1}$. If $x = 5$, then $x \in K_n$. Otherwise $|x| \le \frac{1}{n+1} < \frac{1}{n}$, so

$x \in I_n \subseteq K_n$. Therefore $K_{n+1} \subseteq K_n$ for every $n \in \mathbb{N}$. Hence $\{K_n\}$ is a nested sequence of nonempty, closed, bounded sets, and therefore $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ (Nested Set Property).
Intersection. For every $n \in \mathbb{N}$, $5 \in K_n$ and $0 \in K_n$. Therefore $\{0,5\} \subseteq \bigcap_n K_n$. Conversely, let $x \in \mathbb{R}$ with $x \neq 0$ and $x \neq 5$. Then $|x| > 0$, so by the Archimedean Property there is $n \in \mathbb{N}$ with $\frac{1}{n} < |x|$. Hence $x \notin I_n$; since also $x \neq 5$, $x \notin K_n$. Therefore $x \notin \bigcap_n K_n$. Hence $\bigcap_n K_n = \{0,5\}$. □

$$\boxed{\bigcap_{n=1}^{\infty} K_n = \{0,5\} \neq \emptyset.}$$

Method 2: Sequential Criterion for Closed Sets and the Order Limit Theorem

Proof. **Closed.** Fix $n \in \mathbb{N}$. Let $\{x_m\}$ be a sequence in K_n that converges to some $x \in \mathbb{R}$. For each $m \in \mathbb{N}$, either $x_m = 5$ or $-\frac{1}{n} \leq x_m \leq \frac{1}{n}$, so at least one of these two alternatives holds for infinitely many indices m.
Case 1: $x_m = 5$ for infinitely many m. Listing these indices in increasing order $m_1 < m_2 < \cdots$ gives a subsequence with $x_{m_j} = 5$ for every j, so $x_{m_j} \to 5$. By Subsequential Limits this subsequence also converges to x. Therefore $x = 5 \in K_n$ (uniqueness of limits).
Case 2: $-\frac{1}{n} \leq x_m \leq \frac{1}{n}$ for infinitely many m. Listing these indices as $m_1 < m_2 < \cdots$ gives a subsequence with $x_{m_j} \to x$ (Subsequential Limits). For every j, $x_{m_j} + \frac{1}{n} \geq 0$ and $\frac{1}{n} - x_{m_j} \geq 0$. By the Algebraic Limit Theorem these two sequences converge to $x + \frac{1}{n}$ and $\frac{1}{n} - x$, so the Order Limit Theorem gives $x + \frac{1}{n} \geq 0$ and $\frac{1}{n} - x \geq 0$. Hence $-\frac{1}{n} \leq x \leq \frac{1}{n}$, and $x \in K_n$.
In both cases $x \in K_n$. Therefore K_n is closed (Sequential Criterion for Closed Sets).
Bounded, nonempty, nested. Every $x \in K_n$ satisfies $x = 5$ or $|x| \leq \frac{1}{n} \leq 1$, so $|x| \leq 5$; therefore K_n is bounded. Also $0 \in K_n$, so $K_n \neq \emptyset$. If $x \in K_{n+1}$, then $x = 5 \in K_n$ or $|x| \leq \frac{1}{n+1} < \frac{1}{n}$; therefore $x \in K_n$, and $K_{n+1} \subseteq K_n$. Hence $\{K_n\}$ is nested, nonempty, closed and bounded, and $\bigcap_n K_n \neq \emptyset$ (Nested Set Property).
Intersection. For every $n \in \mathbb{N}$, $0 \in K_n$ and $5 \in K_n$, so $\{0,5\} \subseteq \bigcap_n K_n$. Conversely, let $x \in \bigcap_n K_n$ with $x \neq 5$. Then for every $n \in \mathbb{N}$, $x \in K_n$ and $x \neq 5$, so $|x| \leq \frac{1}{n}$, that is, $b_n := \frac{1}{n} - |x| \geq 0$. Since $\frac{1}{n} \to 0$ (Archimedean Property and def. of limit), the Algebraic Limit Theorem gives $b_n \to 0 - |x| = -|x|$. The Order Limit Theorem then gives $-|x| \geq 0$. Hence $|x| \leq 0$, so $x = 0$. Therefore $\bigcap_n K_n = \{0,5\}$. □

$$\boxed{\bigcap_{n=1}^{\infty} K_n = \{0,5\} \neq \emptyset.}$$

Integrated Example 6: which test? conditional convergence of a mixed series (Lectures 3, 5, 10–12)

Define $d_n := \dfrac{n}{n^2+1}$ for $n \in \mathbb{N}$. Determine whether $\sum_{n=1}^{\infty}(-1)^n d_n$ is absolutely convergent, conditionally convergent, or divergent.
Method 1: Alternating Series Test, then Comparison with $\sum \frac{1}{2n}$

Proof. **Step 1 (decreasing).** For each $n \in \mathbb{N}$, $d_n > 0$. Expanding the numerator of $d_n - d_{n+1}$,

$$n\left[(n+1)^2+1\right] - (n+1)(n^2+1) = (n^3+2n^2+2n) - (n^3+n^2+n+1) = n^2+n-1.$$

Hence

$$d_n - d_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} = \frac{n^2+n-1}{(n^2+1)\left[(n+1)^2+1\right]}.$$

Since $n \geq 1$, $n^2+n-1 \geq 1 > 0$, and the denominator is positive. Therefore $d_n - d_{n+1} > 0$ for every $n \in \mathbb{N}$, so $\{d_n\}$ is decreasing.

Step 2 ($d_n \to 0$). For each $n \in \mathbb{N}$, $n^2+1 > n^2$, so $0 < d_n < \dfrac{n}{n^2} = \dfrac{1}{n}$. Let $\varepsilon > 0$ be given. By the Archimedean Property there is $N \in \mathbb{N}$ with $\dfrac{1}{N} < \varepsilon$, and for every $n \geq N$, $\dfrac{1}{n} \leq \dfrac{1}{N} < \varepsilon$. Hence $\dfrac{1}{n} \to 0$ (def. of limit). Since $0 < d_n < \dfrac{1}{n}$ and both bounding sequences tend to 0, $d_n \to 0$ (Squeeze Theorem).
Step 3 (convergent). $d_n \geq 0$, $\{d_n\}$ is decreasing, and $d_n \to 0$. Therefore $\sum(-1)^{n-1}d_n$ is convergent (Alternating Series Test). Since $(-1)^n d_n = (-1)\cdot(-1)^{n-1}d_n$ for every n, the Algebraic Limit Theorem for Series with $c=-1$ shows that $\sum(-1)^n d_n$ is convergent.
Step 4 ($\sum \frac{1}{2n}$ diverges). Let $N \in \mathbb{N}$. The block from $n = N+1$ to $n = 2N$ has N terms, and for $N+1 \leq n \leq 2N$ we have $2n \leq 4N$, so each term satisfies $\dfrac{1}{2n} \geq \dfrac{1}{4N}$. Hence

$$\sum_{n=N+1}^{2N} \frac{1}{2n} \geq N \cdot \frac{1}{4N} = \frac{1}{4}.$$

Therefore, with $\varepsilon = \frac{1}{4}$, for every $N \in \mathbb{N}$ the indices $2N > N \geq N$ give a block sum that is at least ε. So the partial sums of $\sum \frac{1}{2n}$ do not satisfy the Cauchy condition, and $\sum \frac{1}{2n}$ diverges (Cauchy Criterion for Series).
Step 5 (not absolutely convergent). For each $n \in \mathbb{N}$, $n^2 \geq 1$ gives $n^2+1 \leq 2n^2$, so

$$d_n = \frac{n}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2n} \geq 0.$$

Suppose, for contradiction, that $\sum|(-1)^n d_n| = \sum d_n$ converges. Then $0 \leq \frac{1}{2n} \leq d_n$ for every n, and the Comparison Test gives that $\sum \frac{1}{2n}$ converges, contradicting Step 4. Therefore $\sum|(-1)^n d_n|$ diverges. Hence $\sum(-1)^n d_n$ is convergent but not absolutely convergent, that is, conditionally convergent (def. of conditional convergence). □

$$\boxed{\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1} \text{ is conditionally convergent.}}$$

Method 2: Splitting $d_n = \frac{1}{n} - \frac{1}{n(n^2+1)}$ into the alternating harmonic series and an absolutely convergent series

Proof. **Splitting.** For each $n \in \mathbb{N}$,

$$\frac{1}{n} - \frac{1}{n(n^2+1)} = \frac{(n^2+1)-1}{n(n^2+1)} = \frac{n^2}{n(n^2+1)} = \frac{n}{n^2+1} = d_n.$$

Write $e_n := \dfrac{1}{n(n^2+1)}$. Then $d_n = \dfrac{1}{n} - e_n$ and $(-1)^n d_n = \dfrac{(-1)^n}{n} - (-1)^n e_n$ for every $n \in \mathbb{N}$.
$\sum e_n$ **converges.** For each $n \in \mathbb{N}$, $n^2 \geq n$, so $n^2+1 \geq n+1$ and

$$0 < e_n \leq \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Let $t_N := \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right)$. By induction, $t_N = 1 - \frac{1}{N+1}$: $t_1 = 1 - \frac{1}{2}$, and if $t_k = 1 - \frac{1}{k+1}$ then $t_{k+1} = 1 - \frac{1}{k+1} + \frac{1}{k+1} - \frac{1}{k+2} = 1 - \frac{1}{k+2}$. The sequence $\left\{\frac{1}{N+1}\right\}$ is a subsequence of $\left\{\frac{1}{n}\right\}$, and $\frac{1}{n} \to 0$ (Archimedean Property and def. of limit), so $\frac{1}{N+1} \to 0$ (Subsequential Limits). Therefore $t_N \to 1$

(Algebraic Limit Theorem), and $\sum \frac{1}{n(n+1)}$ converges. Hence $\sum e_n$ converges (Comparison Test). Since $|(-1)^n e_n| = e_n$, $\sum (-1)^n e_n$ converges (Absolute Convergence Test).
$\sum \frac{(-1)^n}{n}$ **converges.** $\frac{1}{n} \geq 0$, $\frac{1}{n+1} < \frac{1}{n}$ for every n, and $\frac{1}{n} \to 0$. Therefore $\sum \frac{(-1)^{n-1}}{n}$ converges (Alternating Series Test), and so does $\sum \frac{(-1)^n}{n}$ (Algebraic Limit Theorem for Series, $c = -1$).
Convergent. By the Algebraic Limit Theorem for Series (sum rule, and $c = -1$),

$$\sum_{n=1}^{\infty} (-1)^n d_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + (-1) \sum_{n=1}^{\infty} (-1)^n e_n$$

is convergent.
Not absolutely convergent. $|(-1)^n d_n| = d_n$ for every n. Suppose, for contradiction, that $\sum d_n$ converges. Since $\frac{1}{n} = d_n + e_n$ and $\sum e_n$ converges, the Algebraic Limit Theorem for Series gives that $\sum \frac{1}{n}$ converges. But for every $N \in \mathbb{N}$, each of the N terms $\frac{1}{n}$ with $N + 1 \leq n \leq 2N$ is at least $\frac{1}{2N}$, so

$$\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{2N} \geq N \cdot \frac{1}{2N} = \frac{1}{2}.$$

Hence, with $\varepsilon = \frac{1}{2}$, the Cauchy condition fails for the indices $2N > N \geq N$, for every $N \in \mathbb{N}$, and $\sum \frac{1}{n}$ diverges (Cauchy Criterion for Series). This is a contradiction. Therefore $\sum d_n$ diverges, and $\sum (-1)^n d_n$ is convergent but not absolutely convergent, that is, conditionally convergent. □

$$\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1} \text{ is conditionally convergent.}}$$

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7.5 Midterm Revision Checklist

Checklist Part 1: can you, without notes, ... (Lectures 1–9)

- ☐ state the definitions of upper/lower bound, supremum/infimum, and maximum/minimum, and the ε-Criterion for Suprema and Infima;
- ☐ state the Axiom of Completeness and explain why it fails for $\mathbb{Q}$ (e.g. $\{q \in \mathbb{Q} : q^2 < 2\}$ has no rational supremum);
- ☐ prove the Nested Interval Property and the Archimedean Property from the Axiom of Completeness;
- ☐ write the ε-N definition of $a = \lim a_n$ from memory, with quantifiers in the correct order;
- ☐ prove Uniqueness of Limits and Boundedness of Convergent Sequences directly from the definition of limit;
- ☐ state all four parts of the Algebraic Limit Theorem and all three parts of the Order Limit Theorem, and recall which parts were actually proved on the slides (part 4 of ALT and parts 2–3 of OLT were not);
- ☐ apply the Squeeze Theorem, and recall it does *not* need the Order Limit Theorem;
- ☐ write both equivalent forms of the Cauchy sequence definition, and prove the Cauchy Criterion in both directions;
- ☐ construct a subsequence explicitly via a strictly increasing index map n_k, and prove Subsequential Limits;
- ☐ state the Divergence Criterion correctly, *including* the boundedness hypothesis, and use it to prove a specific bounded sequence diverges;
- ☐ sketch the supremum-based construction behind the Bolzano–Weierstrass Theorem;
- ☐ state the Monotone Convergence Theorem and the Monotone Convergence Lemma, and know which one needs boundedness.

Checklist Part 2: can you, without notes, ... (Lectures 10–17, and exam craft)

- ☐ distinguish a series' terms from its partial sums, and explain why $a_n \to 0$ is necessary but not sufficient for $\sum a_n$ to converge;
- ☐ explain why the Cauchy Criterion for Series is a direct transport of the (sequence) Cauchy Criterion to the partial sums;
- ☐ apply the Alternating Series Test, the Comparison Test, and the Absolute Convergence Test, and sketch each one's Cauchy-Criterion-based proof;
- ☐ distinguish absolute from conditional convergence, with an example of each;
- ☐ state the Sequential Criterion for Limit Points, for Closures, and for Closed Sets, and use each to compute the limit points, closure, and closedness of a given set;
- ☐ prove that $\overline{\overline{A}} = \overline{A}$ and that the limit points of $\overline{A}$ equal the limit points of A;
- ☐ state and use the Closure Criterion for Closed Sets (F closed iff $\overline{F} \subseteq F$);
- ☐ state the definition of (sequential) compactness and the Sequential Compactness of Closed and Bounded Sets theorem, and use it both to prove *and* to disprove compactness of a given set;
- ☐ prove the Extrema of Closed and Bounded Sets theorem;
- ☐ prove the Nested Set Property using Sequential Compactness;
- ☐ answer a True/False item in under a minute using the mental library of stock (counter)examples;
- ☐ for any Q3/Q4-style proof, identify which named theorems the question bans *before* writing, and produce a first-principles proof instead.

7.6 Master Memory Sheet

Master Memory Sheet 1: Real Numbers and Completeness (Lectures 1–3)

b upper bound for $A : \forall a \in A,\ a \le b$. Similarly lower bound.
$s = \sup A : s$ upper bound for A, and s no greater than any upper bound for A.
$m = \max A : m \in A$ and m upper bound for A. $\max A$ (if exists) $= \sup A$.
ε-Criterion: $s = \sup A \iff s$ upper bound and $\forall \varepsilon > 0\ \exists a \in A :\ s - \varepsilon < a$.
Axiom of Completeness: $A \neq \emptyset$, bounded above $\implies A$ has a supremum.
Nested Interval Property: $I_n = [a_n, b_n]$ closed, bdd., nested $\implies \bigcap_n I_n \neq \emptyset$.
Archimedean Property: $\forall a \in \mathbb{R}\ \exists n \in \mathbb{N} :\ n > a$. Equiv.: $\forall b > 0\ \exists n \in \mathbb{N} :\ \frac{1}{n} < b$.

Master Memory Sheet 2: Sequences and Limits (Lecture 4)

Sequence: a function $\mathbb{N} \to \mathbb{R}$, $\{a_n\}$. Tail: $a_N, a_{N+1}, \ldots$ Eventually: some tail satisfies it.
$a = \lim a_n : \forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \geq N :\ |a_n - a| < \varepsilon$.
Uniqueness of Limits: a convergent sequence has a unique limit.
$\{a_n\}$ bounded: $\exists M > 0\ \forall n \in \mathbb{N} :\ |a_n| \leq M$.
Boundedness of Convergent Sequences: convergent $\implies$ bounded (converse false).

Master Memory Sheet 3: Limit Theorems and Cauchy Sequences (Lectures 5–6)

ALT: $a_n \to a,\ b_n \to b \implies ca_n \to ca;\ a_n + b_n \to a + b;\ a_n b_n \to ab;\ 1/a_n \to 1/a\ (a \neq 0$, unproved).
OLT: $a_n \geq 0$ ev. $\implies \lim a_n \geq 0$ (proved on slides).
$a_n \leq b_n$ ev. $\implies \lim a_n \leq \lim b_n$; $a_n \leq c$ ev. $\implies \lim a_n \leq c$ (not proved on slides).
Squeeze: $a_n \leq b_n \leq c_n$ ev., $\lim a_n = \lim c_n = l \implies b_n \to l$.
$\{a_n\}$ Cauchy: $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \geq N :\ |a_n - a_m| < \varepsilon$.
Cauchy Criterion: $\{a_n\}$ convergent $\iff$ $\{a_n\}$ Cauchy.

Master Memory Sheet 4: Subsequences and the Big Theorems (Lectures 7–9)

Subsequence: $a_{n_1}, a_{n_2}, \ldots,\ n_1 < n_2 < \cdots$ strictly increasing in $\mathbb{N}$.
Subsequential Limits: $a_n \to a \implies$ every subsequence $a_{n_k} \to a$.
Divergence Criterion ($\{a_n\}$ bounded): divergent $\iff$ two subseq. with different limits.
Bolzano–Weierstrass: $\{a_n\}$ bounded $\implies$ has a convergent subsequence.
Monotone: $a_n \leq a_{n+1}\ \forall n$ (increasing); similarly decreasing.
Monotone Conv. Lemma: eventually monotone + convergent subseq. $\implies$ convergent.
Monotone Conv. Thm.: eventually monotone + bounded $\implies$ convergent.

Master Memory Sheet 5: Series (Lectures 10–12)

Series: $\sum a_n$, partial sum $s_n := a_1 + \cdots + a_n$. $\sum a_n$ convergent : $\{s_n\}$ convergent.
Terms of Conv. Series: $\sum a_n$ convergent $\implies a_n \to 0$ (not sufficient!).
Cauchy Crit. for Series: $\sum a_n$ convergent $\iff \forall \varepsilon > 0\ \exists N\ \forall n > m \geq N :\ |a_{m+1} + \cdots + a_n| < \varepsilon$.
Alternating Series Test: $b_n \geq 0$ decreasing, $b_n \to 0 \implies \sum (-1)^{n-1} b_n$ convergent.
Comparison Test: $0 \leq a_n \leq b_n$, $\sum b_n$ conv. $\implies \sum a_n$ conv.
Absolute Conv. Test: $\sum |a_n|$ conv. $\implies \sum a_n$ conv.
$\sum a_n$ absolutely convergent: $\sum |a_n|$ convergent. Conditionally: convergent but not absolutely.

Master Memory Sheet 6: Limit Points and Closure (Lectures 13–14)

a limit point of $A : \forall \varepsilon > 0\ \exists x \in A \setminus \{a\} :\ |x - a| < \varepsilon$.
a isolated point of $A : a \in A,\ a$ not a limit point of A.
Seq. Crit. Limit Pts.: a limit pt. of $A \iff \exists \{a_n\} \subset A \setminus \{a\} :\ a_n \to a$.
$\overline{A} := A \cup \{\text{limit points of } A\}$. ε-Crit. Closure: $a \in \overline{A} \iff \forall \varepsilon > 0\ \exists x \in A :\ |x - a| < \varepsilon$.
Seq. Crit. Closures: $a \in \overline{A} \iff \exists \{a_n\} \subset A :\ a_n \to a$.
Limit Pts. of Closures: a limit pt. of $\overline{A} \iff a$ limit pt. of A. Hence $\overline{\overline{A}} = \overline{A}$.

Master Memory Sheet 7: Closed Sets and Compactness (Lectures 15–17)

F closed: contains all its limit points. Closure Crit.: F closed $\iff \overline{F} \subseteq F$.
Seq. Crit. Closed Sets: F closed $\iff$ every conv. seq. in F has its limit in F.
K (sequentially) compact: every sequence in K has a convergent subseq. with limit in K.
Extrema of Closed & Bdd. Sets: $K \neq \emptyset$, closed, bounded $\implies K$ has a max and a min.
Sequential Compactness: K closed and bounded $\iff K$ compact.
Nested Set Property: K_n nonempty, closed, bdd., nested $\implies \bigcap_n K_n \neq \emptyset$.

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A Theorem & Definition Reference

Chapter 1: Real Numbers and Completeness (Lectures 1–3)

Definitions

- Upper bound / bounded above: $b \in \mathbb{R}$ is an *upper bound* for $A \subseteq \mathbb{R}$ if $\forall a \in A\ a \leq b$; A is *bounded above* if it has an upper bound. *Lower bound / bounded below* are defined symmetrically; A is *bounded* if it is bounded above and below.
- Supremum: $s = \sup A$ if s is an upper bound for A and s is not greater than any other upper bound for A. *Infimum* $\inf A$ is defined symmetrically.
- Maximum: $m = \max A$ if $m \in A$ and m is an upper bound for A. *Minimum* $\min A$ is defined symmetrically.

Results

- Axiom of Completeness: every nonempty $A \subseteq \mathbb{R}$ that is bounded above has a supremum. [axiom; no proof]
- Theorem (Axiomatic Definition of the Real Numbers): all ordered fields satisfying the Axiom of Completeness are isomorphic to each other; $\mathbb{R}$ is the ordered field satisfying it. [stated on slides, not proved in the course (deferred to Spivak); cite by name only]
- Theorem (ε-Criterion for Suprema and Infima): for $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$, $s = \sup A$ iff s is an upper bound for A and $\forall \varepsilon > 0\ \exists a \in A\ s - \varepsilon < a$ (symmetric criterion for $\inf A$). [proved in L2]
- Theorem (Nested Interval Property): if $I_n = [a_n, b_n]$ is a closed and bounded interval for each $n \in \mathbb{N}$ with $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$. [proved in L3]
- Theorem (Archimedean Property): $\forall a \in \mathbb{R}\ \exists n \in \mathbb{N}\ n > a$ (equivalently, $\forall b > 0\ \exists n \in \mathbb{N}\ 1/n < b$). [proved in L3]

Procedures

- ε-Criterion template: unpack $s = \sup A$ via a chain of iffs turning "$s - \varepsilon$ is not an upper bound" into "$\exists a \in A\ a > s - \varepsilon$".
- Nested Interval Property template: let $A = \{a_n : n \in \mathbb{N}\}$ (the left endpoints); the Axiom of Completeness gives $c = \sup A$; show $a_n \leq c$ (as c is an upper bound) and $c \leq b_n$ (as c is the *least* upper bound), so $c \in I_n$ for every n.
- Archimedean Property template: let $A = \{n \in \mathbb{N} : n \leq a\}$; if $A = \emptyset$ then any n works; else the Axiom of Completeness gives $s = \sup A$; $s - 1$ is not an upper bound, so $\exists n \in A$ with $n > s - 1$; then $n + 1 \notin A$, so $n + 1 > a$.

Chapter 2: Sequences, Limits, and Cauchy Sequences (Lectures 4–6)

Definitions

- Sequence: a function from $\mathbb{N}$ to $\mathbb{R}$, written $\{a_n\}$. A *tail* of $\{a_n\}$ is $a_N, a_{N+1}, a_{N+2}, \ldots$ for some $N \in \mathbb{N}$; $\{a_n\}$ satisfies a property *eventually* if some tail satisfies it.
- Limit of a sequence: $a = \lim a_n$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n \geq N \implies |a_n - a| < \varepsilon)$. $\{a_n\}$ is *convergent* if it has a limit, otherwise *divergent*.
- Bounded sequence: $\{a_n\}$ is bounded if $\exists M > 0\ \forall n \in \mathbb{N}\ |a_n| \leq M$.
- Cauchy sequence: $\{a_n\}$ is Cauchy if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ (n \geq N \implies |a_n - a_N| < \varepsilon)$; equivalently, $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \in \mathbb{N}\ (n, m \geq N \implies |a_n - a_m| < \varepsilon)$.

Results

- Theorem (Uniqueness of Limits): a convergent sequence $\{a_n\}$ has a unique limit. [proved in L4]
- Theorem (Boundedness of Convergent Sequences): every convergent sequence is bounded. [proved in L4]
- Theorem (Algebraic Limit Theorem): if $a_n \to a$ and $b_n \to b$, then $ca_n \to ca$ for all $c \in \mathbb{R}$; $a_n + b_n \to a + b$; $a_n b_n \to ab$; and, provided $a \neq 0$, $1/a_n \to 1/a$. [parts 1–3 proved in L5; part 4 (reciprocal) stated on slides; QRS proof in Chapter 2]
- Theorem (Order Limit Theorem): for convergent $\{a_n\}, \{b_n\}$: if eventually $a_n \geq 0$ then $\lim a_n \geq 0$; if eventually $a_n \leq b_n$ then $\lim a_n \leq \lim b_n$; if eventually $a_n \leq c$ then $\lim a_n \leq c$. [part 1 proved in L5; parts 2–3 stated on slides; QRS proof in Chapter 2]
- Theorem (Squeeze Theorem): if eventually $a_n \leq b_n \leq c_n$ and $\lim a_n = \lim c_n = l$, then $\{b_n\}$ converges to l. [proved in L5]
- Theorem (Cauchy Criterion): $\{a_n\}$ is convergent iff it is Cauchy. [proved in L6]

Procedures

- Uniqueness of Limits template: show $|a - a'| < 2\varepsilon$ for every $\varepsilon > 0$, then substitute $\varepsilon = |a - a'|/2$ to force $a = a'$.
- Boundedness of Convergent Sequences template: apply the limit definition at the specific value $\varepsilon = 1$ to get $\{a_n\}$ eventually inside $(a - 1, a + 1)$, hence bounded.
- Algebraic Limit Theorem (sum/product) template: split the target ε into controllable pieces (e.g. $\varepsilon/2$ each, or a $\min\{\sqrt{\varepsilon/2}, \ldots\}$ combination) and use the add-and-subtract identity $a_n b_n - ab = (a_n - a)b_n + a(b_n - b)$ to bound $|a_n b_n - ab|$.
- Cauchy Criterion ("if") template: from finitely many ε_k-neighbourhoods build running max/min sequences c_m, d_m bracketing the tail terms, giving nested intervals $[c_m, d_m]$; apply the Nested Interval Property to obtain the limit candidate, then ε-chase.

Chapter 3: Subsequences, Bolzano-Weierstrass, and Monotone Convergence (Lectures 7–9)

Definitions

- Subsequence: $\{a_{n_k}\}$ where $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence in $\mathbb{N}$; the subsequence is *proper* if $n_k \neq k$ for some $k \in \mathbb{N}$.
- Monotone sequence: $\{a_n\}$ is *increasing* if $\forall n \in \mathbb{N}\ a_n \leq a_{n+1}$ (*strictly increasing* if $<$); *decreasing* symmetrically ($\geq$, strictly $>$). Either case is *monotone* (strictly monotone). $\{a_n\}$ is *eventually monotone* if some tail is monotone.

Results

- Theorem (Subsequential Limits): if $a_n \to a$ and $\{a_{n_k}\}$ is a subsequence of $\{a_n\}$, then $a_{n_k} \to a$. [proved in L7]
- Theorem (Divergence Criterion): a *bounded* sequence $\{a_n\}$ is divergent iff it has two subsequences converging to different limits. [proved in L7 ("if"); L8 ("only if")]
- Theorem (Bolzano-Weierstrass Theorem): every bounded sequence has a convergent subsequence. [proved in L8]
- Lemma (Monotone Convergence Lemma): an eventually monotone sequence with a convergent subsequence is convergent. [proved in L9]

- Theorem (Monotone Convergence Theorem): an eventually monotone and bounded sequence is convergent. [proved in L9]

Procedures

- Subsequential Limits template: reuse the original N as K, using $n_k \geq k$.
- Divergence Criterion ("if") template: contrapositive of Subsequential Limits (boundedness not needed for this direction).
- Bolzano-Weierstrass template: let $a = \sup\{x \in \mathbb{R} : x \leq a_n \text{ for infinitely many } n\}$ (Axiom of Completeness); use the ε-Criterion to show infinitely many terms lie within any ε_k of a; recursively pick the next qualifying term to build $\{a_{n_k}\}$.
- Divergence Criterion ("only if") template: apply Bolzano-Weierstrass to $\{a_n\}$ to get $a_{n_k} \to a$; the terms at distance $\geq \varepsilon$ from a are infinite and still bounded, so Bolzano-Weierstrass applies a second time to extract $\{a_{m_k}\}$ converging away from a.
- Monotone Convergence template: pass to a tail to assume $\{a_n\}$ monotone; sandwich an arbitrary term between two subsequence terms already ε-close to the limit via $|a_{m'} - a| \leq \max\{|a_m - a|, |a_{m''} - a|\}$ for $m \geq m' \geq m''$; chain Bolzano-Weierstrass with the Monotone Convergence Lemma to get the full theorem.

Chapter 4: Series (Lectures 10–12)

Definitions

- Series: $\sum_{n=1}^{\infty} a_n$ for a sequence $\{a_n\}$; the n-th partial sum is $s_n := a_1 + a_2 + \cdots + a_n$.
- Convergence of series: $\sum_{n=1}^{\infty} a_n$ *converges* (to s) if $\{s_n\}$ converges (to s), and then s is the *sum*; otherwise $\sum_{n=1}^{\infty} a_n$ is *divergent*.
- Absolute / conditional convergence: $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if $\sum |a_n|$ converges; *conditionally convergent* if $\sum_{n=1}^{\infty} a_n$ converges but $\sum |a_n|$ diverges.
- Rearrangement: $\sum b_n$ is a rearrangement of $\sum_{n=1}^{\infty} a_n$ if there is a bijection $f : \mathbb{N} \to \mathbb{N}$ with $b_{f(n)} = a_n$.

Results

- Theorem (Terms of Convergent Series): if $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. [stated on slides; QRS proof in Chapter 4]
- Theorem (Algebraic Limit Theorem for Series): if $\sum a_n \to s$ and $\sum b_n \to t$, then $\sum ca_n \to cs$ for all $c \in \mathbb{R}$, and $\sum(a_n + b_n) \to s + t$. [stated on slides; QRS proof in Chapter 4]
- Theorem (Cauchy Criterion for Series): $\sum_{n=1}^{\infty} a_n$ converges iff $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n, m \in \mathbb{N}\ (n > m \geq N \implies |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon)$. [stated on slides; QRS proof in Chapter 4]
- Theorem (Alternating Series Test): if $\{b_n\}$ is decreasing and $b_n \to 0$, then $\sum(-1)^{n-1} b_n$ converges. [proved in L11]
- Theorem (Comparison Test): if $0 \leq a_n \leq b_n$ for all $n \in \mathbb{N}$ and $\sum b_n$ converges, then $\sum a_n$ converges. [proved in L12]
- Theorem (Absolute Convergence Test): if $\sum |a_n|$ converges, then $\sum a_n$ converges. [proved in L12]

Procedures

- Cauchy-tail-estimate template (replaces a Monotone-Convergence-Theorem argument): bound the target series' tail sum by a reference series' tail sum, via $0 \le a_n \le b_n$ or the triangle inequality, then invoke the Cauchy Criterion for Series.
- Alternating Series Test template: group consecutive terms of the tail into nonnegative pairs (using $\{b_n\}$ decreasing) to bound the tail sum by a single term b_{N+1}, then use $b_n \to 0$ together with the Cauchy Criterion for Series.

Chapter 5: Limit Points, Closures, and Closed Sets (Lectures 13–15)

Definitions

- Limit point: $a \in \mathbb{R}$ is a *limit point* of $A \subseteq \mathbb{R}$ if $\forall \varepsilon > 0\ \exists x \in A \setminus \{a\}\ |x - a| < \varepsilon$ (also called a *cluster point* or *accumulation point*).
- Isolated point: $a \in A$ is an *isolated point* of A if it is not a limit point of A.
- Closure: the closure of $A \subseteq \mathbb{R}$ is $\overline{A} := A \cup \{\text{limit points of } A\}$.
- Closed set: $F \subseteq \mathbb{R}$ is *closed* if it contains all its limit points (equivalently, F has no limit points outside F).

Results

- Theorem (Sequential Criterion for Limit Points): a is a limit point of A iff there is a sequence $\{a_n\}$ in $A \setminus \{a\}$ with $a_n \to a$. [proved in L13]
- Theorem (ε-Criterion for Closures): $a \in \overline{A}$ iff $\forall \varepsilon > 0\ \exists x \in A\ |x - a| < \varepsilon$. [stated on slides; QRS proof in Chapter 5]
- Theorem (Sequential Criterion for Closures): $a \in \overline{A}$ iff there is a sequence $\{a_n\}$ in A with $a_n \to a$. [stated on slides; QRS proof in Chapter 5]
- Theorem (Limit Points of Closures): a is a limit point of $\overline{A}$ iff a is a limit point of A. [proved in L14]
- Corollary (Closures of Closures): $\overline{\overline{A}} = \overline{A}$. [stated on slides; QRS proof in Chapter 5]
- Theorem (Closure Criterion for Closed Sets): F is closed iff $\overline{F} \subseteq F$. [proved in L15]
- Theorem (Sequential Criterion for Closed Sets): F is closed iff every convergent sequence contained in F has its limit in F. [proved in L15]

Procedures

- ε_n-trick: turn a $\forall \varepsilon$ statement into a witnessing sequence using a specific null sequence $\{\varepsilon_n\}$; conversely, extract a single ε-witness from a sequence limit by taking $x = a_N$.
- "Closure adds nothing new" template: given $x \in \overline{A}$ within ε of a, use the ε-Criterion for Closures to find $y \in A$ with $|y - x| < \min\{|x - a|, \varepsilon - |x - a|\}$ (so $y \neq a$), then bound $|y - a| \le |y - x| + |x - a| < \varepsilon$ via the triangle inequality.
- "Identify-and-substitute" template: prove a new Criterion by identifying both of its sides with sets already shown equal via earlier Criteria, e.g. identify {limits of convergent sequences in F} with $\overline{F}$, and "F closed" with "$\overline{F} \subseteq F$".

Chapter 6: Compact Sets (Lectures 16–17)

Definitions

- (Sequentially) compact set: $K \subseteq \mathbb{R}$ is *(sequentially) compact* if every sequence contained in K has a convergent subsequence with limit in K.

Results

- Theorem (Extrema of Closed and Bounded Sets): a nonempty, closed, and bounded set $K \subseteq \mathbb{R}$ has a maximum and a minimum. [proved in L16]
- Theorem (Sequential Compactness of Closed and Bounded Sets): $K \subseteq \mathbb{R}$ is closed and bounded iff every sequence in K has a convergent subsequence with limit in K. [proved in L16]
- Theorem (Nested Set Property): if $K_n \subseteq \mathbb{R}$ is nonempty, closed, and bounded for each $n \in \mathbb{N}$ with $K_n \supseteq K_{n+1}$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$. [proved in L17]

Procedures

- Extrema template: the Axiom of Completeness gives $\sup K$; the (unproved, exercise) fact $\sup A \in \overline{A}$ together with the Closure Criterion for Closed Sets ($\overline{K} \subseteq K$) place $\sup K \in K$, making it the maximum (symmetrically for the minimum, via $-K$).
- Sequential Compactness ("if", boundedness) template: prove the contrapositive by constructing an explicit unbounded witness sequence $a_n \in K$ with $|a_n| > M_n$ for an unbounded $\{M_n\}$, showing every subsequence of $\{a_n\}$ is unbounded, hence divergent (Boundedness of Convergent Sequences).
- Nested Set Property template: pick $a_n \in K_n$; Sequential Compactness of K_1 gives a convergent subsequence $\{a_{n_k}\}$; since $n_k \geq n$ for $k \geq n$, every tail of $\{a_{n_k}\}$ lies in K_n, so its (unchanged) limit lies in K_n by the Sequential Criterion for Closed Sets, for every $n \in \mathbb{N}$.

Dependency chain

- Axiom of Completeness $\implies$ ε-Criterion for Suprema and Infima.
- Axiom of Completeness $\implies$ Nested Interval Property.
- Axiom of Completeness $\implies$ Archimedean Property.
- Nested Interval Property $\implies$ Cauchy Criterion (sequences, "if" direction).
- Axiom of Completeness, ε-Criterion for Suprema and Infima $\implies$ Bolzano-Weierstrass Theorem.
- Bolzano-Weierstrass Theorem $\implies$ Divergence Criterion ("only if").
- Bolzano-Weierstrass Theorem $\implies$ Monotone Convergence Theorem (with the Monotone Convergence Lemma).
- Bolzano-Weierstrass Theorem $\implies$ Sequential Compactness of Closed and Bounded Sets.
- Subsequential Limits $\implies$ Divergence Criterion ("if").
- Subsequential Limits $\implies$ Sequential Compactness of Closed and Bounded Sets ("if", closedness direction).
- Subsequential Limits $\implies$ Nested Set Property.
- Cauchy Criterion (sequences) $\implies$ Cauchy Criterion for Series.
- Cauchy Criterion for Series $\implies$ Alternating Series Test, Comparison Test, Absolute Convergence Test.

- Sequential Criterion for Limit Points $\implies$ Sequential Criterion for Closures.
- Sequential Criterion for Closures $\implies$ Limit Points of Closures $\implies$ Closure Criterion for Closed Sets.
- Closure Criterion for Closed Sets $\implies$ Sequential Criterion for Closed Sets.
- Closure Criterion for Closed Sets, Axiom of Completeness $\implies$ Extrema of Closed and Bounded Sets.
- Sequential Criterion for Closed Sets, Bolzano-Weierstrass Theorem, Subsequential Limits, Boundedness of Convergent Sequences $\implies$ Sequential Compactness of Closed and Bounded Sets.
- Sequential Compactness of Closed and Bounded Sets, Subsequential Limits, Sequential Criterion for Closed Sets $\implies$ Nested Set Property.

B Notation

Throughout these notes $\mathbb{N} = \{1, 2, 3, \ldots\}$ excludes 0; when 0 must be included the course writes $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Sequences are written $\{a_n\}$ (curly braces, not the tuple notation (a_n) used in some textbooks), with full index range $\{a_n\}_{n=1}^{\infty}$ used once at first definition and abbreviated thereafter. Limits are stated value-first: $a = \lim a_n$ (equivalently $a_n \to a$), read "a equals the limit of a_n". A sequence has a property *eventually* if some *tail* $a_N, a_{N+1}, a_{N+2}, \ldots$ has that property. A series $\sum_{n=1}^{\infty} a_n$ is understood via its partial sums $s_n := a_1 + a_2 + \cdots + a_n$, and closure of a set A is written $\overline{A}$. Quantifiers $\forall, \exists$ and the connectives $\implies$, $\iff$ are used in run-on, colon-free chains (e.g. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n \in \mathbb{N}\ \ n \geq N \implies |a_n - a| < \varepsilon$); in written proofs these symbols must always be surrounded by words ("for every", "there exists", "which implies", "if and only if"), not left to stand alone in a sentence.

Number systems

Symbol	Meaning	Context / first appears
$\mathbb{N}$	natural numbers $\{1, 2, 3, \ldots\}$ (starts at 1)	L1
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$, nonnegative integers	L1
$\mathbb{Z}$	integers	L1
$\mathbb{Z}^+$, $\mathbb{Z}_0^+$	positive integers ($= \mathbb{N}$); nonnegative integers ($= \mathbb{N}_0$)	L1
$\mathbb{Z}^-$, $\mathbb{Z}_0^-$	negative integers; nonpositive integers	L1
$\mathbb{Q}$	rational numbers	L1
$\mathbb{Q}^+$, $\mathbb{Q}_0^+$, $\mathbb{Q}^-$, $\mathbb{Q}_0^-$	positive / nonnegative / negative / nonpositive rationals	L1
$\mathbb{R}$	real numbers	L1
$\mathbb{R}^+$, $\mathbb{R}_0^+$, $\mathbb{R}^-$, $\mathbb{R}_0^-$	positive / nonnegative / negative / nonpositive reals	L1

The "0" subscript consistently marks the version of a sign-restricted set that includes zero; the same pattern applies to $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.

Logic, sets and bounds

Symbol	Meaning	Context / first appears
$\forall$, $\exists$	for every; there exists	L2 (bounds), passim
$\Longrightarrow$, $\Longleftrightarrow$	implies; if and only if	L2 (bounds), passim
$A \subseteq \mathbb{R}$	A is a set of real numbers	L2
upper bound b / lower bound	$\forall a \in A\ a \leq b$ / $\forall a \in A\ a \geq c$ for A	L2
bounded above / below / bounded	A has an upper bound / lower bound / both	L2
$\sup A$, $\inf A$	supremum (least upper bound); infimum (greatest lower bound) of A	L2
$\max A$, $\min A$	maximum; minimum of A (an element of A that is also $\sup A$ / $\inf A$)	L2
$\sup \emptyset = -\infty$	every real number is an upper bound for $\emptyset$, so $\emptyset$ has no supremum; the slides write $\sup \emptyset = -\infty$	L2
$\sup A = \infty$	convention: A not bounded above	L2
$\forall \varepsilon > 0\ \exists a \in A\ s - \varepsilon < a$	ε-Criterion for Suprema and Infima ($s = \sup A$)	L2
$I_n = [a_n, b_n]$	closed and bounded interval, indexed for nested-interval arguments	L3
$I_1 \supseteq I_2 \supseteq \cdots$	nested sequence of closed and bounded intervals	L3

Sequences and series

Symbol	Meaning	Context / first appears
$\{a_n\}$	sequence, written with curly braces; short for $\{a_n\}_{n=1}^{\infty}$	L4
$a_1, a_2, a_3, \ldots$	terms of a sequence listed directly	L4
$a = \lim a_n$ (or $\lim_{n\to\infty} a_n$)	limit, value written first; equivalently $a_n \to a$	L4
$\lvert a_n - a \rvert < \varepsilon$	absolute value / distance-to-limit, workhorse of every ε-proof	L4
tail $a_N, a_{N+1}, a_{N+2}, \ldots$	used to define "eventually" (some tail has the property)	L4
$M > 0$, $-M \leq a_n \leq M$	bounded sequence	L4
$\{a_{n_k}\}$	subsequence of $\{a_n\}$ ($n_1 < n_2 < \cdots$); *proper* if $n_k \neq k$ for some k	L7
$\{a_{n_k}\}_{k=n}^{\infty}$	tail of a subsequence	L17
$\sum a_n$, $\sum_{n=1}^{\infty} a_n$	series; sum may also start at $n = 0$	L10
$s_n := a_1 + \cdots + a_n$	partial sum of a series	L10
$s = \sum_{n=1}^{\infty} a_n$	sum of a convergent series	L10
$\sum \lvert a_n \rvert$ convergent	absolute convergence	L12
$\sum a_n$ conv. but $\sum \lvert a_n \rvert$ div.	conditional convergence	L12
$f : \mathbb{N} \to \mathbb{N}$ bijection, $b_{f(n)} = a_n$	rearrangement of a series, giving $\sum b_n$	L12

Sets of real numbers

Symbol	Meaning	Context / first appears
$\forall \varepsilon > 0\ \exists x \in A \setminus \{a\}\ \lvert x-a\rvert < \varepsilon$	a is a limit point (cluster / accumulation point) of A	L13
$\overline{A}$	closure of A: $A \cup \{\text{limit points of } A\}$	L14
$\overline{\overline{A}}$	closure of the closure (equals $\overline{A}$)	L14
$\forall \varepsilon > 0\ \exists x \in A\ \lvert x-a\rvert < \varepsilon$	ε-Criterion for Closures: $a \in \overline{A}$	L14
$F \subseteq \mathbb{R}$	letter reserved for a closed set (vs. A for a general set)	L15
F closed	F contains all its limit points; $\overline{F} \subseteq F$	L15
(sequentially) compact	every sequence in the set has a convergent subsequence with limit in the set	L16
$-K := \{-a : a \in K\}$	reflection of a set K	L16
$K_1 \supseteq K_2 \supseteq \cdots \supsetneq \emptyset$	nested sequence of nonempty closed and bounded sets (the slides write $\supsetneq \emptyset$ for "nonempty")	L17

C Exam & Admin Information

Midterm Test

Item	Details
Coverage	Lectures 1–17: the real numbers (Axiom of Completeness) → sequences and series → sets of real numbers (limit points, closures, closed sets, compact sets). Lectures 18–20 (function limits, continuity) are *not* examined.
Format	4 questions on 8 printed pages, answered in the boxes provided (use the facing page for more space). On the sample paper each question carries 6 marks (total 24) and the time allowed is 60 minutes.
Materials	Closed book; no help sheet. The sample paper allows calculators.
Citing results	"You may cite any results proved in the lectures and tutorials, unless you are asked by the question to prove the result." Questions often forbid a named shortcut (e.g. "Do not use the ε-Criterion for suprema and infima").
Question styles	True/false statements (no justification required) and "write down the definition of ..., and use the definition to show ..." proofs.
Weighting	Not stated in the materials available to QRS; check the official course outline.

How Proofs Are Marked

The course states three writing expectations. They apply to every proof you hand in.

1. **Complete sentences.** Write "b is an upper bound for A if for all $a \in A$, $a \le b$.", not "$a \in A$, $a \le b$."
2. **Logical connectives join related statements.** Write "$\lvert a\rvert \le b \implies -b \le a \le b$", not two unconnected formulas.
3. **Inference is signposted** with "therefore", "hence", "thus", "so" or $\therefore$: write "$\lvert a\rvert \le b$. Therefore $-b \le a \le b$."

In these notes, every **Proof** paragraph is written to this standard. The **Idea** paragraph that precedes it is scratch work and would not be handed in.

Resources

- Main reference: the MH3100 lecture slides (A/P Chua Chek Beng, AY2026/27), Lectures 1–17 for the Midterm.
- Textbook: S. Abbott, *Understanding Analysis* (the slides follow its notation and theorem names).
- Tutorials 1–5 and their official hints; the sample midterm.
- QRS@NTU materials: enhanced tutorial solutions, the Midterm Memory Sheet, and practice papers at qrsntu.org.

Revision Checklist Before the Midterm

☐ State every definition in Lectures 1–17 from memory, with the quantifiers in the right order.

☐ Prove $s = \sup A$ from the definition (upper bound + least), and again with the ε-Criterion.

☐ Write an ε–N proof for a limit, including the Idea (find N) and the clean Proof.

☐ Use the definition of a Cauchy sequence directly (no Cauchy Criterion, no Algebraic Limit Theorem).

☐ Know one counterexample for each true/false trap in Tutorials 2–5.

☐ Find limit points, isolated points and closures of standard sets; decide closedness and compactness.

☐ Reread the three writing expectations above and check your proofs against them.

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